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Making Maps and Keeping Logs

Quantum Gravity from Classical Viewpoints

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ACTA
UNIVERSITATIS
UPSALIENSIS
UPPSALA
2009

ISSN 1651-6214
ISBN 978-91-554-7510-9
urn:nbn:se:uu:diva-100504

Dissertation presented at Uppsala University to be publicly examined in Polhemsalen, Ångström Laboratory, Lägerhyddsvägen 1, Uppsala, Wednesday, May 20, 2009 at 13:15 for the degree of Doctor of Philosophy. The examination will be conducted in English.

Abstract
Johansson, N. 2009. Making Maps and Keeping Logs. Quantum Gravity from Classical Viewpoints. Acta Universitatis Upsaliensis. *Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology* 638. 140 pp. Uppsala. ISBN 978-91-554-7510-9.

This thesis explores three different aspects of quantum gravity. First we study D3-brane black holes in Calabi-Yau compactifications of type IIB string theory. Using the OSV conjecture and a relation between topological strings and matrix models we show that some black holes have a matrix model description. This is the case if the attractor mechanism fixes the internal geometry to a conifold at the black hole horizon. We also consider black holes in a flux compactification and compare the effects of the black holes and fluxes on the internal geometry. We find that the fluxes dominate. Second, we study the scalar potential of type IIB flux compactifications. We demonstrate that monodromies of the internal geometry imply as a general feature the existence of long series of continuously connected minima. This allows for the embedding of scenarios such as chain inflation and resonance tunneling into string theory. The concept of monodromies is also extended to include geometric transitions: passing to a different Calabi-Yau topology, performing its monodromies and then returning to the original space allows for novel transformations. All constructions are performed explicitly, using both analytical and numerical techniques, in the mirror quintic Calabi-Yau. Third, we study cosmological topologically massive gravity at the chiral point, a prime candidate for quantization of gravity in three dimensions. The prospects of this scenario depend crucially of the stability of the theory. We demonstrate the presence of a negative energy bulk mode that grows logarithmically toward the AdS boundary. The AdS isometry generators have non-unitary matrix representations like in logarithmic CFT, and we propose that the CFT dual for this theory is logarithmic. In a complementing canonical analysis we also demonstrate the existence of this bulk degree of freedom, and we present consistent boundary conditions encompassing the new mode.

Keywords: String theory, black holes in string theory, Calabi-Yau geometry, flux compactifications, three-dimensional gravity

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ISSN 1651-6214
ISBN 978-91-554-7510-9
urn:nbn:se:uu:diva-100504 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-100504)

Till morfar

List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Ulf H. Danielsson, Niklas Johansson, Magdalena Larfors, Martin E. Olsson and Marcel Vonk, 4D black holes and holomorphic factorization of the 0A matrix model, *J. High Energy Phys.* JHEP10(2005)046. hep-th/0506219.

II Ulf H. Danielsson, Niklas Johansson and Magdalena Larfors, Stability of flux vacua in the presence of charged black holes, *J. High Energy Phys.* JHEP09(2006)069. hep-th/0605106.

III Ulf H. Danielsson, Niklas Johansson and Magdalena Larfors, The world next door - Results in landscape topography, *J. High Energy Phys.* JHEP03(2007)080. hep-th/0612222.

IV Diego Chialva, Ulf H. Danielsson, Niklas Johansson, Magdalena Larfors and Marcel Vonk, Deforming, revolving and resolving - New paths in the string theory landscape, *J. High Energy Phys.* JHEP02(2008)016. arXiv:0710.0620 [hep-th].

V Daniel Grumiller and Niklas Johansson, Instability in cosmological topologically massive gravity at the chiral point, *J. High Energy Phys.* JHEP07(2008)134. arXiv:0805.2610 [hep-th].

VI Daniel Grumiller, Roman Jackiw and Niklas Johansson, Canonical analysis of cosmological topologically massive gravity at the chiral point, Contribution to “Fundamental Interactions - A Memorial Volume for Wolfgang Kummer”, to appear in World Scientific. arXiv:0806.4185 [hep-th].

VII Daniel Grumiller and Niklas Johansson, Consistent boundary conditions for cosmological topologically massive gravity at the chiral point *IJMPD*, Vol. **17**, Nos. 13&14 (2008), 2367-2372. arXiv:0808.2575 [hep-th].

Contents

1. Introduction

Февраль. Достать чернил и плакать.
Boris Pasternak

February. On the whole not a good month in Uppsala. Outside it is cold and dark, biking paths are bumpy and hostile, and an unholy alliance of calicivirus and the Eurovision Song Contest tryouts sweeps over the country. And time has come. Time to put four and a half years of graduate studies into a thesis. No choice but to get the ink.

No choice.

Choices and parameterizing the possible

Choices are what make life interesting. Without them the world, including ourselves, would be just a ready-made piece of art for us to perceive, but not to affect or take part in. Still, there are many important aspects of the universe not affected by choice. The sunlight always heats the sea sand, and the apple always falls when the twig breaks. It cannot choose to fly.

Physics is concerned with the part of the world that has no choice. Its aim is to *predict*. Given the status of a system at a certain time, any self-respecting physicist will try to tell you what you can predict, and how to do it. The predictions are made by physical *laws*. Newton's gravitational law is an excellent example. It states that two massive bodies will attract each other with a force proportional to the product of their masses and inversely proportional to the square of their distance:

$$F = G_N \frac{m_1 m_2}{r_{12}^2}.$$

The constant of proportionality G_N is called *Newton's constant*.

A certain law can predict the time evolution of many different-looking systems. Newton's gravitational law is applicable to the motion of planets in the solar system as well as to a downhill skier. The motion of the planets or the skier are examples of different *solutions* to the same *theory*. As physicists, we use the same laws and the same mathematics to describe the two systems, but the initial conditions are very different. In one case we have a set of big balls of frozen gas, in the other a small but brave piece of flesh standing on the top of a slippery slope. In this way, physical laws can be viewed as parameterizing the possible, the parameters being the initial conditions. Often there are parameters in the laws themselves — in this case Newton's constant. Formulating physical laws allows us to understand the choice-less part of the universe.

The laws of physics have different ranges of validity. During the last millennia new laws have been found, old ones refined, and apparently unrelated laws have been understood to be different manifestations of the same, more general law. This unification is one of the greatest achievements of physics. As of today, all experiments can be understood by using tools from two frameworks: *General Relativity* and *Quantum Field Theory*.

In the course of this development toward more fundamental and more widely applicable laws, mathematical beauty and philosophical appeal have been important complements to experiments as guiding principles. For instance, when Albert Einstein formulated his theory of General Relativity, he was lead to it by consistency and beauty — not by unexplained experiments.

The fact that a human(?) mind can *guess* or "derive" the laws of nature may suggest that, at a basic level, there is only one way to formulate them, and that we are approaching that level. In Einstein's own words:

> *What really interests me is whether God had any choice in the creation of the world.*

If there is only one consistent way to formulate physical laws describing a universe at least remotely similar to ours, then these laws parameterize the possible in a much stronger way. Namely, letting the parameters be both the initial conditions and the parameters in the laws themselves, they parameterize what is possible in *any* world.

The Quantum Field Theory that describes (much of) our universe perhaps does not look as unique as General Relativity. It is called the *Standard Model* and it requires specification of gauge groups and particle content. After these are fixed to the observed values it involves 28 additional continuous parameters which must be empirically determined. This is to say that the theory would make logical sense for any values of these parameters.

It has been (and still is) a dream of many physicists to explain or derive gauge groups, field content and the numerical values of these parameters from "first principles".

Let us pause here and ask the following question: What does it mean if it is possible to derive all Standard Model parameters and Newton's constant? In such a scenario, a world with physical laws similar to ours would be logically impossible if the laws were not *identical* to ours. This *does not* mean, however, that *the world itself* would be identical to ours. Indeed, a physical system is described by the laws and the initial conditions. Two worlds with identical laws can be as different as a solar system and a downhill skier. Slightly more technically put, while the theories are identical, the specific solutions describing the worlds need not be. Also, the hypothetical uniqueness of the Standard

Model says *a priori* nothing about the choice-full part of worlds described by it.

Showing uniqueness of the Standard Model would also imply a puzzle. Despite that there are many solutions, many possible initial conditions, the values of the Standard Model parameters heavily influence the properties of the world. For many sets of values, life would not be possible. It is of course an enormous task to judge whether life can evolve or not, but there are some solid claims that can be made. For instance, raising the mass of the up quark would eventually make all protons unstable. Raising the electromagnetic coupling constant would destabilize nuclei. It is hard to imagine formation of life under any of these circumstances, and there are many more examples. Why would logic demand that these parameters are fine-tuned to make up a world suitable to host life?

The brilliant success of unification of physical laws, using logic and beauty, has thus put us before philosophical questions of immense depth. With General Relativity in one hand and the Standard Model in the other we are amazed with the simplicity, but uneased by the free parameters. But if the parameters are fixed, we are left with the puzzle of the friendliness of the resulting universe.

Can a further unification shed light on these questions? What happens if we can unify General Relativity and Quantum Field Theory? A lot happens.

Quantum gravity and cosmic LEGO

To consistently quantize gravity has become something of a holy grail of theoretical physics, and is therefore a subject of intense study and debate. The reasons for its status are manifold. Apart from the uniqueness question described above, understanding quantum gravity is likely to change radically our understanding of space and time. It will also give us tools to study the birth and death of our universe and to understand mysterious objects such as black holes. Furthermore quantizing gravity is a necessary requirement for having a theory of everything.

Because of the high energies involved, quantum gravity is haunted by a chronic lack of experimental and observational results. Physicists therefore must look to other principles, like consistency, correct low-energy limits and symmetry, to make progress. This is tricky business, but as exemplified by Einstein above, by no means impossible.

On the other hand, many beautiful explanations have turned out to be wrong when subjected to experimental tests. Famous examples include Kepler's trying to explain the sizes of planetary orbits by Platonic solids and string theory as a theory of the strong interaction.

The poor experimental connection has caused some mild (and some not so mild) skepticism toward studies of quantum gravity. Perhaps adding to this skepticism is the fact that the preliminary results are mind-blowing. The by far most developed candidate for quantum gravity is string theory. In the earliest developments, string theory was intended as describing the strong force. As such it was abandoned because of the discovery of Quantum Chromodynamics (QCD), and also because of some embarrassing features. One of these embarrassing features — a spin two particle — turns into a lump of gold if viewed from another angle. Any theory of quantum gravity needs a spin two particle: the graviton, i.e., the force quantum of gravity.

According to string theory everything in the universe is made up of the same basic material: string. The string is like a rubber band with tension of the Planck scale. Different particles are described as different vibrational modes of the string. One of the vibrational patterns is interpreted as the graviton, whose existence puts gravity on the same footing as all other fundamental forces.

But string theory does not stop at unifying all forces and matter. It also strongly suggests that there is really only one way[1] to build the universe. At least if you want to build it from strings. And if we want it to contain quantum mechanics and gravity, we do not know of any other way to build it. For instance there are *no free parameters* in string theory. Everything is instead fixed from very weak assumptions. Without completely molesting the truth, it is fair to say that assuming that

- strings are the basic degrees of freedom
- the world is stable (at least on short terms)
- the world is rotationally invariant

leaves you with only one candidate theory. This fact has lead to an almost religious belief in string theory as *the* fundamental theory, answering Einstein's question in the negative.

In string theory, there is even no choice for the number of spacetime dimensions: there must be ten of them. Whether this is a feature or a bug depends on your point of view. Indeed, extreme pessimists would say that string theory, when compared to the real world in the simplest of aspects, failed, and failed miserably. Let us not join that crowd however, but instead try to fit the world around us into the newly discovered framework.

One way to get rid of six extra dimensions is to curl them up real tight. The surface of a straw is two-dimensional. One dimension along the straw, and one around it. If the world would be two-dimensional, and curled up into a straw, then big things would not care about the curled up direction[2]. For instance, two hands clutching the straw could not pass each other on different sides of the straw.

Furthermore, in quantum mechanics the momentum of an object is proportional to how fast its structure varies in space. Anything varying over a very small distance will therefore carry a lot of momentum, and thus a lot of energy.

[1]There are actually five theories, but in the modern view they are merely different aspects of the same, underlying theory.

[2]The curled up dimension is often referred to as "compact" or "internal".

So if people living on a thin straw do not have high energy particle accelerators available, they will not be able to probe the compact direction, and they will think that their world only has one dimension. String theory suggests that we are much like the straw people.

Thus we propose that at every point in spacetime there are really six secret directions in which we could move provided we had the energy. With the present funding for basic research we do not, so we perceive the world as four-dimensional. From our lower-dimensional perspective physics is excellently described by General Relativity and Quantum Field Theory. The idea of compact dimensions is not new, but was suggested by Kaluza and Klein in the 1920s in an ingenious attempt to unify gravity and electromagnetism.

There is a major difference between curling up or *compactifying* six dimensions and curling up only one. One direction can only be curled up into a circle while six of them can be compactified in an infinite number of ways, and the four-dimensional physics that results depends crucially on the geometry of the curling. When string theory is compactified, the geometry[3] of the compact dimensions determines the whole defining data set of the resulting four-dimensional General Relativity and Quantum Field Theory: gauge groups, particle content, coupling constants (including Newton's constant!), masses and so on.

Not any compact geometry is allowed by string theory. It must solve the equations of motion and fulfill some consistency requirements. A convincing amount of evidence suggests that there are many solutions. Typical numbers cited in the literature are 10^{500} and ∞.

This is a remarkable insight. In the quest for quantum gravity we found a "unique" theory. It is not a surprise that this theory has many solutions; it would be a very boring theory otherwise. In some solutions part of spacetime is compact. The shape of the compact part is an initial condition on the solution. But different choices for the compact part look like different *theories* from a four-dimensional perspective. Couplings, masses and the like depend on the geometry. In this language, the parameters in the four-dimensional theory are really initial conditions! Thus, the parametrization has been changed from "initial conditions and parameters in the theory" to just "initial conditions". Put differently, string theory gives Einstein the answer

> *To use strings was compulsory and they obey strict rules. They are still very versatile though. Not unlike LEGO.*

Note also that even if the theory is devoid of free parameters, string theory avoids the puzzle of why the "unique" theory would contain life. At low-energy, there are many solutions of string theory. Some will contain life, and some will not.

[3] A note to the experts. The geometry alone is not enough to fully determine the four-dimensional physics. Data such as the presence of fluxes, branes and orientifold planes also contributes.

The set of four-dimensional solutions to string theory has been named the *string theory landscape*. Identifying and exploring this landscape is a huge technical task, and much effort has been put into it. A small part of this effort is the basis for some of the later chapters of this thesis. The string theory landscape provides, for the first time in the history of science, a quantitative framework for parameterizing the set of possible low energy effective theories. From a four-dimensional point of view, string theory is a theory of theories.

Let us end this string theory commercial with noting that the excitement over the landscape expressed above is not shared by the whole physics community. The main logical reason for this is our insofar incomplete understanding of string theory. We do know a lot at the perturbative level but the full, non-perturbative nature of the theory remains to some extent elusive. Therefore, it is fair to say that we cannot be completely sure that the solutions we find with our present tools correspond to solutions to the full theory. Maybe there are consistency requirements not yet discovered that rule out many (all but one?) candidate solutions. Moreover, despite a lot of work, an embedding of the Standard Model into string theory, including all parameters, is yet to be achieved. There is also an emotional reason for "landskepticism". Many physicists would just really like the Standard Model to be derivable. If this is a repetition of the anthropocentric mistake[4], only time will tell.

String theory is not the only framework in which quantum gravity can be analyzed. (Even if it is possible that string theory *contains* any framework in which it can be analyzed.) It is also important and interesting to find technically simple toy-models that still contain interesting aspects of quantum gravity.

One way to make things simpler is to reduce the number of dimensions. For instance, taking away two of our spatial dimensions renders gravity trivial when left on its own. Coupling gravity to other fields like the dilaton however, leads to interesting physics, e.g., black holes and Hawking radiation. Still, the models admit quantization by standard canonical or path integral approaches.

In one dimension higher the situation is less clear, but the potential upside is huge. Three-dimensional gravity contains black holes that are very similar to four-dimensional ones. As opposed to their two-dimensional cousins they have spin and a good analog of horizon area. If the microscopic structure of these black holes is understood, it is likely to tell us a great deal about the microscopic structure of black holes in our universe. The last chapter in this thesis is concerned with gravity in three dimensions.

[4]The anthropocentric mistake is to fallaciously put ourselves in the center of things. Believing that the earth, the sun or the Milky Way is the center of the universe are examples. Perhaps also believing that our Standard Model is the only possibility.

The papers

The research presented in this thesis addresses some (quite diverse) aspects of quantum gravity. It is contained in seven papers and I would like to devote a couple of sentences to the contents of each.

In paper I, we study a particularly simple limit of a string theory compactification: a *conifold* limit. In particular we study a four-dimensional black hole in this theory. By combining two previous results we show that it is possible to understand the thermodynamics of this black hole as the thermodynamics of a much simpler system: a *matrix model*.

Paper II studies a similar black hole in a similar compactification. The internal geometry of this compactification is kept in place by *fluxes*. It turns out that not only the fluxes, but also the black hole affects the internal geometry. We try to find out which effect is the stronger one. If the black hole effect is stronger, one could imagine that the world looks very different close to the horizon of a black hole. Perhaps black holes can even destabilize the compactification! We find that the effect from the fluxes is stronger and that the compactification will be stable even in the presence of a black hole.

The two papers III and IV take us on excursions in the string theory landscape. We study a specific corner, and show that in this corner a generic feature is long sequences of closely related four-dimensional effective theories. We explain how it is possible to continuously move between these theories. The papers make use of quite complicated technical machinery to describe the compact geometry.

Finally, papers V, VI and VII deal with a specific form of gravity in three-dimensions known as *topologically massive gravity*. This theory contains both gravitons and black holes. The gravitons carry a mass originating from a so-called Chern–Simons term in the action. The mass is a parameter that can be varied freely. Some of these gravitons seem to have negative energy, meaning that the theory is unstable. However, for a certain value of the graviton mass parameter the negative energy gravitons seemingly disappear. If they do (or if they do not, but can be consistently ignored) topologically massive gravity would be an ideal candidate for trying to quantize gravity in three dimensions.

We show in V that the gravitons are really there, even for the special value. However they have different asymptotic behavior. Paper VI explicitly counts, without any approximations, the degrees of freedom in the theory. We find exactly one graviton. The short note VII demonstrates that the different asymptotic behavior found in V in itself is not a problem, and leads to no inconsistencies.

Outline of the thesis

The thesis is divided into two parts. The first three chapters review standard, textbook material, needed for understanding on a technical level the questions addressed in the papers. The level is intended to be such that a beginning graduate student in theoretical physics catches the main ideas without consulting further literature. Chapter 2 gives an introduction to *Conformal Field Theory* or CFT — a subject of great importance both in string theory and three-dimensional gravity. *String theory* is introduced in Chapter 3, and we immediately put CFT to good use. Focusing on the bosonic string we learn why the number of spacetime dimensions is constrained. Superstrings are treated less thoroughly, focusing on the resulting low-energy effective theory: *supergravity*. Concluding the prerequisites part, Chapter 4 describes how type IIB superstring theory compactifications work. Dimensional reduction and moduli stabilization are explained.

The second part is devoted to putting the different papers into context. After completing a chapter, the reader is encouraged to read the corresponding papers, which are reprinted at the end of the thesis. Papers I and II, both dealing with black holes are described in Chapter 5, and the two landscape papers III and IV in Chapter 6. The three-dimensional gravity tale of V, VI and VII is told in Chapter 7.

Part I:

Prerequisites

2. Conformal field theory

So close, no matter how far...
James Hetfield

Conformal field theory (CFT) lies at the heart of two main topics of this thesis. It is the technical framework of string theory, and its role in three-dimensional gravity can hardly be overestimated. In our story, we will need several CFT results. It is the purpose of this chapter to introduce these and to provide some intuition for the non-expert. The CFTs relevant to string theory and three-dimensional gravity are both two-dimensional. We therefore put the emphasis on such theories in this chapter. They display many features not shared by their higher-dimensional counterparts.

There are several nice introductions to CFT available. Almost all material presented below is contained in two excellent reviews, Ref. [Sch96] by A.N. Schellekens and Ref. [Car05a] by S. Carlip, and in the big wonderful yellow book [DFMS] by P. Di Francesco, P Mathieu and D. Sénéchal.

2.1 Conformal invariance

Conformal symmetry is a concept that is intimately connected to distances and angles, but that does not require any deformation of the metric; indeed it plays its most important part in non-gravitational theories with fixed metric. To define conformal invariance in an intuitive way however, it is useful to first discuss its cousin Weyl invariance which is formulated in terms of the metric. In some (quite specific) sense, conformal invariance is the part of Weyl invariance that can be translated into theories with fixed metric.

2.1.1 Weyl or conformal

Consider a field theory containing a set of fields ϕ living on a $(d+1)$-dimensional manifold with coordinates x^α and metric $\gamma_{\alpha\beta}(x)$. This theory is said to be *Weyl invariant* if its action is left unchanged under the Weyl rescaling

$$\gamma_{\alpha\beta}(x) \to e^{\omega(x)}\gamma_{\alpha\beta}(x). \tag{2.1}$$

The transformation above is a local scaling of lengths. All angles are preserved, but distances are transformed in an x-dependent way. Infinitesimal things, i.e, small "lumps of field", are made bigger or smaller dependent on location, but their shape is unaltered.

Since the stress-energy tensor $T_{\alpha\beta}$ is defined as the functional derivative of the action with respect to the metric, Weyl invariance leads directly to off-shell tracelessness $T^\alpha_\alpha = 0$. To see this, vary the action by the infinitesimal version of (2.1) $\delta\gamma_{\alpha\beta} = \omega(x)\gamma_{\alpha\beta}$, yielding

$$0 = \delta S = -\frac{1}{2}\int d^{(d+1)}x\sqrt{-\gamma}T^{\alpha\beta}\delta\gamma_{\alpha\beta} = \int d^{(d+1)}x\sqrt{-\gamma}T^\alpha_\alpha\omega(x). \quad (2.2)$$

Since this is true for any function $\omega(x)$, we must have $T^\alpha_\alpha = 0$.

The explicit form of the metric is subject to coordinate changes. If our Weyl invariant theory also has general coordinate invariance, changing coordinates $x^\alpha \to x^\alpha + \xi^\alpha$ transforms the metric according to

$$\gamma_{\alpha\beta}(x) \to \gamma_{\alpha\beta}(x) + \delta\gamma_{\alpha\beta}(x) = \gamma_{\alpha\beta}(x) + \nabla_{(\alpha}\xi_{\beta)} \quad (2.3)$$

if ξ is infinitesimal. Note that, contrary to Weyl rescalings, not only the metric transforms under coordinate changes. All fields do, with transformation laws according to their spin.

Under some coordinate changes, the metric transforms by a Weyl rescaling. All Weyl rescalings cannot be achieved this way. For the ones that can, however, we can define a transformation that combines a change of coordinates and a Weyl rescaling that cancels the transformation of the metric. Under such a transformation the fields transform, but the metric stays the same! Thus, they make sense even for theories with fixed metrics. These transformations are the conformal transformations, and invariance under these is the defining property of conformal field theories.

Before analyzing conformal field theory in two dimensions in some detail, let us make a note that applies to any dimension. In a world described by a CFT, two states that differ by a conformal transformation are as similar as two states differing by a rotation in a world (like ours) described by a Lorentz invariant theory. Since conformal transformations can be used to scale distances, this means that *there is no notion of length* in CFTs. The transformation $x \to x' = 2x$, for instance, changes a scalar field ϕ according to $\phi(x) \to \phi'(x') = \phi(x'/2)$, but leaves the metric identical in the x and x' coordinates. This has fundamental implications when the field theory is quantized. If conformal invariance is left unbroken by quantum effects, there is no scale in the theory. This means that coupling constants are scale independent and do not get renormalized.

2.2 Conformal field theory in two dimensions

In this section we describe some properties of two-dimensional CFTs at the quantum level. This topic is completely dependent of that quantum mechanical effects *break* conformal invariance. Namely, any unitary two-dimensional CFT whose symmetries remain unbroken after quantization is trivial!

The breaking of conformal invariance is parameterized by two single positive numbers: the left- and right-moving *central charges*. These numbers will play an important role in both string theory and in applications to three-dimensional gravity. In string theory they determine the critical dimension and in three-dimensional gravity they measure the number of states that we ultimately would like to interpret as black hole microstates. The relation between the central charges and the asymptotic number of states is expressed by the Cardy formula presented last in the section.

The way we investigate the breaking of conformal symmetry at the quantum level is to find the charges that generate the symmetry. These will be integrals of the field variables in the theory. Promoting the fields to operators will then possibly induce ordering ambiguities, which alter the symmetry algebra. In the following subsections we first explore the conformal group in two dimensions. Then we find the generators of the conformal symmetry and study their quantum realization in a specific all-important example.

2.2.1 Conformal transformations in two spacetime dimensions

If $D = 2$ the conformal group turns out to be infinite dimensional. This makes two-dimensional CFT quite different from higher dimensions where the group is finite dimensional. Let us find the group of conformal transformations and the associated algebra.

For our applications keeping a cylindrical topology in mind will be most useful. Consider therefore a two-dimensional space with coordinates τ and σ and the fixed flat metric $ds^2 = \eta_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = -d\tau^2 + d\sigma^2$, where $\sigma \sim \sigma + \pi$, In light-cone variables $\sigma^\pm = \tau \pm \sigma$ the metric is $ds^2 = -d\sigma^+ d\sigma^-$.

Let us find the infinitesimal conformal transformations. For the metric to transform by a Weyl rescaling under the transformation

$$\sigma^\alpha \to \sigma^\alpha + \xi^\alpha(\sigma) \tag{2.4}$$

we must require that $\delta\gamma_{\alpha\beta}$ in (2.3) is proportional to $\gamma_{\alpha\beta}$. In light-cone coordinates this translates to

$$\begin{aligned} \delta\gamma_{++} &= 2\partial_+\xi_+ = -\partial_+\xi^- = 0 \\ \delta\gamma_{+-} &= \partial_-\xi_+ + \partial_+\xi_- = -\frac{1}{2}\left(\partial_-\xi^- + \partial_+\xi^+\right) = -\omega(\sigma^+, \sigma^-) \\ \delta\gamma_{--} &= 2\partial_-\xi_- = -\partial_-\xi^+ = 0. \end{aligned} \tag{2.5}$$

We see immediately that the allowed transformations are parameterized by two arbitrary real functions of one variable: $\xi^-(\sigma^-)$ and $\xi^+(\sigma^+)$. The global form of these transformations are $\sigma^+ \to f(\sigma^+)$ and $\sigma^- \to g(\sigma^-)$ for any functions f and g.

The conformal transformations are generated by the Lie-algebra elements

$$\xi^\alpha \partial_\alpha = \xi^+(\sigma^+)\partial_+ + \xi^-(\sigma^-)\partial_-. \tag{2.6}$$

To specify the arbitrary functions $\xi^+(\sigma^+)$ and $\xi^-(\sigma^-)$ one needs infinitely many parameters. Therefore the corresponding Lie algebra is infinite dimensional. One choice of basis, suitable for our periodic σ is formed by the Fourier modes of $\xi^\pm$:

$$L_n = \frac{1}{2}e^{2in\sigma^+}\partial_+ \qquad \bar{L}_n = \frac{1}{2}e^{2in\sigma^-}\partial_-. \tag{2.7}$$

satisfying the algebra

$$[L_m, L_n] = i(m-n)L_{m+n} \tag{2.8}$$

and similarly for $\bar{L}_n$. The algebra in equation (2.8) is the celebrated Virasoro algebra. Thus, the conformal algebra in two dimensions consists of two commuting copies of the Virasoro algebra.

2.2.2 An infinite set of conserved quantities

Its algebra being infinite dimensional, conformal symmetry in two dimensions leads to an infinite set of conserved quantities. More specifically, each Fourier mode in (2.7) can be seen as a generator of a one-parameter symmetry, each with its associated Noether charge. Instead of using the Noether procedure we identify these quantities somewhat indirectly, following the book [GSWa] by M.B Green, J.H. Schwarz and E. Witten.

From general coordinate invariance it follows that the stress energy tensor is conserved, $\nabla_\alpha T^{\alpha\beta} = 0$. In light cone coordinates and with our flat metric conservation of a current $\nabla_\alpha J^\alpha = 0$ is just $\partial_+ J_- + \partial_- J_+ = 0$. Thus we have for $\beta = +$

$$\partial_+ T_{--} + \partial_- T_{+-} = 0. \tag{2.9}$$

The $\beta = -$ equation is similar but with $(+ \leftrightarrow -)$. Now our theory also enjoys conformal invariance, and we saw that this implies tracelessness of $T_{\alpha\beta}$. In our coordinate system

$$T^\alpha_\alpha = -2T_{+-} = 0. \tag{2.10}$$

Therefore the conservation of energy momentum reduces to

$$\partial_+ T_{--} = 0, \tag{2.11}$$

and similarly for $(+ \leftrightarrow -)$. Hence T_{--} is a function only of σ^-. But this means that we can, given *any function* $f(\sigma^-)$ construct a conserved current J^f_α:

$$J^f_- = fT_{--} \qquad J^f_+ = 0. \tag{2.12}$$

The corresponding Noether charge is

$$Q_f = \int_0^\pi d\sigma J^\tau_f = -\int_0^\pi d\sigma J^f_- = -\int_0^\pi d\sigma f T_{--}. \tag{2.13}$$

Exactly the same game can be played with the component T_{++}.

Thus we have found two sets of conserved charges, each parameterized by an arbitrary function of one variable. That these are conserved depends crucially on conformality. In fact they are the Noether charges of the conformal transformations.

With appropriate identification of the function, they generate the conformal transformations (2.6):

$$\begin{aligned} Q[\xi^-] &= \int_0^\pi d\sigma \xi^- T_{--} \quad \text{generates} \quad \xi^-\partial_- \quad \text{and} \\ Q[\xi^+] &= \int_0^\pi d\sigma \xi^+ T_{++} \quad \text{generates} \quad \xi^+\partial_+. \end{aligned} \tag{2.14}$$

For $\xi^\pm = e^{2in\sigma^\pm}/2$ we obtain the Virasoro generators L_n and $\bar{L}_n$

$$\begin{aligned} L_n &= \frac{1}{2}\int_0^\pi d\sigma e^{2in\sigma} T_{--}(\tau = 0) \\ \bar{L}_n &= \frac{1}{2}\int_0^\pi d\sigma e^{-2in\sigma} T_{++}(\tau = 0). \end{aligned} \tag{2.15}$$

We chose to perform the integration at $\tau = 0$. For a given field theory with a given action, we can now express the generators L_n and $\bar{L}_n$ in the fields and study how their commutation relations change upon quantization.

2.2.3 Conformal anomaly and the central charge

To understand how conformal invariance is broken upon quantization we study a specific system: D free scalar fields in two dimensions. This is the system relevant for string theory, so the results obtained here will be directly applicable in Chapter 3. In string theory the scalar fields will be interpreted as the spacetime coordinates of a propagating string. Anticipating this interpretation we denote the scalars X^μ with $\mu = 0, \ldots, D-1$.

2.2.3.1 Classical analysis

The action describing the scalars is

$$S = -\frac{T}{2}\int d\sigma d\tau\, \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X, \tag{2.16}$$

where the $\cdot$ means scalar product with respect to some flat metric $g_{\mu\nu}$ in the space of fields on which the X^μ are coordinates. $\eta^{\alpha\beta}$ is our usual two-dimensional flat metric, and T is a constant that in the string interpretation represents the tension of the string.

Finding and solving the equations of motion of this system is an easy task, and one finds free waves traveling in direction of either increasing or decreasing σ. Let us parameterize these solutions in a way that makes the transition to quantum mechanics as smooth as possible.

When σ is periodic (as we assume) it makes sense to expand X^μ in Fourier modes:

$$X^\mu(\sigma,\tau) = \sum_{n=-\infty}^{\infty} e^{2in\sigma} x_n^\mu(\tau). \tag{2.17}$$

Keeping it real requires $x_n^\mu = (x_{-n}^\mu)^\dagger$. After performing the σ-integration our action becomes

$$S = \frac{T\pi}{2}\int d\tau \sum_n [\dot{x}_n \cdot \dot{x}_{-n} - (2n)^2 x_n \cdot x_{-n}]. \tag{2.18}$$

Nicely enough, this is just the action for an infinite number of decoupled harmonic oscillators! The real and imaginary parts of x_n^μ for $n > 0$ are the corresponding displacement coordinates, and the angular frequencies are $\omega_n = 2n$. x_0^μ are not really oscillators since the potential is flat. These zero modes will play an important role later.

To prepare for a good old-fashioned canonical quantization, let us pass to a Hamiltonian description of the system. We choose the x_n^μ as canonical coordinates. The momentum π_n^μ conjugate to x_n^μ is

$$\pi_n^\mu = T\pi\dot{x}_{-n}^\mu, \qquad [\pi_m^\mu, x_n^\nu]_{\text{P.B.}} = \delta_{mn}g^{\mu\nu}, \tag{2.19}$$

where $g^{\mu\nu}$ is the inverse metric on X-space and the index P.B. stands for "Poisson bracket". The Hamiltonian becomes

$$H = \frac{1}{2\pi T}\sum_n [\pi_n \cdot \pi_{-n} + (2\pi n T)^2 x_n \cdot x_{-n}]. \tag{2.20}$$

For each mode with $n \neq 0$ we can define creation and annihilation coordinates. We choose the normalization

$$\alpha_n^\mu = -in\sqrt{\pi T}x_n^\mu + \frac{1}{\sqrt{4\pi T}}\pi_{-n}^\mu \quad \tilde{\alpha}_n = in\sqrt{\pi T}x_{-n}^\mu - \frac{1}{\sqrt{4\pi T}}\pi_n^\mu. \tag{2.21}$$

In these coordinates the reality condition is $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$ and $(\tilde{\alpha}_n^\mu)^\dagger = \tilde{\alpha}_{-n}^\mu$. The Poisson brackets read

$$\begin{aligned} &[\alpha_m^\mu, \alpha_n^\nu]_{\text{P.B.}} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{\text{P.B.}} = im\delta_{m+n}g^{\mu\nu}, \\ &[\tilde{\alpha}_m^\mu, \alpha_n^\nu]_{\text{P.B.}} = 0. \end{aligned} \tag{2.22}$$

and the Hamiltonian is

$$H = \frac{1}{2\pi T}\pi_0^2 + \sum_{n\neq 0}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n). \tag{2.23}$$

By commuting H with α_n^μ, $\tilde{\alpha}_n^\mu$ and x_0^μ it is now simple to solve the Hamiltonian equations of motion. We get

$$X^\mu = x_0^\mu(0) + \frac{1}{T\pi}\pi_0^\mu\tau + \frac{i}{2\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^\mu(0)e^{-2in\sigma^-} + \tilde{\alpha}_n^\mu(0)e^{-2in\sigma^+}\right], \tag{2.24}$$

where π_0^μ is constant. We see that α ($\tilde{\alpha}$) are exactly counterclockwise (clockwise) moving waves traveling around the compact direction σ. We adopt the standard terminology right-movers (left-movers) for α ($\tilde{\alpha}$).

This way of solving the equations of motion for free waves in (1+1)-dimensions is like shooting mosquitoes with a cannon. However, the stage is now completely set for quantizing the system. Before we do this, let us express the generators of conformal transformations in terms of the α. The relevant components of the stress energy tensor are obtained by varying the covariantized version of action (2.16) with respect to the metric: $T_{++} = T\partial_+ X\cdot\partial_+ X$ and $T_{--} = T\partial_- X\cdot\partial_- X$. Adopting the convenient notation $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \pi_0^\mu/\sqrt{4\pi T}$ we get

$$\begin{aligned} L_n &= \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m \\ \bar{L}_n &= \frac{1}{2}\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m. \end{aligned} \tag{2.25}$$

Note that the Hamiltonian is $H = 2(L_0 + \bar{L}_0)$.

Having expressed the Virasoro generators in terms of the fields we can now study their quantum commutation relations.

2.2.3.2 Quantization

Quantization of the free scalars is obtained by promoting the fields to operators and replacing Poisson brackets with commutators:

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n}g^{\mu\nu} \qquad [\pi_m^\mu, x_n^\nu] = -i\delta_{mn}g^{\mu\nu}. \tag{2.26}$$

The α and $\tilde{\alpha}$ are now harmonic oscillator creation and annihilation operators. α^μ_{-n} creates a right-moving quantum with angular frequency $2n$.

When fields become operators, one must be careful how to interpret the product between two fields that do not commute. For our Virasoro operators there is an ambiguity for $n = 0$, since α^μ_n and α^μ_{-n} do not commute. Therefore we define L_0 and $\bar{L}_0$ to be

$$\begin{aligned} L_0 &\equiv \frac{1}{2}\alpha_0^2 + \sum_{m\geq 1} \alpha_{-m}\cdot\alpha_m \\ \bar{L}_0 &\equiv \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{m\geq 1} \tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m. \end{aligned} \tag{2.27}$$

This definition is *normal ordered* in the oscillator coordinates. Since the commutator $[\alpha^\mu_n, \alpha^\nu_{-n}]$ is just a number, the only ambiguity in the definition is really a constant. It is *not* obvious that the above definition gives the physical L_0 and $\bar{L}_0$. The quantities appearing in the Hamiltonian for instance can be shifted by a normal ordering constant.

Let us now study the commutator algebra. For simplicity we focus on the L algebra. The $\bar{L}$ story is identical. If the conformal symmetry is preserved, then

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{2.28}$$

For $m + n \neq 0$ no problem arises, and (2.28) really holds. The computation goes through exactly as it would for Poisson brackets. The subtlety occurs for $n = -m$. Commuting L_m and L_{-m} as given by (2.25) using (2.26) gives

$$[L_m, L_{-m}] = \frac{1}{2}\sum_r [r\alpha_{m-r}\cdot\alpha_{-(m-r)} + (m-r)\alpha_{-r}\cdot\alpha_r]. \tag{2.29}$$

Now, had $\alpha_{\pm r}$ commuted, we could just have substituted $r \to m + r$ in the first sum to arrive at $2mL_0$. But we need to see to it that the operators are in the order $\alpha_{-|k|}\cdot\alpha_{|k|}$. Otherwise we do not get back L_0 the way we defined it. Indeed, the first term is in the wrong order for $r < m$ and the second is wrong for $r < 0$. The price for changing the order is the commutator:

$$\alpha_{|k|}\cdot\alpha_{-|k|} = \alpha_{-|k|}\cdot\alpha_{|k|} + |k|g^\mu_\mu = \alpha_{-|k|}\cdot\alpha_{|k|} + D|k|, \tag{2.30}$$

where D is the number of fields. Thus, what we obtain in the quantized theory is

$$\begin{aligned}[L_m, L_{-m}] &= 2mL_0 + \frac{1}{2}\sum_{r=-\infty}^{m-1} rD(m-r) - \frac{1}{2}\sum_{r=-\infty}^{-1} r(m-r)D = \\ &= 2mL_0 + \frac{D}{2}\sum_{r=0}^{m-1} r(m-r) = 2mL_0 + \frac{D}{12}(m^3-m).\end{aligned} \tag{2.31}$$

In its full glory the quantum corrected Virasoro algebra is therefore

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n}\frac{D}{12}(m^3-m). \tag{2.32}$$

In the derivation above we took the difference of two infinite sums. This might seem dangerous, and there are safer ways to evaluate the commutator. E.g., as described in [GSWa], requiring the quantum operators to fulfill Jacobi's identity determines the form of the anomaly to $\sim c_3 m^3 + c_1 m$ for some constants c_1 and c_3. The constant c_3 can then be fixed by computing a suitable matrix element. Note that there is no invariant meaning to the constant c_1. It can be absorbed into the definition of L_0, where a normal ordering constant may well be inserted.

The last term on the right-hand side of (2.32) is just the identity operator multiplied with a number. For the algebra to be closed we extend the Lie-algebra to include the identity operator. Since it commutes with all elements in the algebra, it is called the *central extension*. The same phenomenon occurs in any CFT, and the coefficient in front of m^3 is usually denoted $c/12$ where c is called the *central charge*.

When the left- and right-movers represent independent degrees of freedom (as for our D free bosons living on a circle) there is one central charge for the L-algebra and one for the $\bar{L}$-algebra. They are denoted c_R and c_L, respectively. For our bosons $c_R = c_L = D$, but if the right- and left-movers behave differently, c_R and c_L may be different numbers.

Note that the Virasoro algebra has a closed sub-algebra generated by L_{-1}, L_0 and L_1. This sub-algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, and is not affected by the central charge. In the three-dimensional gravity context this subalgebra is the isometry algebra of the AdS_3 background.

2.2.4 Virasoro representation theory and partition functions

In conformal field theory, states in Hilbert space are classified according to their transformation properties under the conformal group. We will encounter both unitary and non-unitary representations.

2.2.4.1 Unitary representations

In unitary representations the Virasoro generators satisfy $L_n^\dagger = L_{-n}$ with respect to the Hilbert space inner product, implying hermiticity of the energy momentum tensor. More specifically, the hermiticity of L_0 (and $\bar{L}_0$) implies that the Hamiltonian is Hermitian, and hence that time evolution is unitary. That L_0 is Hermitian also means that state vectors can be decomposed in eigenvectors of L_0.

The most important representations are so-called *highest weight* representations. Let $n > 0$, and suppose that $|h, \bar{h}\rangle$ is an L_0 and $\bar{L}_0$-eigenstate, i.e., that $L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle$ and $\bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle$. The eigenvalues $(h, \bar{h})$ are called the *conformal weights* of the state and their sum $\Delta = h + \bar{h}$ is called the *conformal dimension*.

The commutation relation

$$[L_0, L_n] = -nL_n \tag{2.33}$$

tells us that L_n lowers the eigenvalue h to $h - n$:

$$L_0[L_n|h, \bar{h}\rangle] = (-nL_n + L_nL_0)|h, \bar{h}\rangle = (h - n)[L_n|h, \bar{h}\rangle]. \tag{2.34}$$

Conversely, L_{-n} raises the L_0-eigenvalue by n. A highest weight representation is a representation containing a state with a lowest eigenvalue of L_0 (and $\bar{L}_0$). Since $H \sim L_0 + \bar{L}_0$ this means that the energy of such a representation is bounded from below.

The state with the lowest L_0 and $\bar{L}_0$ eigenvalues is called the highest weight state. It is annihilated by all $L_{n>0}$ and $\bar{L}_{n>0}$. All states in the representation can then be obtained by action of the raising operators $L_{n<0}$ and $\bar{L}_{n<0}$. In this sense highest weight representations work pretty much the same way as ordinary SO(3) representations do, L_0 playing the role of J_z and $L_{\pm n}$ generalizing $J_\pm$. The highest weight states are also known as *primaries* and the states obtained by acting with raising operators are called *descendants*.

2.2.4.2 Non-unitary representations

The only non-unitary representations we will need are *logarithmic* representations. Logarithmic conformal field theories (LCFTs) were recognized as useful physical models in the 1990s. For reviews see e.g. the papers by M. Flohr [Flo03] and M. Gaberdiel [Gab03]. These theories are applicable to a number of physical systems including turbulence, the quantum Hall effect and critical polymers.

In LCFT L_0 and $\bar{L}_0$ are not Hermitian. Some representations can be characterized by a highest weight state $\psi_{h,\bar{h}}$ that is not an eigenstate of L_0. Rather

$$L_0\psi_{h,\bar{h}} = h\psi_{h,\bar{h}} + \phi_{h,\bar{h}}, \tag{2.35}$$

with

$$L_0\phi_{h,\bar{h}} = h\phi_{h,\bar{h}}, \tag{2.36}$$

and similarly for $\bar{L}_0$. The operators L_0 and $\bar{L}_0$ therefore have the matrix representations

$$L_0 \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \quad \bar{L}_0 \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} \bar{h} & 1 \\ 0 & \bar{h} \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{2.37}$$

from which it is clear that neither L_0 nor $\bar{L}_0$ can be diagonalized. The field ψ is called the *logarithmic partner* of ϕ.

We will not delve deeper into Virasoro representation theory. Unitary and logarithmic highest weight representations are enough to be equipped at least for the rest of the thesis.

2.2.4.3 Partition functions

Let us now comment briefly on the torus partition function of a CFT. The moduli space of the torus plays an important part here, and a reader not familiar with it may want to consult Section 6.1, or skip to subsection 2.2.5.

The only thing we need for our story is some understanding when the partition function can factorize holomorphically. Compactifying time in our CFT, $\tau \sim \tau + \pi \mathrm{Im}\,\omega$, and allowing for a shift $\sigma \to \sigma + \pi \mathrm{Re}\,\omega$ when making the identification, results in a CFT defined on a torus defined by the modular parameter ω. [1] Let us study the partition function

$$Z(q, \bar{q}) = \int \mathcal{D} e^{-S_{\text{Eucl.}}} \tag{2.38}$$

on the torus. This function depends on the parameter q related to ω as $q = e^{2\pi i\omega}$. The partition function (2.38) subject to the periodicity conditions can be computed as the trace

$$Z(q, \bar{q}) = \mathrm{Tr}\, e^{-\pi \mathrm{Im}\,\omega H + \pi i \mathrm{Re}\,\omega P} \tag{2.39}$$

where H is the Hamiltonian generating the evolution from τ to $\tau + \pi\mathrm{Im}\,\omega$, and P is the σ-momentum generating the twist $\sigma \to \sigma + \pi\mathrm{Re}\,\omega$. At this point the normal ordering constant that we could have inserted in the definition of L_0 and $\bar{L}_0$ becomes important. Actually, there is a Casimir energy and angular momentum when a cylindrical or toroidal topology is considered. Fixing L_0 and $\bar{L}_0$ by demanding that the vacuum have zero conformal weights shifts $L_0 \to L_0 - c_R/24$ and $\bar{L}_0 \to \bar{L}_0 - c_L/24$. Thus we have

$$H = 2(L_0 + \bar{L}_0 - \frac{c_L + c_R}{24}) \quad P = 2(L_0 - \bar{L}_0 + \frac{c_L - c_R}{24}). \tag{2.40}$$

[1] In an attempt to follow many standard conventions at once ω is called τ in Section 6.1.

Inserting this into (2.39) produces

$$Z(q,\bar{q}) = \mathrm{Tr}\, q^{L_0 - c_R/24} \bar{q}^{\bar{L}_0 - c_L/24} = q^{-c_R/24} \bar{q}^{-c_L/24} + \ldots \tag{2.41}$$

The partition function is a power expansion in q and $\bar{q}$ whose coefficients depend on the number of primaries at each pair of conformal weights. As indicated in the equation the series starts with the contribution from the unique vacuum state with $h = \bar{h} = 0$. Note that to make sense as a partition function on the torus Z must be invariant under the modular group generated by $\omega \to \omega + 1$ and $\omega \to -1/\omega$.

A particularly simple class of CFTs are those when the partition function factorizes in a holomorphic and antiholomorphic piece:

$$Z(q,\bar{q}) = \zeta(q)\bar{\zeta}(\bar{q}), \tag{2.42}$$

where each of the factors, $\zeta(q)$ and $\bar{\zeta}(\bar{q})$ can be interpreted as the partition functions of *holomorphic* CFTs. These are theories in which one of the Virasoro algebras acts trivially. The ζ must then have an expansion

$$\zeta(q) = q^{-c_R/24} + \ldots \tag{2.43}$$

Since $q^{-c_R/24} \to e^{-2\pi i c_R/24} q^{-c_R/24}$ under the transformation $\omega \to \omega + 1$ it is clear that this function can only be modular invariant and thus work as a CFT partition function if $c_R = 24k_R$ for some integer k_R. The same goes for c_L, so holomorphic factorization is conceivable only if both central charges are integer multiples of 24. As we will see in Chapter 7 these are exactly the central charges relevant in three-dimensional gravity.

Let us now turn to the last subject of this chapter: the density of states in CFTs and the Cardy formula.

2.2.5 States and the Cardy formula

In conformal field theory, the density of states $\rho(h)$ at L_0-level h has a universal behavior for $h \to \infty$. This is the famous formula due to J.L. Cardy [Car86]:

$$\rho(h) \sim \exp\left(2\pi\sqrt{\frac{(c - 24h_0)h}{6}}\right)\rho(h_0) \tag{2.44}$$

Here h_0 is the smallest L_0 eigenvalue in the spectrum of the theory, when defined on the plane. Thus, the asymptotic growth of the number of states is the same, in any two CFTs sharing the values c and h_0. The quantity $c^{\text{eff}} = c - 24h_0$ is called the effective central charge. For the density of states at level

$(h, \bar{h})$ we get

$$\rho(h, \bar{h}) \sim \exp\left(2\pi\sqrt{\frac{c_R^{\text{eff}} h}{6}} + 2\pi\sqrt{\frac{c_L^{\text{eff}} \bar{h}}{6}}\right) \rho(h_0, \bar{h}_0), \tag{2.45}$$

or for the logarithm

$$S = \log \rho \sim 2\pi\sqrt{\frac{c_R^{\text{eff}} h}{6}} + 2\pi\sqrt{\frac{c_L^{\text{eff}} \bar{h}}{6}}. \tag{2.46}$$

We will not attempt a derivation of the Cardy formula here, but just check that it hold for a single free scalar. This theory has a vacuum state $|0\rangle$ with

$$\alpha_n |0\rangle = \tilde{\alpha}_n |0\rangle = 0 \qquad n \geq 0. \tag{2.47}$$

This is just the ground state of all the oscillators. From the commutation relations $[L_0, \alpha_{-k}]$ it is clear that a general state at level $(h, \bar{h})$ is obtained by action of a sequence of creation operators:

$$\alpha_{-n_1} \alpha_{-n_2} \ldots \alpha_{-n_N} \tilde{\alpha}_{-m_1} \tilde{\alpha}_{-m_2} \ldots \tilde{\alpha}_{-m_M} |0\rangle \tag{2.48}$$

with n_i and m_j satisfying

$$\sum_{i=1}^{N} n_i = h \qquad \sum_{j=1}^{M} m_j = \bar{h}. \tag{2.49}$$

The number of ways to partition the number h into a sum of integers is called the partition function $p(h)$ of h. This function was studied by S. Ramanjuan and G.H. Hardy [RH18], who found its asymptotic behavior

$$p(h) \sim \frac{1}{4h\sqrt{3}} \exp\left(2\pi\sqrt{\frac{h}{6}}\right). \tag{2.50}$$

Thus, for a single scalar,

$$\rho(h, \bar{h}) = p(h)p(\bar{h}) \sim \frac{1}{4h\sqrt{3}} \exp\left(2\pi\sqrt{\frac{h}{6}} + 2\pi\sqrt{\frac{\bar{h}}{6}}\right). \tag{2.51}$$

This matches the Cardy formula for $c_R = c_L = 1$.

Let us end this chapter with a brief note on the case $c = 0$. From the Cardy formula it seems that the leading contribution to the number of states vanishes in such a theory. For a unitary theory there is actually a much stronger statement. A unitary CFT with $c = 0$ contains only the vacuum state, and is thus trivial. The argument to show this is somewhat involved and the interested reader is referred to Ref. [Sch96].

3. String theory

String, or nothing!
Gollum

As advertised in the introduction, string theory aims to unify all forces and matter. Different particles are described as different vibrational modes of the string. Being a quantum theory of gravity, the natural mass scale for string theory is the Planck mass. A naive expectation would therefore be that the energy quantum of excited strings is Planckian and that all excited strings would have masses of the Planck scale. This expectation turns out to be erroneous due to subtle quantum effects. A magical "-1" appears as both a blessing and a curse: it renders the first excited states massless, but the ground state tachyonic. Incorporating supersymmetry lifts the curse and keeps the blessing: the tachyon can be consistently eliminated.

In this chapter we introduce strings and describe their quantum dynamics. We start with bosonic strings to keep technical clutter at a minimum. The main goals here are to establish the critical dimension and present the spectrum. These results are not explicitly needed in the later chapters of the thesis, but are included as a motivation for why we study the low-energy theories, and as an illustration to the deep and rich structure that lies beneath them.

Superstrings are described in less detail, much by analogy to the bosonic case. Important differences are highlighted and the main goal is to arrive at the low energy theory of type IIB superstrings that plays an important role in the subsequent chapters. The chapter ends with a section on D-branes — objects on which open strings can end.

Everything presented in this chapter is nowadays textbook material. The textbooks in question include the books by M.B. Green, J.H. Schwarz and E. Witten [GSWa, GSWb] and the ones by J. Polchinski [Pola, Polb]. These volumes enjoy a well-deserved biblical status. For a complement, containing clear, pedagogical and down-to-earth explanations, the reader should consult the book by B. Zwiebach [Zwi].

3.1 The bosonic string

The bosonic string is a freely moving string, relativistic and quantized. Given the simplicity of this starting point the richness of the structure that emerges is amazing. In describing the bosonic string we make use of our knowledge from

Chapter 2. All conventions and definitions in the description of the D free scalars in that chapter carry over to the description of strings by the Polyakov action to be presented below.

3.1.1 Classical string dynamics

A string that moves in spacetime sweeps out a two-dimensional *world-volume*. Let us consider a D-dimensional spacetime coordinatized by $X^0, \ldots X^{D-1}$. The degrees of freedom of the string are the embedding coordinates $X^\mu(\sigma, \tau)$ which are functions of the two coordinates σ and τ parameterizing the world-volume.

The action for a classical relativistic string is called the *Nambu–Goto action*. It is proportional to the relativistic area of the world-volume:

$$S_{\text{NG}} = -T \int d\text{Vol} = -T \int d\sigma d\tau \sqrt{G}. \tag{3.1}$$

Here G is short for the determinant of the induced metric:

$$G = |\det G_{\alpha\beta}| = |\det \partial_\alpha X \cdot \partial_\beta X|. \tag{3.2}$$

where $\partial_\alpha X \cdot \partial_\beta X = g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$ and $g_{\mu\nu}$ is the spacetime metric. The parameter T is the string tension. It appears in a number of guises, so let us comment on it briefly. First, it may look like a free parameter in a theory advertised to have none. Being dimensionful however (it has the dimension M^2) this is not the case. It is nonsense to try to compare two theories with "different" values of T. It rather sets the scale of "stringy" effects. For solutions to the theory that have low energy compared to T, stringy effects will be unimportant. More specifically, one can make an expansion in

$$\frac{\text{Energy}}{\sqrt{T}}, \tag{3.3}$$

where for low energies only the first term is important. Instead of T often one of the two parameters ℓ_s and α' is used. ℓ_s is the string length and α' is the Regge slope parameterizing how fast the mass of the string states grows with their spin. The parameters are related as

$$\ell_s^2 = 2\alpha' = \frac{1}{\pi T}. \tag{3.4}$$

The expansion in energies is called the α' expansion.

It is useful to think of the coordinates σ and τ as coordinates on an abstract two-dimensional surface called the *world-sheet* rather than on the world-volume itself. The X^μ are then interpreted as fields living on the world-sheet. Note that the theory is invariant under reparameterizations of σ and τ. Two field configurations related by a reparameterization describe the same physical situation, so this symmetry is a gauge symmetry and should be modded out.

While conceptually simple and physically well-motivated, the square root makes the Nambu–Goto action a difficult starting point for a quantum theory of strings. Luckily, there is another action that is classically equivalent to (3.1). In this formulation the X^μ are free scalars coupled to a dynamical metric $\gamma_{\alpha\beta}$ on the world-sheet. The action is the *Polyakov action*

$$S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{\gamma}\,\gamma^{\alpha\beta}\,\partial_\alpha X\cdot\partial_\beta X = -\frac{T}{2}\int d^2\sigma\,\sqrt{\gamma}\,\mathrm{Tr}G \tag{3.5}$$

where the trace is with respect to the world-sheet metric $\gamma_{\alpha\beta}$. To see that the actions are equivalent we solve the equations of motion obtained by varying $\gamma_{\alpha\beta}$. These are just the vanishing of the stress-energy tensor. A straightforward computation gives

$$0 = \frac{T_{\alpha\beta}}{T} = -\frac{2}{T\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}} = G_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\mathrm{Tr}G. \tag{3.6}$$

Taking the determinant of this equation we obtain

$$G = \frac{\gamma}{4}(\mathrm{Tr}G)^2, \tag{3.7}$$

where $\gamma = |\det\gamma_{\alpha\beta}|$. Inserting the square root of this into (3.5) gives back the Nambu–Goto action (3.1).

We thus want to use the Polyakov action as a starting point for our study of strings. Let us make a few comments on this action and on the theory it defines. First, note that the action is the covariant version of (2.16): it is a bunch of free scalar coupled to two-dimensional gravity. Second, even if we must vary the action with respect to the metric $\gamma_{\alpha\beta}$, it should not really be a physical degree of freedom. From the spacetime point of view, all we see are the embedding coordinates X^μ. Therefore, solutions differing only in the metric should be viewed as physically equivalent.

There are many such equivalent solutions, as can be realized from the fact that the Polyakov action is invariant under the Weyl rescaling (2.1) affecting only the metric. Therefore, to have the correct physical degrees of freedom, we must view the Weyl symmetry of (3.5) as a gauge symmetry. In addition to this we also have the reparameterization gauge symmetry.

In total there are three gauge transformations with three independent parameters. The metric $\gamma_{\alpha\beta}$ has three independent components, so these are enough to locally set $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the standard flat metric. There are even some transformations left afterward! Indeed, as we learned in Chapter 2 there are combinations of reparameterizations and Weyl rescalings that leave the metric invariant: the conformal transformations. Thus, if we fix the metric to $\eta_{\alpha\beta}$ there is a residual conformal symmetry that should be divided out. There are three standard approaches to deal with the gauge fixing, and we briefly describe each of them in the next subsection.

Even if we fix the metric in the Polyakov action we must take into account the equations of motion obtained by varying it. This means that we need to impose as constraints the vanishing of the stress energy tensor (3.6). Vanishing of the whole tensor implies vanishing of all its Fourier components, which are nothing but the Virasoro operators L_n and $\bar{L}_n$. These constraints are implemented in different ways in the three approaches of the next subsection. Note that the constraints are intimately connected to the residual conformal symmetry. Requiring $L_n|\psi\rangle = 0$ means that the state $|\psi\rangle$ is invariant under conformal symmetry and thus that it makes sense in a theory where it has been divided out.

A simple but extremely important point that we have so far not touched upon is the fact that strings can have two different topologies: they can be open or closed. Open strings are described by either of two boundary conditions:

$$\begin{aligned} &\text{Neumann:} \qquad \partial_\sigma X^\mu(0,\tau) = \partial_\sigma X^\mu(\pi,\tau) = 0 \\ &\text{Dirichlet:} \qquad X^\mu(0,\tau) = f_0^\mu(\tau), \quad X^\mu(\pi,\tau) = f_\pi^\mu(\tau). \end{aligned} \tag{3.8}$$

The Neumann boundary conditions assert that no momentum is flowing off the string. They describe freely moving strings. Dirichlet conditions mean that the string endpoints are attached to an object whose motion is described by the functions $f^\mu_{0,\pi}$. These object are called D-branes (D for Dirichlet) and will be described later. In this case momentum can flow off the string and onto the D-brane.

Closed strings satisfy the natural periodic boundary conditions

$$X^\mu(0,\tau) = X^\mu(\pi,\tau). \tag{3.9}$$

In Chapter 2 we analyzed the free boson action assuming periodicity. We found two sets of independent oscillators, α and $\tilde{\alpha}$. For the reader's convenience we repeat the mode expansion here (eliminating T for ℓ_s)

$$X^\mu = x_0^\mu + \ell_s^2 \pi_0^\mu \tau + \frac{i\ell_s}{2} \sum_{n\neq 0} \frac{1}{n} \left[\alpha_n^\mu e^{-2in\sigma^-} + \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \right]. \tag{3.10}$$

We use light cone coordinates $\sigma^\pm = \tau \pm \sigma$, and the quantity π_0^μ is the spacetime center of mass momentum of the string.

For open strings the boundary conditions relate the left- and right-movers by forcing them to form standing waves. In the Neumann case the mode expansion becomes

$$X^\mu = x_0^\mu + \ell_s^2 \pi_0^\mu \tau + i\ell_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma, \tag{3.11}$$

i.e, the oscillators are set equal, $\alpha = \tilde{\alpha}$. Similarly, the conformal transformations that preserve the boundary of the world-sheet are only the subset of (2.6) satisfying $\xi^+(x) = \xi^-(x)$ for a general argument x. Open strings are therefore described by just one Virasoro algebra. We denote the generators of this algebra L_n, but remember that they do not generate the same "right-moving" transformations as the original closed string homonym L_n.

Accordingly, the closed string Virasoro generators are given exactly as in (2.25) and (2.27), while the single set of generators for the open strings looks exactly the same as the L_n for the closed strings:

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m} \cdot \alpha_m. \tag{3.12}$$

A numerical peskiness is that $\alpha_0^\mu = \pi_0^\mu/\sqrt{\pi T}$ in the open case but $\alpha_0^\mu = \pi_0^\mu/\sqrt{4\pi T}$ in the closed. Because of the linguistic simplicity of having just one algebra, we focus on open strings

Before dealing with the gauge-fixing let us make a comment on the operator L_0. We saw in Chapter 2 that the physical L_0 includes a normal ordering constant. This constant plays an important role in string theory, and is fixed to a certain value. The Virasoro constraints imply that physical states must be annihilated by the physical L_0, differing from our definition by a constant a. Thus we require

$$(L_0 - a)|\psi\rangle = 0 \tag{3.13}$$

for all physical $|\psi\rangle$. Fleshing this out yields

$$\left[\alpha'\pi_0^2 - a + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m\right] |\psi\rangle = 0. \tag{3.14}$$

Since π_0 is the spacetime momentum the above equation gives an expression for the mass M of the state $|\psi\rangle$:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m - a\right). \tag{3.15}$$

We see that our naive expectations were correct inasmuch as the masses of the string states are Planckian ($\mathcal{O}(1/\sqrt{\alpha'})$), but also that if the correct normal ordering constant a is positive, there may be a nontrivial massless representation in the space of states. On the other hand, if none of the oscillators is excited, a positive a means that the corresponding state is tachyonic.

3.1.2 Modding out the conformal group

Let us now turn to the subtle issue of gauge-fixing the Polyakov action. We view the problem form three different angles. In the first two methods the world-sheet metric is integrated out already at the classical level, and the constraints are put in by hand when treating the gauge fixed action. In the Faddeev–Popov method the metric is integrated out at the quantum level.

In all three approaches problems arise if $D \neq 26$ or $a \neq 1$. The technical appearance of the problems varies. In fact, one way of viewing the situation is that, apart from being inconsistent one by one, the three approaches are not equivalent unless $D = 26$ and $a = 1$.

3.1.2.1 Classical covariant quantization

The most straightforward quantization procedure starts by fixing the world-sheet metric to $\eta_{\alpha\beta}$, leaving the action in (2.16)

$$S = -\frac{T}{2} \int d\sigma d\tau \, \eta^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X. \tag{3.16}$$

We have already dealt with this theory at some length in Chapter 2 and know that the residual conformal invariance is broken by an anomaly at the quantum level. Namely, its central charge is $c = D$. This smells like bad news, since it is impossible to divide out a broken symmetry.

Put differently, the anomaly affects the way we can impose the constraints. Imposing $L_n|\psi\rangle = 0$ for all n becomes impossible. Assume, for instance that $L_n|\psi\rangle = 0$ for $n > 0$. Then

$$||L_{-n}|\psi\rangle||^2 = \langle\psi|L_n L_{-n}|\psi\rangle = 2n\langle\psi|L_0|\psi\rangle + \frac{D}{12}(n^3 - n), \tag{3.17}$$

where we used (2.32) and that $L_n^\dagger = L_{-n}$. Taking n large, the last term dominates and it is evident that we cannot demand $L_{-n}|\psi\rangle = 0$ for all n.

The solution is to impose

$$L_{n>0}|\psi\rangle = 0 \quad \text{and} \quad (L_0 - a)|\psi\rangle = 0 \tag{3.18}$$

with no constraints from the negative Virasoro operators. The rationale behind this is that $\langle\psi|L_n - a\delta_n|\psi\rangle$ then vanishes *for all* n. We thus let (3.18) be the physical state condition.

Constructing states by choosing a π_0^μ eigenvalue and acting on them with the Virasoro algebra one can get some feeling for the Hilbert space of states. The fact that D enters in the algebra and that a enters in the physical condition makes the structure of the Hilbert space dependent on them. Playing around a bit (see Section 2.2 in [GSWa]) reveals that there are negative norm states in the spectrum if $D > 26$. In lower dimensions there are both positive and zero norm states. Zero norm states are, if they decouple, signals of gauge invariance.

For the particular values $D = 26$ and $a = 1$ the number of zero norm states increases dramatically. This is a first hint of the special status of these values.

Pursuing this covariant method even further reveals unitarity problems if $D < 26$.

3.1.2.2 Light-cone gauge quantization

Let us start again by gauging away the metric as in (3.16). It is possible to go even further and to fix also the residual conformal symmetry of the Polyakov action. The conformal symmetry affects only the coordinate fields, so the gauge-fixing condition must be a requirement on these. A very convenient choice is the *light-cone gauge*. This approach has the great advantage that the Hilbert space is composed only of physical states. No zero norm states can arise. The downside is that Lorentz symmetry is not manifest anymore. Actually, as we shall see, only if $D = 26$ and $a = 1$ does Lorentz symmetry persist to the quantum level.

To fix the gauge we choose one of the spacelike directions $X^1, \ldots, X^{D-1}$ at random, say X^{D-1}, and define the light-cone coordinates

$$\begin{aligned} X^+ &= \frac{1}{\sqrt{2}}(X^0 + X^{D-1}) \\ X^- &= \frac{1}{\sqrt{2}}(X^0 - X^{D-1}). \end{aligned} \tag{3.19}$$

Denote the rest of the coordinates X^i with $i = 1, \ldots, D-2$. Now using a conformal transformation $\sigma^+ \to f(\sigma^+)$ and $\sigma^- \to g(\sigma^-)$ it is simple to show that τ can be taken to an arbitrary solution of the two-dimensional free wave equation: $\tau \to (f(\sigma^+) + g(\sigma^-))/2$. Since all coordinate fields satisfy the wave equation we can choose τ to be any of these! In particular let us choose

$$X^+(\sigma, \tau) = x^+ + \pi_0^+ \tau. \tag{3.20}$$

This is the light-cone gauge condition. One of its virtues is that the Virasoro constraints can be solved explicitly. Indeed with the metric fixed to $\eta_{\alpha\beta}$ the condition (3.6) reads

$$\partial_+ X \cdot \partial_+ X = \partial_- X \cdot \partial_- X = 0, \tag{3.21}$$

which in our gauge becomes

$$\begin{aligned} \partial_+ X^- &= \frac{1}{\ell_s^2 \pi_0^+} \partial_+ X^i \partial_+ X^i \\ \partial_- X^- &= \frac{1}{\ell_s^2 \pi_0^+} \partial_- X^i \partial_- X^i. \end{aligned} \tag{3.22}$$

This precisely (save for an integration constant) determines X^- in terms of the X^i. In particular, all creation operators α_n^- become functions of the α_n^i and are not considered as independent degrees of freedom. In this formalism it is clear that the number of physical oscillators corresponds to a transverse set of degrees of freedom. The only subtlety appears in determining α_0^-, since we then encounter normal ordering ambiguities on the right-hand side. This is the way the normal ordering constant a enters in the light-cone gauge approach. Indeed, solving for π_0^- we obtain

$$\pi_0^- = \frac{1}{4\alpha' \pi_0^+} \left(\sum_{m=-\infty}^{\infty} : \alpha_{-m}^i \alpha_m^i : -a \right), \tag{3.23}$$

translating into the mass-shell condition

$$M^2 = (2\pi_0^- \pi_0^+ - \pi_0^i \pi_0^i) = \frac{1}{\alpha'} \left(\sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i - a \right) \tag{3.24}$$

Thus we see that this a plays exactly the same role as the a in (3.13) and (3.15). In the light-cone gauge, however there are less oscillators to excite. We are also provided with an intuitive way to determine the correct condition for a. Since the manifest Lorentz symmetry is broken down to SO$(D-2)$, all states in this formalism will furnish representations of this group. Consider for instance states obtained from the action of α_{-1}^i on the vacuum. They must transform as vectors of SO$(D-2)$. This is the correct transformation property for a *massless* vector in $SO(D-1,1)$ and not for a massive one. But we hope that the Lorentz symmetry we hid by choosing the gauge is still there. It surely is in the classical theory. So this state really must be a massless Lorentz vector. Since the state only has one oscillator quantum, computing the mass from (3.24) gives $\alpha' M^2 = (1-a)$ which is canceled only if $a = 1$.

The statement about when the theory remains Lorentz invariant quantum mechanically can be made precise by computing the Lorentz algebra. The generators of the algebra can be found as Noether charges corresponding to the Lorentz symmetry of the action. Thus their commutation relation can be studied through those of the oscillator operators α_n^μ. Since the α_n^- are expressed in terms of the α_n^i, problems can arise.

The actual computation is slightly technical, but uses the same techniques we used when finding the anomalous term in the Virasoro algebra. Careful evaluation of a suitable commutator shows that the theory is Lorentz invariant if and only if $D = 26$ and $a = 1$. The curious reader will find the computation e.g. in Chapter 2 of [GSWa].

3.1.2.3 A ghost story

Our last approach is much more fancy than the two first. Not wanting to introduce too many new tools we keep the discussion quite superficial. Consider the Euclidean path integral of the Polyakov action

$$Z = \int \mathcal{D}\gamma \, \mathcal{D}X \, e^{-S_{\mathrm{P}}[\gamma, X]} \tag{3.25}$$

where we suppress the indices of $\gamma_{\alpha\beta}$ and X^μ. This functional integral includes each physical field configuration many times. Indeed we want to count configurations differing by a Weyl transformation and a reparameterization as one and the same. An often used procedure to gauge-fix path integrals was invented by L. Faddeev and V. Popov [FP67].

The basic problem is the following. Just fixing the symmetry by some gauge condition is alright as far as the action is concerned. The *measure*, in this case $\mathcal{D}\gamma\mathcal{D}X$, however also contributes to the path integral, and does so in a way that depends on the chosen gauge condition. This must be compensated for by inserting a determinant in the path integral measuring how the condition depends on transformations in the gauge group.

What Faddeev and Popov found was that there is a neat way of formulating this determinant in terms of a new factor in the path integral, dependent on new fields. These fields are called *ghosts* because they are not representing physical degrees of freedom.

Using this method and fixing the reparameterization symmetry by the condition

$$\gamma_{\alpha\beta} = e^{\phi(\sigma,\tau)} \eta_{\alpha\beta} \tag{3.26}$$

yields the following path integral

$$Z = \int \mathcal{D}\phi \, \mathcal{D}X \, \mathcal{D}b \, \mathcal{D}c \, e^{-S_{\mathrm{P}}[\eta, X] - S_{\mathrm{g}}}, \tag{3.27}$$

where S_{g} is the ghost action

$$S_{\mathrm{g}} = -\frac{1}{\pi} \int d^2\sigma \left(c^- \partial_+ b_{--} + c^+ \partial_- b_{++} \right). \tag{3.28}$$

Since the actions are independent of ϕ, we have managed to gauge-fix exactly as much as we did in the covariant approach. The ϕ-integral has become a trivial factor. Now, however we have new fields! The conformal symmetry is still unfixed so the total action $S = S_{\mathrm{P}} + S_{\mathrm{g}}$ is a conformal field theory. The Virasoro generators now obtain a term that generates the conformal transformations of the ghosts:

$$L_n^{\mathrm{tot}} = L_n + L_n^{\mathrm{ghost}} - a\delta_n. \tag{3.29}$$

We also included the normal ordering constant in the definition of L_0 here.

Computing the quantum corrected Virasoro algebra of the full theory is analogous to the free scalar case of Chapter 2 and produces

$$[L_m^{\text{tot}}, L_n^{\text{tot}}] = (m-n)L_{m+n}^{\text{tot}} + \left(\frac{D-26}{12}m^3 - \frac{D-2-24a}{12}m\right)\delta_{m+n}. \tag{3.30}$$

So the theory is really conformally invariant, even at the quantum level, if $D = 26$ and $a = 1$! Put differently, the central charge of the combined system of scalars and ghosts vanishes: $c = 0$.

What remains, is to define what is meant by physical states. Clearly, we should not have more degrees of freedom (b and c) after the gauge fixing than before. The method for this is the BRST procedure, named after C. Becchi, A Rouet, R. Stora and I.V. Tyutin [BRS76, Tyu75]. The method depends on that the total action $S = S_{\text{P}} + S_{\text{g}}$ after gauge-fixing possesses a new fermionic symmetry generated by a nilpotent charge Q. The physical states are *cohomology classes* with respect to Q, meaning that $Q|\psi\rangle = 0$ and that $|\psi\rangle \sim |\psi\rangle + Q|\phi\rangle$ for any $|\phi\rangle$.

In the present case the BRST charge is constructed from the Virasoro generators and it *fails to be nilpotent* if there is a central charge in the algebra. Thus, to be able to define a physical spectrum the theory must be conformally invariant, i.e, D and a must have their magic values.

This concludes our treatment of the gauge fixing of string theory, but before presenting the spectrum of physical states let us recollect what the three approaches taught us. The conformal anomaly that exists if $D \neq 26$ and $a \neq 1$ manifests itself differently in the different cases. This is the common situation with anomalies: they can be shuffled around, but they never go away[1].

Thus, in the classical covariant approach the anomaly surfaces as negative norm states for $D > 26$ and as unitarity problems at $D < 26$. The theory is manifestly Lorentz invariant in all dimensions though. In light cone gauge on the other hand, there are no zero or negative norm states. Instead the anomaly breaks Lorentz invariance unless $D = 26$ and $a = 1$. Finally, in the Faddeev–Popov approach where the gauge fixing takes place at the quantum level the anomaly appears as the conformal anomaly of the resulting theory if $D \neq 26$ and $a \neq 1$. The absence of this anomaly is equivalent to the existence of a nilpotent BRST charge, and thus to the consistency of the gauge-fixing procedure. If we are in 26 spacetime dimensions and choose $a = 1$, all three approaches are equivalent.

3.1.3 Spectrum

In the last subsection we put quite some effort into convincing ourselves that strings have strong opinions on the values of D and a. If $D = 26$ is an ap-

[1]For gauge anomalies, this is an instance of a more general principle: conservation of misery.

pealing answer is left for the reader to judge, but $a = 1$ excellent! It precisely allows for one level of massless vibrational states of the string.

Let us explore this level starting with open strings. The most convenient framework is the light-cone gauge. The string Fock space is built up of simple harmonic oscillator Hilbert spaces. We have 24 infinite sets of oscillator creation operators α^i_{-n}. As described implicitly in Chapter 2, and as is evident from the commutation relations (2.26) the operator α^i_{-n} raises the eigenvalue of the number operator

$$N = \sum_{m \geq 1} \alpha^i_{-m} \alpha^i_m \tag{3.31}$$

by n. The mass is given simply in terms of N:

$$\alpha' M^2 = N - 1. \tag{3.32}$$

The oscillator ground state should be annihilated by all $\alpha_{n>0}$. However, α_0 is not an oscillator but describes the spacetime momentum of the string. It can have any eigenvalue as long as the mass shell condition is satisfied. Thus, there is a continuous family of oscillator ground states labeled by their π^i_0 and π^+_0 eigenvalues, i.e., their spacetime momentum. Let us call these states $|0; p\rangle$ where p replaced π_0 for brevity and the zero indicates that it is an oscillator vacuum $\alpha^i_{n>0}|0; p\rangle = 0$. This state has mass $M^2 = -p^2 = -1$ and is the feared for tachyon.

Moving up to the next level we have the states $\alpha^i_{-1}|0; p\rangle$ representing a massless vector with 24 polarizations. This is the only massless excitation of the open string.

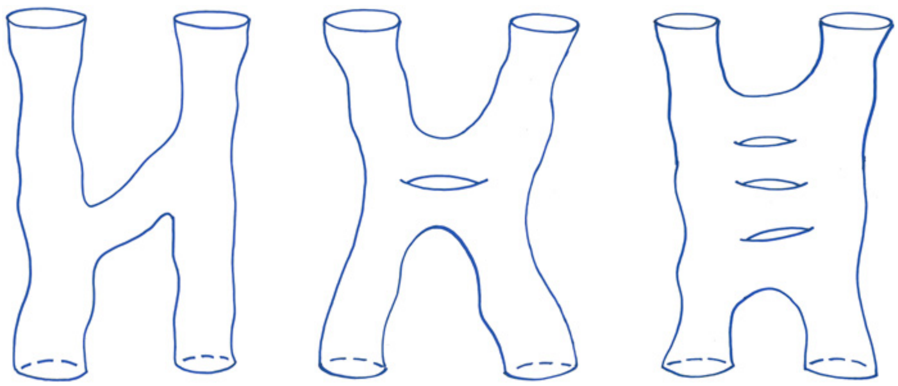
To see the connection of string states to spacetime fields, note that we can construct the superposition

$$|A, \tau\rangle = \int dp\, A_i(\tau, p) \alpha^i_{-1}|0; p\rangle. \tag{3.33}$$

The coefficient $A_i(p)$ should be viewed as the momentum space wave function of a spacetime field, and imposing the world-sheet Schrödinger equation $i\partial_\tau = H$ shows that A fulfills the light-cone gauge Maxwell equations. Thus the massless excitation of the open string is a photon.

Let us turn to closed strings. In this case both L_0 and $\bar{L}_0$ must be shifted by the same constant $a = 1$. This imposes a relation between the α and $\tilde{\alpha}$ operators: $N = \tilde{N}$. Thus, the total mass must split up in equal amounts coming from left- and right-movers. However, this energy need not be distributed evenly on the levels. E.g., $\tilde{\alpha}^i_{-2}\alpha^j_{-1}\alpha^k_{-1}|0; p\rangle$ is perfectly allowed. The closed strings satisfy a mass shell condition similar to that of open strings:

$$\alpha' M^2 = 8(N - 1) = 8(\tilde{N} - 1). \tag{3.34}$$



(a) (b) (c)

Figure 3.1: Three different world-sheet topologies all describing two closed strings that scatter against each other. The different topologies correspond to different loop order: (a) is at tree-level, (b) at one loop and (c) at three loops.

Consequently, the ground state is again tachyonic. The massless states are of the form

$$|M_{ij}\rangle = M_{ij}\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;p\rangle. \tag{3.35}$$

The matrix M_{ij} splits up in three SO(24) irreducible representations: a trace part, an antisymmetric tensor B_{ij} and a traceless symmetric tensor g_{ij}. These three fields (and their superstring cousins) all play fundamental roles. The trace part corresponds to a spacetime field $\Phi(X^\mu)$ called the *dilaton*. As we shall see, it sets the strength of string interactions. The antisymmetric tensor B_{ij} is a two-form analog of the Maxwell field called the Kalb–Ramond field. In Chapters 4, 5 and 6 we put the corresponding field strength through compact cycles in the internal geometry to stabilize it. Last, but certainly not least, the traceless tensor g_{ij} is the massless spin two particle that transmits the gravitational interaction — the graviton.

3.1.4 Interactions and genus expansion

The way interactions are described in string theory is at the same time beautiful and unsatisfactory. Its beauty lies in the geometric nature, and the unsatisfaction in the absence (except in very simple cases) of an underlying principle, like the path integral of quantum field theory.

If we want to study two closed strings that scatter again into two closed strings we have to specify that the world-sheet looks something like one of the examples in Figure 3.1. In fact any of the variants in (a)–(c) could do. String theory proposes that these different topologies of the world-sheet are the counterparts of the Feynman diagrams of quantum field theory. Thus (a) corresponds to a tree diagram, (b) to a one-loop diagram and (c) to a three-loop diagram. We should therefore *sum* over all different topologies to get the total amplitude. This sum is called the *genus expansion*.

By using gauge invariance the local geometry of the different world-sheets can be fixed to a "reference shape" much like when we chose $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. When the world-sheet topology is nontrivial this procedure is slightly more complicated. Not all geometries are gauge equivalent and we must integrate over the inequivalent ones. Luckily the number of parameters in these integrations is finite. The parameters are called world-sheet moduli. The gauge freedom is used to put the external legs into points where the initial (final) states are encoded by insertion of so-called *vertex operators*, whose positions should also be integrated over. Thus, formally,

Here the crosses symbolize vertex operator insertions, and an integration over moduli and insertion points should be carried out for each term.

In field theory there is a coupling constant weighing the amplitudes in the Feynman expansion, allowing for a well-defined perturbation theory. Strings have no free parameters, so what should this be? It can only be a field that is already present in the theory. The field doing the job is the dilaton. Namely, it turns out that the correct way to couple strings to a (possibly spacetime dependent) background dilaton $\Phi(X^\mu)$ is to include the Einstein–Hilbert like term

$$S_{\mathrm{d}} = \frac{1}{4\pi}\int d^2\sigma\sqrt{\gamma}\,\Phi(X^\mu)R \tag{3.36}$$

in the string action. Here R is the world-sheet Ricci scalar obtained from the metric $\gamma_{\alpha\beta}$. In two-dimensions the Einstein–Hilbert term is topological and its integral gives 4π times the *Euler characteristic* χ of the world-sheet, which is a topological invariant. It is related to the *genus* h of the surface by $\chi = 2(1-h)$. The genus is simply the number of holes in the surface. A constant vacuum expectation value Φ_0 for the dilaton therefore sets the coupling constant. Consider for instance an amplitude with a world-sheet of genus h:

$$Z_h = \int \mathcal{D}\gamma\,\mathcal{D}X e^{-S_{\mathrm{P}}-S_{\mathrm{d}}} = e^{-\chi\Phi_0}Z_h[\Phi=0] = g_s^{2h-2}Z_h[\Phi=0], \tag{3.37}$$

where $g_s \equiv e^{\Phi_0}$ is called the *string coupling*. Thus, the genus h amplitude is weighed by g_s^{2h-2}. Rescaling the vertex operators by a factor g_s, this is the power expected from the point particle limit.

To end this section on the bosonic string let us make an additional comment regarding the spacetime fields that correspond to the energy eigenstates of the string. Vacuum expectation values of the fields Φ, B_{ij} and g_{ij} can all be incorporated naturally in the world-sheet descriptions of the string. The dilaton was described above, and the correct way to incorporate the graviton is just to allow for a spacetime dependent (i.e. field dependent) metric[2] $g_{\mu\nu}(X)$ in the Polyakov action

$$S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{\gamma}\,\gamma^{\alpha\beta}\,g_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu. \tag{3.38}$$

The B-field couples to the X^μ through a similar term, but with $\gamma^{\alpha\beta}$ replaced by the two-dimensional antisymmetric tensor $\varepsilon^{\alpha\beta}$.

Quantum consistency (read no Weyl anomaly) gives conditions on these background fields. These conditions look exactly as equations of motion. For instance, to lowest order in α' the condition on the metric is just Einstein's equations for $g_{\mu\nu}$! Strings knows about general relativity. At this point it is dazzling to remember how little we put in from the start: action equals volume times tension.

In this way, and through studying scattering amplitudes, it is possible to formulate the low energy dynamics of string theory in terms of *effective field theory*. This is the approach used in Chapters 5 and 6.

Including background fields also allows other exciting effects. For instance the whole analysis leading to the critical dimension is affected when a non-trivial background is considered. There are examples of consistent string models in other dimensions than 26, but the price paid is always Lorentz non-invariance of the background fields.

3.2 Superstrings

The bosonic string suffers from two major drawbacks. It has a tachyonic ground state, and it lacks fermions. Including fermions in a way that leads to spacetime supersymmetry actually solves both these problems.

3.2.1 World-sheet fermions and the critical dimension

In the Ramond–Neveu–Schwarz approach to superstrings one begins by including fermionic fields ψ^μ propagating on the world-sheet. These are Majorana spinors with two real components

$$\psi^\mu = \begin{pmatrix}\psi^\mu_- \\ \psi^\mu_+\end{pmatrix} \tag{3.39}$$

[2]Since the world–sheet action is covariant, what appears is the covariant manifestation $g_{\mu\nu}$ of the light-cone field g_{ij}.

that transform as spacetime vectors. With the two-dimensional "γ"-matrices

$$\rho^\tau = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^\sigma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{3.40}$$

the supersymmetric action reads

$$S_\mathrm{P} = -\frac{T}{2}\int d^2\sigma\ \left(\partial^\alpha X \cdot \partial_\alpha X - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right). \tag{3.41}$$

This action can be treated very analogously to the bosonic case, and having analyzed that in some detail we just present the results and main differences.

One important new feature is the possibility for two different boundary conditions for the fermions. Indeed, varying with respect to ψ^μ in (3.41) and keeping the boundary terms reveals that one must require

$$\psi_+ \cdot \delta\psi_+ = \psi_- \cdot \delta\psi_- \tag{3.42}$$

at $\sigma = 0, \pi$ for the open strings. This is satisfied for either $\psi_+^\mu = \psi_-^\mu$ or $\psi_+^\mu = -\psi_-^\mu$ at the two boundaries. Since we are free to rename the fields $\psi_+ \to -\psi_+$, we can always choose $\psi_+^\mu(0) = \psi_-^\mu(0)$ so that the distinct physical possibilities are parameterized by the $\sigma = \pi$ condition. The two boundary conditions are known as Ramond (R) and Neveu–Schwarz (NS) boundary conditions:

$$\begin{aligned} \mathrm{Ramond:} \qquad & \psi_+^\mu(\pi) = \psi_-^\mu(\pi), \\ \mathrm{Neveu-Schwarz:} \qquad & \psi_+^\mu(\pi) = -\psi_-^\mu(\pi). \end{aligned} \tag{3.43}$$

For closed strings one can choose periodicity (R) or anti-periodicity (NS) for the left- and right-moving fermions, respectively. This results in four sectors: NS–NS, NS–R, R–NS and NS–NS.

Proceeding as in the bosonic case, i.e, identifying the constraints[3] and implementing them in the light-cone gauge one finds the condition on D. It turns out that the critical superstring dimension is $D = 10$.

The spectrum is generated by both bosonic creation operators α_n^μ, and fermionic counterparts ψ_r^μ playing an almost identical role. Because of the boundary conditions r takes integer values in the R-sector and half-integer values in the NS–NS sector. Both closed and open superstring theories contain a tachyon in the NS–NS and the NS sectors, respectively.

3.2.2 GSO projection and the fantastic five

Even if we did not show it, it seems plausible that the inclusion of world-sheet fermions also leads to states that transform as spinors under the spacetime

[3]Now these come from gauge-fixing a supergravity action. The corresponding algebra is called the super-Virasoro algebra.

Lorentz group. This is indeed the case. We have therefore met the goal to make strings that describe spacetime fermions. To wrap things up we just need to get rid of the tachyon.

Wonderfully enough, superstring theory admits a consistent truncation known as the *GSO projection* after its inventors F. Gliozzi, J. Scherk and D.I. Olive. Basically it projects out the states with an even number of fermionic excitations and keeps those with an odd number. In the NS sector, the projection is really as simple as that: the projection operator P is given by

$$P = \frac{1}{2}(1 - (-1)^F) \tag{3.44}$$

where F is the fermion number operator

$$F = \sum_{r>0} \psi_{-r} \cdot \psi_r . \tag{3.45}$$

In the R sector it turns out that one has to do a bit more. One has to correlate the fermion number projected out with the spacetime chirality of the state. There are two choices for how to do this:

$$P = \frac{1}{2}(1 \pm \Gamma^{11}(-1)^F) \tag{3.46}$$

where Γ^{11} is the spacetime chirality operator. The choice between $+$ and $-$ is just conventional for open strings, but it becomes important in the R–R sector of closed strings where it is possible to use different GSO projections for left- and right-movers.

If the projections above are carried out, only half of the mass levels survive. In particular, the tachyonic NS mode is eliminated. Left is a theory which is unitary, Lorentz invariant, free of tachyons, spacetime supersymmetric and offers a perturbative description of quantum gravity. This is a truly remarkable achievement.

Let us now try to collect the possible consistent string theories. It turns out that there are two more pieces of information needed before we can do so. First, in the open string case one can actually obtain a consistent theory by choosing the left-movers to be a set of $D = 26$ bosonic oscillators while the right-movers are superstring variables. The additional 16 bosonic degrees of freedom are interpreted as internal. The resulting theory is called *heterotic* string theory. Second, it is possible to add non-abelian gauge invariance to open strings. The quantum theory turns out to be very restrictive on the possible gauge groups, allowing only SO(32) in the “pure superstring” case and allowing either of SO(32) or $E_8 \times E_8$ in the heterotic case.

When the dust settles, there are five known consistent superstring theories:

- *Type I* superstrings are open strings with fermionic excitations traveling in both directions. The only allowed gauge group is SO(32). The theory has $\mathcal{N} = 1$ supersymmetry in spacetime.
- *Type IIA* superstrings are closed superstrings where the *opposite* GSO projections are chosen for left- and right-movers. The theory is $\mathcal{N} = 2$ spacetime supersymmetric.
- *Type IIB* superstrings are closed superstrings where the *same* GSO projections are chosen for left- and right movers. Also this theory has $\mathcal{N} = 2$ supersymmetry.
- The *heterotic SO*(32) theory consists of open strings with left-movers from bosonic string theory. The gauge group is SO(32) and the theory has $\mathcal{N} = 1$ supersymmetry.
- The *heterotic* $E_8 \times E_8$ theory consists of open strings with left-movers from bosonic string theory. The gauge group is $E_8 \times E_8$ and the theory has $\mathcal{N} = 1$ supersymmetry.

One can now go on to construct the massless spectra of these theories. The procedure is identical to that for bosonic strings. We will only be concerned with type II theories in this thesis, so we present their spectra only.

Spacetime fermions reside in the mixed (NS–R and R–NS) sectors. These are gravitini (Ψ and Ψ') and dilatini (λ and λ'). In our story fermions will always be set to zero. Bosons, on the other hand will play an important part. In the NS–NS sector we find the massless bosonic fields present already in the bosonic theory: the graviton $g_{\mu\nu}$, the Kalb–Ramond two-form field[4] B_2 with corresponding field strength $H_3 = dB_2$ and the dilaton Φ. In the R–R sector type IIA has gauge potentials of odd degree: C_k with $k = 1, 3, 5, 7$ whereas type IIB has gauge potentials of even degree C_k with $k = 0, 2, 4, 6, 8$. The field strength of a k-form gauge potential is a $(k + 1)$-form given by the external derivative: $F_{k+1} = dC_k$.

These R–R fields turn out to be electromagnetic dual to each other:

$$F_{k+1} = *F_{10-(k+1)}, \tag{3.47}$$

where $*$ is the Hodge star operator. Thus only $C_{1,3}$ and $C_{0,2,4}$ represent independent degrees of freedom and the five-form field strength of type IIB is self dual:

$$F_5 = *F_5. \tag{3.48}$$

The bosonic field content of type II string theories is summarized in Table 3.1. Let us now turn to the low energy dynamics of the type IIB fields.

[4]To avoid cluttering the notation with indices we use the language of differential forms to describe gauge fields and their field strengths. Readers feeling uneasy with forms should without delay consult the excellent lecture notes on topological strings [Von05] by M. Vonk.

Sector:	IIA	IIB
NS–NS	$g_{\mu\nu}, B_2, \Phi$	$g_{\mu\nu}, B_2, \Phi$
NS–R	$\Psi_\mu, \quad \lambda$	$\Psi_\mu, \quad \lambda$
R–NS	$\Psi'_\mu, \quad \lambda'$	$\Psi'_\mu, \quad \lambda'$
R–R	$C_1, C_3, \ldots, C_7$	$C_0, C_2, \ldots, C_8$

Table 3.1: *Massless spectra of the type IIA and IIB theories. The NS–NS sector consists of the graviton $g_{\mu\nu}$, the Kalb–Ramond field B_2 and the dilaton Φ. The mixed sectors contain gravitini (Ψ and Ψ') and dilatini (λ and λ'). In the R–R sector type IIA has gauge potentials of odd degree, whereas type IIB has gauge potentials of even degree. Electromagnetic duality relates C_k to C_{8-k} in each theory. In particular $F_5 = dC_4$ is self dual.*

3.2.3 Type IIB supergravity

Since string theory is technically very complicated, and computations in the full formalism are hard, a useful approach is to identify situations where "stringy" corrections are small. These situations can be analyzed in terms of a *low energy effective theory*. For string theory such theories are *supergravity theories*, i.e., theories with local supersymmetries.

There are a number of ways to obtain information about the low energy field theories. One was illustrated in Section 3.1: consistent quantization requires background fields to fulfill certain equations. One can also use tree-level scattering amplitudes or, as is most efficient in the present case, rely on supersymmetry. Type IIB string theory is maximally supersymmetric and possesses 32 supersymmetry generators. In ten dimensions there are only two maximal supergravities. These, not coincidentally, go under the names type IIA and type IIB supergravity.

The bosonic part of the action of type IIB supergravity is[5]

$$S_{\text{IIB}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}} \tag{3.49}$$

where

$$\begin{aligned}
S_{\text{NS}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{|g|}\, e^{-2\Phi} \left(R + 4\partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2 \cdot 3!} |H_3|^2 \right) \\
S_{\text{R}} &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{|g|} \left(|F_1|^2 + \frac{1}{3!} |\tilde{F}_3|^2 + \frac{1}{2 \cdot 5!} |\tilde{F}_5|^2 \right) \\
S_{\text{CS}} &= -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3.
\end{aligned} \tag{3.50}$$

[5]In taking the square of a p-form, we differ by a factor $p!$ form the conventions of Ref. [Polb]. We take $|F_p|^2 = F_{\mu_1 \cdots \mu_p} F^{\mu_1 \cdots \mu_p}$.

The $\tilde{F}_3$ and $\tilde{F}_5$ are just combinations of the fields and are given by

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3, \qquad \tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3, \tag{3.51}$$

and

$$\kappa_{10}^2 = \frac{1}{2}(2\pi)^7 (\alpha')^4. \tag{3.52}$$

Nothing in this action is really that surprising. There is an Einstein–Hilbert term for the metric. It is multiplied by the dilaton, but this coupling can be eliminated by redefining the metric $g_{\mu\nu}^E = e^{-\Phi/2} g_{\mu\nu}$. In string lingo this is known as changing from "string frame" to "Einstein frame". Then there is a kinetic term for the dilaton Φ and Maxwell terms $|F|^2 = F \wedge *F$ for each of the field strengths F_1, $\tilde{F}_3$, $\tilde{F}_5$ and H_3. Regarding these so-called *improved* field strengths $\tilde{F}_{p+1}$ it is noteworthy that the gauge transformations that leave the action invariant are not the naively expected ones. Instead they are

$$C_p \to C_p + d\Lambda_{p-1} + H \wedge \Lambda_{p-3}. \tag{3.53}$$

This transformation leaves $\tilde{F}_p$ invariant, but not $F_p = dC_{p-1}$. Note that the field strengths $\tilde{F}_{p+1}$ are not closed. Finally there is a Chern–Simons term S_{CS}. The action (3.49) almost describes the low-energy dynamics, but not quite. The self-duality condition of the five-form field strength does not follow from varying the fields but must be put in by hand at the level of equations of motion.

This concludes our discussion of perturbative superstring theory. The last section of this chapter treats other extended objects in string theory: D-branes.

3.3 D-branes

Toward the end of the last section we found that the type II string theories contain a number of p-form gauge fields. It would be a shame to have all these gauge fields with nothing to source them. Let us now turn to this issue. As a warm-up let us deal with a simple system but in fancy language.

Consider a one-form $U(1)$ gauge field $C_1 = C_\mu dx^\mu$ in D-dimensional spacetime. A test-particle is electrically charged under A if there is a term

$$S_{\text{charge}} = q_e \int_{\Sigma^1} C_1 \tag{3.54}$$

in the action. Here Σ^1 is the particle's world-line and q_e is the charge. Note that it is essential that the dimensions match. The one-form is integrated over a one-dimensional world-volume. Including also a Maxwell term

$$S_{\text{Maxwell}} = \frac{1}{2} \int_{\Sigma^1} F_2 \wedge *F_2 \tag{3.55}$$

the equations of motion read

$$d * F_2 = q_e \delta^{(D-1)}(\Sigma^1). \tag{3.56}$$

They imply the Gauss law

$$\int_{S^{D-2}} *F_2 = q_e \int_{B^{D-1}} \delta^{(D-1)}(\Sigma^1) = q \tag{3.57}$$

for any spacelike $S^{D-2} = \partial B^{D-1}$ enclosing the point charge. For a magnetically charged object, there is no term in the action corresponding to (3.54), because the gauge potential is not well defined on a magnetic monopole. Namely, a magnetic charge q_m is a source for F_2 rather than $*F_2$:

$$dF_2 = q_m \delta^3(\Sigma^{D-3}) \neq 0. \tag{3.58}$$

This implies

$$\int_{S^2} F_2 = q_m \int_{B^3} \delta^3(\Sigma^{D-3}) = q_m. \tag{3.59}$$

Note that a magnetic monopole is not point-like if $D \neq 4$, but rather a $(D-4)$-dimensional object with $(D-3)$-dimensional world-volume.

For higher dimensional gauge fields the story is completely analogous. A $(p+1)$-form gauge field C_{p+1} couples electrically to a p dimensional object with $(p+1)$-dimensional world-volume:

$$S_{\text{object}} = q_e \int_{\Sigma^{p+1}} C_{p+1}, \tag{3.60}$$

and magnetically to a $(D-(p+4))$-dimensional object with $(D-(p+3))$-dimensional world-volume:

$$\int_{S^{p+2}} F_{p+2} = q_m \int_{B^{p+3}} \delta^{(p+3)}(\Sigma^{D-(p+3)}) = q_m. \tag{3.61}$$

From the perturbative point of view there is only one extended object in string theories: the string itself. It should be electrically charged under a two-form gauge field. This gauge field turns out to be the B-field.

For the R–R fields C_{p+1} of the type II theories there is no such world-sheet coupling. Indeed these fields couple to perturbative strings only through their field strengths and not through the potentials. Strings are therefore not charged even under the C_2 of type IIB for which the dimensions match.

The nature of the R–R charges was first analyzed from a supergravity perspective. The charged objects were found to be solitonic solutions, not unlike the ordinary magnetic monopole, and were called p-branes.

It was J. Polchinski who in 1995 explained the microscopic nature of the p-branes from string theory. They are objects on which open strings can end. Recall the possible boundary conditions for open strings (3.8). One can *mix* Neumann and Dirichlet boundary conditions, having different conditions in different directions:

$$\begin{array}{ll} \partial_\sigma X^\mu(0,\tau) = \partial_\sigma X^\mu(\pi,\tau) = 0 & \mu = 0 \dots p \\ X^\mu(0,\tau) = f_0^\mu(\tau), \quad X^\mu(\pi,\tau) = f_\pi^\mu(\tau) & \mu = p+1 \dots D. \end{array} \tag{3.62}$$

These boundary conditions mean that the strings are free to move in p spatial dimensions but constrained in the other $D-(p+1)$. It is exactly as if the endpoints were attached to some objects whose motion is described by the functions $f_{0,\pi}^\mu$.

What Polchinski did was to compute the R–R charge of such an object, finding that it corresponds exactly to that of a p-brane! This discovery was one of the key ingredients in the second superstring revolution of the mid 90s, the other being the web of dualities and M-theory[6].

There are different types of gauge fields in the two type II theories and therefore different types of D-branes: type IIA has even dimensional branes, and type IIB odd dimensional. For later reference we write down the value of the electric R–R charge of a μ_p of a Dp-brane (i.e., its coupling to C_{p+1}):

$$\mu_p^2 = \frac{\pi}{\kappa_{10}^2}(4\pi^2\alpha')^{3-p}. \tag{3.63}$$

Branes are nowadays viewed as our most important window into the non-perturbative physics of string theory. They are also of crucial importance in constructing realistic string theory models. In this thesis D-branes will appear as pieces in this cosmic LEGO in Chapter 6 and as microscopic models of black holes in Chapter 5.

[6]We postpone the discussion of string dualities until we need them in Part II of this thesis.

4. Type IIB compactifications

Rakt fram kan man inte gå så långt
Lille prinsen

To obtain phenomenological models from string theory, the number of spacetime dimensions must be reduced. The most popular way of doing this is to compactify six of the dimensions. In this chapter we explain the basic physics of compactifications of type IIB superstrings. We start by a down-to-earth treatment of Kaluza–Klein reduction, explaining how the ten-dimensional fields split up in modes that are massless, and modes with mass inversely proportional to the compactification radius. Then we describe type IIB compactifications with $\mathcal{N} = 2$ spacetime supersymmetry in four dimensions. The corresponding compact spaces are Ricci-flat and Kähler, and have received their own name: *Calabi–Yau* manifolds. For a given Calabi–Yau topology there is a finite-dimensional space of Ricci-flat Kähler metrics called the *moduli space*. Deformations of the metric within this space correspond to unobserved, and hence unwanted, massless scalar fields in four dimensions.

To construct models where all these fields receive masses is crucial to come in contact with real world phenomenology. One effect that gives masses to the moduli fields are R–R and NS–NS fluxes piercing cycles in the internal geometry. These break supersymmetry (at least) to $\mathcal{N} = 1$ in four dimensions and the resulting theories are called *flux compactifications*. Flux compactifications play an important role in this thesis, and we put some effort into reviewing their basic properties in the last section of this chapter.

4.1 Kaluza–Klein reduction

Suppose that the world, let us call it $\mathcal{X}$, is a direct product of Minkowski space $\mathbb{R}^{3,1}$ and a small compact d-dimensional part $\mathcal{M}$:

$$\mathcal{X} = \mathbb{R}^{3,1} \times \mathcal{M}. \tag{4.1}$$

We denote the coordinates on the whole $\mathcal{X}$ by x^M with $M = 0, \ldots d + 3$ with a capital Latin index. The coordinates on $\mathbb{R}^{3,1}$ are denoted x^μ with Greek indices and those on $\mathcal{M}$ are called y^m, $m = 1, \ldots, d$ with small case Latin indices. Note that this differs from Chapter 3, where the ten-dimensional indices were labeled by Greek indices.

We assume that the background metric is block-diagonal in (x^μ, y^m):

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{mn}(y) \end{pmatrix}, \tag{4.2}$$

where $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric.

Suppose now that there is some field theory living on $\mathcal{X}$. We are interested in the low-energy physics, i.e., physics that takes place at length scales much larger than the compactification radius $R = (\mathrm{Vol}(\mathcal{M}))^{1/d}$. Consider first a massless scalar field ϕ obeying the Klein–Gordon equation

$$\nabla^M \nabla_M \phi = \partial^\mu \partial_\mu \phi + \nabla^m \nabla_m \phi = 0. \tag{4.3}$$

Let us expand ϕ in the eigenfunctions of the Laplacian $\Delta^{(d)}$ on $\mathcal{M}^d$:

$$\phi = \sum_n f_n(x^\mu) h_n(y), \qquad \Delta^{(d)} h_n = \nabla^m \nabla_m h_n = -M_n^2 h_n. \tag{4.4}$$

We have denoted the (non-positive) eigenvalues of the Laplacian by $-M_n^2$. They indeed have mass-dimension two, and by inserting the expansion into (4.3) we find

$$\partial^\mu \partial_\mu f_n - M_n^2 f_n = 0. \tag{4.5}$$

The expansion coefficients f_n behave as four-dimensional fields with mass M_n. This infinite tower of massive fields is called the *Kaluza–Klein* (KK) tower after T. Kaluza and O. Klein who studied compactifications in the 1920s [Kal21, Kle26]. Their motivation was to unify electromagnetism and general relativity.

The eigenvalues of the Laplacian go as $M_n \sim 1/R$, so the low energy physics taking place at length scales $\gg R$ will not be able to excite expansion coefficients with nonzero M_n. For the scalar field case this means that the low energy degrees of freedom are *constant* on $\mathcal{M}^d$ since any harmonic function on a compact manifold is constant. Let us, just for fun, put a naive experimental upper bound on the size R of extra dimensions. In the latest particle accelerators we are able to detect particles of masses $M \sim 10^2$ GeV, so we should see the first states of the KK tower if the compact dimensions have a size of order

$$R \sim \frac{\hbar c}{10^2\mathrm{GeV}} \sim 10^{-18}\,\mathrm{m}. \tag{4.6}$$

Not only scalar fields give rise to massless modes and KK towers when compactified. For fields with nonzero spin the analysis is very similar, but a little more interesting. Let us consider the gauge potentials $C_{0,2,4}$ and B_2 of type

IIB for instance. Making a similar expansion as in (4.4) one finds that modes that do not receive KK masses correspond to harmonic forms on the internal space. There are two crucial differences from the scalar case. First, the legs of the form can be split up in several ways into external and internal indices. For definiteness, B_2 will have three components:

$$B_2(x,y) = B_2(x) + B_1(x) \wedge v(y) + b(x)\omega(y). \tag{4.7}$$

Here $B_2(x)$, $B_1(x)$ and $b(x)$ are a two-form, a one-form and a scalar, respectively. These forms have no legs in the compact dimensions, and do not depend on them. The forms v and ω are purely compact forms (v is a one-form and ω is a two-form). The compact forms must be harmonic for the multiplying non-compact form to be massless. Thus a two-form field can give rise to massless two-forms, one-forms and scalars in four-dimensions!

Secondly, the harmonic forms are more interesting than the harmonic scalars that are simply constant. A compact manifold can have many independent harmonic forms of a given degree. The space of harmonic forms is in one-to-one correspondence with de Rham cohomology classes. So (4.7) should really look like this:

$$B_2(x,y) = B_2(x) + B_1^A(x) \wedge v_A + b^a(x)\omega_a(y), \tag{4.8}$$

where v_A is a basis of $H^1_\Delta(\mathcal{M}) \approx H^1_{\mathrm{deRham}}(\mathcal{M})$ and ω_a is a basis for $H^2_\Delta(\mathcal{M}) \approx H^2_{\mathrm{deRham}}(\mathcal{M})$.

Here we encounter the fact that the topology of the compact manifold $\mathcal{M}$ influences the low energy field content of the four dimensional theory. The dimensions of the de Rham cohomology groups — the *Betti numbers* — are topological invariants. They determine the spectrum of four-dimensional form fields.

By diagonalizing the relevant wave operator, fermion fields can be treated in a very similar manner. Treating fluctuations of the metric also follows a closely related routine. Fluctuations $\delta g_{\mu\nu}$ in the non-compact components of the metric are heavy unless they fulfill the linearized four-dimensional Einstein's equations. They are ordinary gravitational waves. Massless off-diagonal fluctuations $\delta g_{m\mu}$ are in one-to-one correspondence with continuous isometries of the compact manifold. These isometries and their generating Killing vectors thus play the role harmonic one-forms did for antisymmetric two-tensors. Their existence directly influences the low-energy field content. Last, a field $z(x)$ multiplying a purely internal fluctuation δg_{mn} is massless only if the fluctuation δg_{mn} keeps the internal geometry Ricci flat. Such fluctuations parameterize the possible background metrics and are called *moduli*. Thus, a continuous degeneracy in the space of compact background metrics g_{mn} results in massless scalars in four-dimensions.

So we learned that in compactifying a higher dimensional field theory, the compact geometry and topology provide the defining data of the low-energy effective field theory. With our newly acquired knowledge of string theory we immediately ask ourselves if there are any special compact geometries that string theory would like, or perhaps even demand. We saw earlier that string theory is picky with a lot of things. Happily or sadly, dependent on your inclination, it seems that string theory has relatively mild opinions regarding the compact space. From a computational and phenomenological point of view however, supersymmetry is very desirable. Let us turn to a class of compactifications having this feature.

4.2 Calabi–Yau spaces

The most obvious requirement that the internal geometry must fulfill is that it solves the equations of motion. If we consider just empty space, these are (to first order in α') Einstein's vacuum equations $R_{mn} = 0$. If we consider more general backgrounds, e.g including fluxes, the equations are more complicated.

Another, albeit not *a priori* necessary, but both phenomenologically and practically attractive feature is supersymmetry. The phenomenological appeal comes from a tentative solution to the hierarchy problem and a more accurate unification of gauge couplings. Supersymmetry is also desirable from a practical perspective. It is much simpler to find supersymmetric solutions to a theory, and supersymmetry can restrict e.g. the form of the resulting low energy action.

Supersymmetry will play a vital role in the rest of this chapter, and also in Chapters 5 and 6. It will mainly do so behind the scenes, but now and again some technicalities will surface. We do not attempt any introduction to supersymmetry here. Instead we try to explain the necessary concepts first when explicitly forced to, starting in the next subsection. A reader completely unfamiliar with supersymmetry will find the explanations far from sufficient though, and should consult e.g. the reviews by J.M. Figueroa-O'Farrill [FO01] and D.G. Cerdeno and C. Munoz [CM] which are both very readable.

4.2.1 Geometry and supersymmetry

Supersymmetry is a symmetry relating fermions and bosons; a supersymmetry transformation takes bosons to fermions and vice versa. Consequently, the infinitesimal parameter in the transformation must be Grassmannian. Denoting bosonic fields collectively by ϕ and fermionic fields by ψ, a very schematic transformation would look like

$$\begin{aligned} \delta_\epsilon \phi &\sim \bar{\epsilon}\psi \\ \delta_\epsilon \psi &\sim \gamma^M \mathcal{D}_M^{(\phi,\psi)} \epsilon. \end{aligned} \tag{4.9}$$

Here γ^M are the Dirac matrices and $\mathcal{D}_M^{(\phi,\psi)}$ is a differential operator possibly depending on the fields. We are concerned with *local* supersymmetry, i.e., we let the transformation parameter ϵ depend on position.

In (4.9) the fermions are implicitly assumed to be spin 1/2. For spin 3/2 the spinor has both a spinor index and a vector index. The corresponding transformation is (still very formally, and with the spinor index still suppressed)

$$\delta_\epsilon \psi_M \sim \mathcal{D}_M^{(\phi,\psi)} \epsilon. \tag{4.10}$$

A supersymmetric theory has an action that is invariant under transformations such as (4.9) and (4.10), and to find a supersymmetric solution means to find a field configuration that is annihilated by them.

Typically one is interested in purely bosonic backgrounds (i.e., $\psi = 0$). For such field configurations the bosonic variations $\delta_\epsilon \phi$ automatically vanish. The relevant conditions for supersymmetry therefore become the vanishing of the fermionic variations: $\delta_\epsilon \psi = 0$. In type IIB supergravity there are four spinors — two dilatini and two gravitini. These are denoted λ, λ' and[1] Ψ_M, Ψ'_M in Table 3.1. Here we collect them into column vectors:

$$\lambda = \begin{pmatrix} \lambda \\ \lambda' \end{pmatrix} \tag{4.11}$$

and similarly for Ψ_M. Setting all fermions to zero, the gravitini and dilatini variations of type IIB read

$$\begin{aligned} \delta_\epsilon \psi_M &= \nabla_M \epsilon - \frac{1}{4} \not{H}_M \sigma^3 \epsilon + \frac{1}{16} e^\phi \sum_{n=1}^{9} \tilde{\not{F}}_n \gamma_M \mathcal{P}_n \epsilon \\ \delta_\epsilon \lambda &= \left(\not{\partial}\phi - \frac{1}{2} \not{H} \sigma^3 \right) \epsilon + \frac{1}{8} e^\phi \sum_{n=1}^{9} (-1)^n (5-n) \tilde{\not{F}}_n \mathcal{P}_n \varepsilon. \end{aligned} \tag{4.12}$$

In these equations ∇_M is the covariant derivative with respect to the Levi-Civita connection, a "/" means a suitable contraction with Dirac γ-matrices and the operators $\mathcal{P}_n$ are either σ^1 or $i\sigma^2$ dependent on n. (Note that these Pauli matrices act not on the individual spinors, but on the column vectors of spinors.) The variations in (4.12) seem horrendously complicated. However, if we are interested in backgrounds where also no bosonic fields are excited, there are big simplifications. Indeed, setting all fields save for the metric to zero yields

$$\begin{aligned} \delta_\epsilon \psi_M &= \nabla_M \epsilon \\ \delta_\epsilon \lambda &= 0. \end{aligned} \tag{4.13}$$

[1]We remind of the difference in conventions: In Chapter 3 ten-dimensional indices were denoted by Greek letters.

Demanding that the gravitino variation vanishes implies $\nabla_M \epsilon = 0$. Therefore empty space can be supersymmetric if and only if there exists a covariantly constant spinor ϵ. Splitting this spinor into compact and non-compact components this implies the existence of a covariantly constant spinor $\eta_+ = (\eta_-)^*$ on the compact space $\mathcal{M}$. The supersymmetry parameter ϵ decomposes as

$$\epsilon = \xi_+ \otimes \eta_+ + (\text{c.c.}), \tag{4.14}$$

where ξ_+ is a four-dimensional spinor. The spinor ξ_+ plays the role of the supersymmetry parameter in the four-dimensional effective theory.

Existence of a covariantly constant spinor turns out to be a condition with quite profound implications. These implications were first analyzed in the seminal paper Ref. [CHSW85] by P. Candelas, G. Horowitz, A. Strominger and E. Witten. Let us follow in their footsteps.

A covariantly constant spinor has constant norm (let us choose $||\eta_+|| = 1$), and is therefore nowhere vanishing. The existence of a nowhere vanishing spinor is a topological condition and $\nabla_m \eta_+ = 0$ is a differential condition, so they restrict both the topology and the metric on $\mathcal{M}$. These conditions can be conveniently formulated in terms of *holonomy*. On any Riemannian manifold one can parallel transport tangent vectors. Doing this along a loop, starting and ending at the same point, the result is in general a new vector, related to the old by a linear transformation. This linear transformation depends on the chosen path and is called a holonomy. The set of holonomies forms a group, depending on both the topology and geometry of the manifold. For a generic d-dimensional Riemannian manifold this group is $\mathrm{SO}(d)$, but it can also be smaller. For flat Euclidean space for instance the group is just the identity element, and for the flat Möbius strip it is $\mathbb{Z}_2$. Let us now heuristically explain the connection between existence of spinors and holonomy.

The spin group of six dimensions is the double cover of $\mathrm{SO}(6)$ which is $\mathrm{SU}(4)$. A spinor transforms in the fundamental $\mathbf{4}$ or the anti-fundamental $\overline{\mathbf{4}}$ of $\mathrm{SU}(4)$ depending on chirality. Let us consider our covariantly constant spinor η_+ at some point p in $\mathcal{M}$. We choose coordinates so that $\eta_+(p) = (1, 0, 0, 0)$. Since a covariantly constant spinor is invariant under holonomy, it means that whatever loop we traverse, η_+ must return to itself. Therefore, in this parameterization, the holonomy group of $\mathcal{M}$ must leave the first component of any spinor invariant, and only act on the following three components. This implies that the holonomy is contained in $\mathrm{SU}(3) \subset \mathrm{SO}(6)$. To have supersymmetry in four dimensions we thus require our compactification manifold to have $\mathrm{SU}(3)$ holonomy.

The U in $\mathrm{SU}(3)$ indicates that it might be convenient to describe the compact geometry using *complex geometry*[2]. This is indeed the case. Using the language of complex geometry it is straightforward to show that the existence

[2]We assume some familiarity with this subject. Recommended places to acquire this familiarity include Vonk's lecture notes [Von05] and the book "Mirror symmetry" [H$^+$] by K. Hori *et al.*

of a covariantly constant spinor also implies that the manifold is Ricci flat, i.e., that the equations of motion are fulfilled. Let us briefly describe this argument.

The spinor η_+ defines an almost complex structure $J_m{}^n$ by

$$J_m{}^n = i(\eta_+)^\dagger \gamma_m{}^n \eta_+. \tag{4.15}$$

Since η_+ is covariantly constant it follows that $J_m{}^n$ also is covariantly constant. This implies directly that we can lose the "almost": $J_m{}^n$ is a complex structure.

We remind the reader that defining a complex structure allows one to split the coordinates in holomorphic z^i and antiholomorphic coordinates $z^{\bar{\imath}}$. In these coordinates the only non-zero components of $J_m{}^n$ are pure: $J_i{}^j = i\delta_i^j$ and $J_{\bar{\imath}}{}^{\bar{\jmath}} = -i\delta_{\bar{\imath}}^{\bar{\jmath}}$. A Hermitian metric has only mixed components: $ds^2 = g_{i\bar{\jmath}} dz^i dz^{\bar{\jmath}}$ and defines a $(1,1)$-form J:

$$J = i g_{i\bar{\jmath}} dz^i \wedge dz^{\bar{\jmath}} = J_{i\bar{\jmath}} dz^i \wedge dz^{\bar{\jmath}}. \tag{4.16}$$

This form is called the *Kähler form*. If the Kähler form is closed $dJ = 0$ the metric $g_{i\bar{\jmath}}$ is said to be Kähler.

The Kähler condition $dJ = 0$ is equivalent to $J_m{}^n$ being covariantly constant. Therefore the existence η_+ with $\nabla_+\eta = 0$ implies that $\mathcal{M}$ is a complex Kähler manifold. We also know that the holonomy must be SU(3). Manifolds with these properties are called *Calabi–Yau manifolds* after the geometers E. Calabi and S-T. Yau who respectively conjectured and proved a deep theorem concerning these spaces. We come to this result shortly. Let us for emphasis write down a formal definition.

Definition 1 *An n complex dimensional* Calabi–Yau manifold *is a compact, Kähler manifold having SU(n) holonomy.*

Note that we (this is a matter of taste) require the holonomy to be *exactly* SU(n) and not a subgroup of it. Thus we do not consider, e.g., flat tori to be Calabi–Yau. We will exclusively deal with the case $n = 3$.

It is an easy exercise to show that for a Kähler metric all Christoffel symbols having mixed indices are zero, meaning that (anti)holomorphic vectors stay (anti)holomorphic under parallel transport. This is equivalent to the holonomy being contained in U(3). From the Christoffel symbols one can work out the Ricci form:

$$\mathfrak{R}_{i\bar{\jmath}} = -\partial_{\bar{\jmath}} \Gamma^k_{ik}. \tag{4.17}$$

The trace part Γ^k_{ik} of the connection generates the U(1)-factor of U(3) and vanishes for SU(3) holonomy. Therefore such a manifold is automatically Ricci flat.

We have now reached the conclusion that supersymmetric compactifications with no non-zero fields (except the metric) are compactifications on Calabi–Yau manifolds. Such manifolds therefore have played a very important part in string phenomenology. In the next subsection we dig deeper into Calabi–Yau geometry and present some concepts that will feature in Chapters 5 and 6.

Let us end this subsection by noting that in type II theories there really two spinors, ϵ and ϵ' corresponding to the two column vector components of (4.13). This means that there are two four-dimensional supersymmetry generators ξ_+ and ξ'_+, and thus that there are two unbroken supersymmetries: $\mathcal{N} = 2$. In type I and heterotic theories Calabi–Yau compactifications result in $\mathcal{N} = 1$ supergravities in four-dimensions.

4.2.2 Calabi–Yau geometry

In the last subsection we found a condition on $\mathcal{M}$ that results in supersymmetric compactifications. To do phenomenology it is crucial to find such manifolds. To construct complex Kähler manifolds is relatively easy, but to construct the Ricci flat metric with $\mathrm{SU}(3)$ holonomy is not. In fact for $n = 3$ there is not a single example of an explicitly known Calabi–Yau metric.

There is however a simple condition which is equivalent to the existence of a Ricci flat metric. Namely, we have

Theorem 1 *Suppose $\mathcal{M}$ is a complex Kähler manifold with Kähler form J. If the first Chern class of $\mathcal{M}$ vanishes, then there is a unique Ricci flat Kähler metric on $\mathcal{M}$, whose Kähler form is in the same cohomology class as J.*

The first Chern class $c_1(\mathcal{M})$ is a cohomology class in $H^2_{\mathrm{deRham}}(\mathcal{M})$. In fact it is the cohomology class of the (suitable normalized) Ricci form: $c_1 = [\mathfrak{R}/2\pi]$. It is therefore obvious that a Ricci flat manifold must have $c_1 = 0$. It is the converse, formalized in the theorem above that is highly nontrivial. E. Calabi showed the uniqueness part of the theorem in 1955 [Cal55]: if the Ricci flat metric exists it is unique. He also conjectured the existence which was proven later by S-T. Yau [Yau78].

Using Yau's theorem one can now construct Kähler manifolds, compute c_1 (which is straightforward) and, if $c_1 = 0$, be certain that there is a metric on the manifold suitable for superstring compactification.

Yau's theorem also hints at the structure of the moduli space of Ricci flat metrics. Recall that metric deformations that preserve Ricci flatness are called moduli, and that they correspond to massless fields in four dimensions. For a Calabi–Yau, given a complex structure and a Kähler class the Ricci flat metric is completely determined. The moduli space of a Calabi–Yau manifold is therefore identical to the combined moduli space of complex structures and Kähler classes. In Chapters 5 and 6 we will be very interested in Calabi–Yau moduli spaces, and in particular in the complex structure moduli space. Therefore we put some effort into describing them here. Dolbeault cohomology theory is essential for this, so let us recall a few facts.

On a complex manifold the exterior derivative can be split up into a sum of a holomorphic and antiholomorphic derivative:

$$d = \partial + \bar{\partial}. \tag{4.18}$$

Forms similarly split up according to the number of holomorphic and antiholomorphic indices: e.g, $dz^i \wedge dz^j \wedge dz^k$ is a $(3,0)$-form, whereas $dz^i \wedge dz^j \wedge dz^{\bar{k}}$ is a $(2,1)$-form. Both ∂ and $\bar{\partial}$ square to zero and can be used to define *Dolbeault cohomology classes* $H^{p,q}_{\bar{\partial}}$ and $H^{p,q}_{\bar{\partial}}$ of (p,q)-forms. For a Kähler manifold, $H^{p,q}_{\bar{\partial}} \approx H^{p,q}_{\partial}$ and the de Rham cohomology is a sum of Dolbeault cohomology:

$$H^k_{\text{deRham}} = H^{k,0}_{\bar{\partial}} \oplus H^{k-1,1}_{\bar{\partial}} \oplus \ldots \oplus H^{0,k}_{\bar{\partial}}. \tag{4.19}$$

The Betti numbers $b^k = \dim(H^k_{\text{deRham}})$ consequently become sums over integers $h^{p,q} = \dim(H^{p,q}_{\bar{\partial}})$ called the *Hodge numbers*:

$$b^k = h^{k,0} + h^{k-1,1} + \ldots + h^{0,k}. \tag{4.20}$$

Using complex conjugation and Poincaré duality, the Hodge numbers satisfy $h^{p,q} = h^{q,p}$ and $h^{p,q} = h^{n-p,n-q}$. A common way to present the Hodge numbers is in a so-called Hodge diamond. For the case $n = 3$ it is

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} . \tag{4.21}$$

The relation between the Hodge numbers implies that this figure is symmetric under reflection in both the horizontal and vertical axes.

On a Calabi–Yau manifold the Hodge numbers are even more constrained. SU(3) holonomy implies the existence of a unique (up to constant rescalings) nowhere vanishing holomorphic three-form Ω. This means that $h^{3,0} = h^{0,3} = 1$. Furthermore, one can show that these are the only nontrivial purely (anti)holomorphic cohomology classes[3]: $h^{1,0} = h^{0,1} = h^{2,0} = h^{0,1} = 0$. We therefore end up with a hodge diamond of the following form for Calabi–Yau

[3]Here it is essential that the holonomy is exactly SU(3). Manifold with further restricted holonomy need e.g, not be simply connected: $h^{1,0}(T^6) = 3 \neq 0$.

spaces:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{4.22}$$

The vanishing of $h^{1,0} = h^{0,1}$ is related to the fact that there are no continuous isometries on a Calabi-Yau manifold. There are no globally defined Killing vectors, and thus the off-diagonal metric perturbations $\delta g_{m\mu}$ are all massive.

Let us return to the holomorphic top-form Ω. This form is of importance since it parameterizes the complex structure of the manifold. To understand this, note that in local coordinates we must have

$$\Omega = f(z) dz^1 \wedge dz^2 \wedge dz^3 \tag{4.23}$$

for some holomorphic function f. Hence, Ω determines the complex coordinates, and thus the complex structure. The Calabi–Yau metric is in this way determined by two forms: the $(3, 0)$-form Ω and the $(1, 1)$-form J.

These two forms (or rather their cohomology classes) parameterize the complex structure and Kähler moduli spaces, respectively. Through studying small deformations of the forms, we obtain local descriptions of these moduli spaces, as we now explain.

For a given complex structure, Yau's theorem tells us that each $(1, 1)$ cohomology class $[J]$ corresponds to a Ricci-flat metric. The only requirement on J is that the corresponding metric $g_{i\bar{\jmath}}$ (recall (4.16)) is positive definite. This leads to some inequalities on $[J]$ forcing it to lie in a cone of maximal dimension in $H^{1,1}$. Therefore the Kähler moduli space is locally described by $H^{1,1}$ — in particular its dimension is $h^{1,1}$. In a Calabi–Yau compactification there are thus $h^{1,1}$ massless scalar fields corresponding to the Kähler moduli.

It is a little more subtle to describe the deformations of complex structure. We can for instance not perform the deformation keeping the Kähler class constant since the notion of $(1, 1)$-form depends on complex structure. The space of deformations of complex structure is in fact isomorphic to $H^{2,1}$. A heuristic way to see this is the following. An infinitesimal deformation of complex structure is a deformation of the complex coordinates z^i. Locally

$$z^i \rightarrow z^i + \epsilon^i_j z^j + \epsilon^i_{\bar{\jmath}} z^{\bar{\jmath}}. \tag{4.24}$$

Expanding Ω to first order in ϵ shows that it changes by a rescaling and a general $(2, 1)$-form. The rescaling does not change the complex structure, so the relevant deformation of the cohomology class $[\Omega]$ is a $(2,1)$-cohomology class. Thus the complex structure moduli correspond to $h^{2,1}$ massless scalar fields in four dimensions.

Chapters 5 and 6 both deal with the physics of complex structure moduli. In the next subsection we introduce the necessary mathematical tools to describe this space.

4.2.3 More on the complex structure moduli space

Consider a Calabi–Yau three-fold $\mathcal{M}$ with complex structure moduli space M. The complex structure is described by the cohomology class of the holomorphic three-form Ω. By the duality between homology and cohomology $H^3 \approx H_3$ such a class is completely described by the integrals over a basis of H_3. Let us choose such a basis C_I with $I = 1, \ldots, b^3 = \dim(H_3)$. The integrals

$$\Pi_I = \oint_{C_I} \Omega \tag{4.25}$$

are called the *periods*, and form a set of coordinates on M. This set is redundant: we learned in the last subsection that $\dim(M) = h^{2,1}$ but $b^3 = 1 + h^{2,1} + h^{1,2} + 1 = 2h^{2+1} + 2$.

There is a useful way of splitting the set $\{C^I\}$ in two, to obtain homogeneous coordinates on M. In a six dimensional manifold, two three-cycles generically intersect each other in a number of points. If these points are counted with a sign corresponding to the orientation of the intersection, this number is a topological invariant of the two cycles. We denote the intersection number between C_I and C_J by $C_I \cap C_J$. The $\cap$ defines an antisymmetric product on H_3. It is the Poincaré dual[4] of the intersection product

$$\langle \gamma_I, \gamma_J \rangle = \int_{\mathcal{M}} \gamma_I \wedge \gamma_J \tag{4.26}$$

between three-cohomology classes γ_I.

We can choose a so-called symplectic basis of cycles A^I, B_I with $I, J = 0, \ldots h^{2,1}$

$$A^I \cap A^J = B_I \cap B_J = 0 \qquad A^I \cap B_J = -B_J \cap A^I = \delta^I_J. \tag{4.27}$$

Letting α_I denote the Poincaré dual of B_I and $-\beta^J$ the dual of A^J we have

$$\oint_{A^I} \beta^J = \oint_{B_I} \alpha^J = 0 \tag{4.28}$$

[4]We remind the reader that the Poincaré dual of a homology class $C \in H_k$ is the unique cohomology class $[\gamma] \in H^{d-k}$ satisfying $\int_C \omega = \int_{\mathcal{M}} \gamma \wedge \omega$ for all $\omega \in H^k$.

and

$$\oint_{A^I} \alpha_J = \oint_{B_J} \beta^I = \oint_{\mathcal{M}} \alpha_J \wedge \beta^I = A^I \cap B_J = \delta^I_J. \tag{4.29}$$

Let us now construct homogeneous coordinates on the complex structure moduli space M. Forming the period integrals

$$\begin{aligned} X^I &= \oint_{A^I} \Omega \\ F_J &= \oint_{B_J} \Omega \end{aligned} \tag{4.30}$$

we have

$$\Omega = X^I \alpha_I + F_J \beta^J. \tag{4.31}$$

The periods X^I and F_I determine the form Ω, and thus the complex structure, i.e. a point in M. There can be only $h^{2,1}$ independent coordinates on M, so half of the periods are enough. Let us take the F_I as functions of the X^I. This is still one coordinate too many, reflecting the fact that multiplying Ω with a constant does not change the complex structure.

We learned in the last subsection that the infinitesimal variation of the $(3,0)$-form Ω is a $(2,1)$-form. Thus the form $\partial_I \Omega$ (∂_I denotes differentiation with respect to X^I) is a $(2,1)$-form. Wedging a $(3,0)$ form with a $(2,1)$-form yields zero since there are only three holomorphic coordinates. Therefore we have

$$\begin{aligned} 0 &= \int_{\mathcal{M}} \Omega \wedge \partial_I \Omega = \int_{\mathcal{M}} (X^K \alpha_K + F_K \beta^K) \wedge (\alpha_I + \partial_I F_L \beta^L) = \\ &= -X^J \partial_I F_J + F_I = 2F_I - \partial_I \left(X^J F_J\right) \end{aligned} \tag{4.32}$$

where we used (4.29). It is clear that the F_I are the X^I-derivatives of a function $F(X^I)$ satisfying

$$F = \frac{1}{2} X^J \partial_J F. \tag{4.33}$$

The function F is called the *prepotential* of the Calabi–Yau and plays an important part in the effective four-dimensional theory of $\mathcal{N} = 2$ compactifications. It is a homogeneous function of degree two in X^I as is realized by performing the rescaling $\Omega \to \lambda\Omega$ under which

$$X^I \to \lambda X^I \qquad F_I \to \lambda F_I \tag{4.34}$$

and thus $F(\lambda X^I) = \lambda^2 F(X^I)$. As noted, such a rescaling does not change the complex structure, hence X^I and λX^I denote the same point on M which is to say that X^I are homogeneous coordinates. In a patch of moduli space where $X^0 \neq 0$, affine coordinates are given by $z^I = X^I / X^0$.

In the way described above, complex structure moduli can be viewed as the relative 'holomorphic volumes' $\int_{A^I} \Omega$ of three-cycles A^I in the Calabi–Yau. It is therefore plausible that objects whose energy depends on three-cycle volumes will create potentials for the moduli. We explore such objects in the next section: fluxes piercing cycles in the internal geometry.

This concludes our introduction to Calabi–Yau manifolds. It has been a quite technical story and it is easy to lose the grip on what we are really doing. Let us therefore conclude with a bulleted summary.

- Compactification with unbroken supersymmetry requires the existence of a covariantly constant spinor on the internal manifold. This constrains both the topology and geometry of the space. Eligible manifolds are *Calabi–Yau* manifolds that have $\mathrm{SU}(3)$ holonomy, and are Ricci flat and Kähler.
- Calabi–Yau manifolds have very special Hodge numbers. The only ones that are not fixed are $h^{1,1}$ and $h^{2,1}$. They measure the number of Kähler and complex structure moduli, respectively.
- The moduli correspond to deformations of (the cohomology classes of) the holomorphic three-form Ω and the Kähler form J. Ω determines what we mean by complex coordinates. When this is specified J determines the metric: $J = g_{i\bar{\jmath}} dz^i \wedge dz^{\bar{\jmath}}$.
- The period integrals $X^I = \int_{A^I} \Omega$ can be used as homogeneous coordinates on the complex structure moduli space. Affine coordinates are given e.g. by $z^I = X^I / X^0$.

In the next section we describe a way to fix the complex structure moduli in type IIB string theory.

4.3 Fluxes and complex structure moduli stabilization

In the late 1980s Calabi–Yau and simpler toroidal compactifications of heterotic strings were used to engineer phenomenological models with Standard Model or GUT gauge groups. Examples of these models are Ref. [Wit86] by E. Witten, Refs. [GKMR87, GKMR86] by B.R. Greene *et al.* and Ref. [AEHN88] by I. Antoniadis *et al.* While the more developed of these models display impressive features such as explaining the number of fermion families from topological data, all of them suffered from the problem of massless geometric moduli.

In the mid 90s it was realized, notably in Ref. [PS96] by J. Polchinski and A. Strominger and in Ref. [Mic97] by J. Michelson, that inclusion of non-zero RR and NS–NS field strengths — *fluxes* — can create potentials for these moduli. The corresponding models are called *flux compactifications* and form a very active area of research. Apart from offering a physical mechanism that stabilizes moduli, fluxes also provide effects that can accommodate the large hierarchy between the Planck and weak scales.

In this section we review the basic physics of flux compactifications. Our focus is almost exclusively on type IIB compactifications, and even more specifically, on conformal Calabi–Yau compactifications of IIB. This is just one kind of beast from the rich flux compactifications fauna. Zoology interested readers should consult (at least) one of the great reviews in the field. Prominent examples include Ref. [Gra06] by M. Graña and Ref. [DK07] by M. R. Douglas and S. Kachru. Both reviews also provide extensive lists of references.

4.3.1 General idea

The reason that fluxes can influence the geometric moduli is that their energy depends on the volume they are confined to. Therefore, fluxes through different cycles in a Calabi–Yau result in a potential energy that depends on the volume of these cycles. And as we learned in the last section, the volumes of three-cycles determine the complex structure moduli.

A more direct way of seeing that fluxes can influence the geometry is by the terms in the action. The Maxwell-like terms in (3.50) all depend on the metric. The metric changes as the moduli vary, and thus fluxes couple to them. In this way fluxes influence the equations of motion and their presence can potentially lift the degeneracy of the solutions.

Before we get into technical detail, let us stress that the flux we want to put on Calabi–Yau cycles is *not* supported by any charge. A non-zero flux through a submanifold ∂X that is the boundary of another submanifold X always implies the existence of a net charge in X by Gauss' law. There is, however no Gauss law for closed submanifolds that is not the boundary of anything, and there is no charge associated with such flux. More fancily, flux through homology trivial cycles is supported by charges, flux through homology nontrivial cycles just is.

An important feature of flux is that it is *quantized*. The integral of a field strength over a closed hypersurface can only take discrete values, even if it is not supported by charge. This means that the set of possible flux configurations is discrete rather than continuous. Let us briefly recall Dirac flux quantization. Consider type IIB string theory and imagine some p-form flux F_p through a closed p-dimensional sphere S^p:

$$f = \int_{S^p} F_p. \tag{4.35}$$

If f is non-zero, it is impossible to define a continuous gauge potential C_{p-1} everywhere on S^p. Splitting it up in northern S_N and southern S_S hemispheres however, we can choose C_{p-1}^N and C_{p-1}^S differing by a gauge transformation on the intersection $\partial S_N = -\partial S_S \approx S^{p-1}$. The flux through S^p can now be written using Stokes' theorem as

$$
\begin{aligned}
f = \int_{S_N} F_p + \int_{S_S} F_p = \int_{\partial S_N} C_{p-1}^N + \int_{\partial S_S} C_{p-1}^S = \\
= \int_{\partial S_N} C_{p-1}^N - \int_{\partial S_N} C_{p-1}^S .
\end{aligned}
\tag{4.36}
$$

Now, taking a probe D$(p-2)$-brane around ∂S_N changes its phase by the corresponding action:

$$
S = \mu_{p-2} \int_{\partial S_N} C_{p-1} . \tag{4.37}
$$

This procedure must be independent of which representative C_{p-1}^N or C_{p-1}^S we choose in (4.37), so the difference in action must be a multiple n of 2π. Therefore (recall Eq. (3.63))

$$
f = \int_{S^p} F_p = \frac{2\pi n}{\mu_{p-2}} = n \left(4\pi^2 \alpha'\right)^{(p-1)/2} . \tag{4.38}
$$

We see that the existence of charged objects forces flux quantization upon us, even if there is no charge inside the S^p. (Or more correctly if there is no "inside S^p".) Note that the field strengths that are quantized are the F_p, not the improved field strengths $\tilde{F}_p$.

How should we now proceed to find models with flux stabilized moduli? We would like to specify a Calabi–Yau topology, and then postulate a fixed amount of flux through some of its (homology classes of) cycles. Then we need to find solutions to the full supergravity equations of motion subject to these "boundary conditions". One might (correctly) anticipate that this is very complicated. For instance, presence of flux will induce a non-zero energy momentum tensor so we expect $R_{mn} \neq 0$ from Einstein's equations. Fluxes also couple to the dilaton and amongst themselves. It *is* however possible to find such solutions. And there are good reasons to expect that there are many of them.

In the next subsection we discuss an interesting feature of flux compactifications: the geometry becomes *warped*. We also present some, at first sight, discouraging results.

4.3.2 Warping and evading no-go theorems

When discussing Kaluza–Klein compactification we used the metric ansatz (4.2). This is however not the most general ansatz compatible with maximal symmetry in four dimensions, which instead is

$$
g_{MN} = \begin{pmatrix} e^{2A(y)} \tilde{g}_{\mu\nu}(x) & 0 \\ 0 & e^{-2A(y)} \tilde{g}_{mn}(y) \end{pmatrix} . \tag{4.39}
$$

Here $\tilde{g}_{\mu\nu}(x)$ can be either of four-dimensional Minkowski, de Sitter (dS_4) or anti-de Sitter (AdS_4) spacetimes. The factor $e^{2A(y)}$ dependent on the internal coordinates is called a *warp factor*, and compactifications on metrics of the form (4.39) are called *warped* compactifications. Including the warp factor also in the compact metric is conventional. In flux-less, supersymmetric compactifications, a nontrivial warp factor is not needed (even forbidden) but when flux is included it plays an important role. If the warping varies appreciably on the compact space, different regions are red-shifted relative each other, allowing for localized physics to have hierarchically different energy scales.

In the quest of finding solutions to the supergravity equations of motions including flux, some no-go theorems were developed. The most important one (and the only one we study here) was found by J. Maldacena and C. Nuñes [MN01]. They proved that including fluxes (and only fluxes) in a compactification makes Minkowski or dS_4 impossible as non-compact geometries. The argument is quite straightforward, and the reader can collect all details in the original paper.

The line of reasoning uses the properties of the energy-momentum tensor from fluxes, and goes as follows. By tracing Einstein's equations the ten-dimensional curvature scalar R can be solved for in terms of the energy-momentum tensor T_{MN}. Reinserting the result into Einstein's equation yields

$$R_{MN} = T_{MN} - \frac{1}{8} g_{MN} T^L_L. \tag{4.40}$$

Separating the contributions from the warp factor and $\tilde{g}_{\mu\nu}$ in the non-compact components of (4.40) and contracting these with $\tilde{g}^{\mu\nu}$ produces

$$2\nabla^2 e^{2A} = e^{2A}(\tilde{R} + e^{2A}\hat{T}). \tag{4.41}$$

Here $\tilde{R}$ is the Ricci scalar of $\tilde{g}_{\mu\nu}$ and $2\hat{T} = (T^m_m - T^\mu_\mu)$. By straightforward analysis of the stress tensors of p-form fluxes it is possible to show that $\hat{T}_{\text{fluxes}} \geq 0$ with equality only for one-form flux.

This poses a problem as is most simply seen by integrating (4.41) over the compact manifold. The integral of the Laplacian of any function vanishes, so the left-hand side vanishes upon integration. But if $\hat{T}$ is positive, then $\tilde{R}$ has to be negative! (Or zero if we only have F_1-flux.) The conclusion for type IIB is that the fluxes $\tilde{F}_3$, H_3 and $\tilde{F}_5$ are all forbidden in dS or Minkowski compactifications. Sadly, $\tilde{F}_3$ and H_3 are exactly the fluxes we are interested in to stabilize complex structure moduli.

This no-go theorem is a very powerful result. It is applicable to *any* solution, not only supersymmetric ones. It furthermore gives a clear-cut criterion for what is needed to evade it. We must include objects having $\hat{T} < 0$ in the compactification. String theory has such objects: orientifold planes. Let us take a moment to describe them.

Suppose the manifold $\mathcal{M}$ has a discrete symmetry I acting on its points $I : \mathcal{M} \to \mathcal{M}$. Let us for simplicity assume that $I^2 = 1$. One can construct a new (possibly singular) manifold $\mathcal{M}/I$ through *orbifolding* by I, i.e., by identifying points: $p \sim I(p)$. The set of fixed-points of I becomes a singular submanifold of the orbifold called the orbifold plane. In an ordinary orbifold projection one keeps the states that are symmetric under I.

*Orienti*folding includes orbifolding the space. However, one also does something more. Let P denote the world-sheet parity operator. This operator sends the string world-sheet coordinate σ to $\pi - \sigma$. Orientifolding divides the space of states by PI. Only states that are even under this combined transformation are kept in an orientifold projection. The resulting singular submanifold is called an orientifold plane or an Op-plane where p is its spatial dimension.

It is possible to determine the RR charges of orientifold planes, and they turn out to be negative. In fact the charge of an Op-plane is $-2^{p-5}\mu_p$. Their presence leads to negative contributions to $\hat{T}$ and thus makes flux compactifications possible.

4.3.3 Solving the equations à la GKP

Let us now review the famous approach [GKP02] to solving the equations of motion introduced by S.B. Giddings, S. Kachru and J. Polchinski, a trio that will henceforth be referred to as GKP. Given the very complicated system, and given that no supersymmetry is assumed, the simplicity of this class of solutions is remarkable.

Our framework is type IIB supergravity, described by the action given in (3.49) and (3.50). It is convenient to make a few field redefinitions, including going to Einstein frame. Defining

$$\begin{aligned} \tau &= C_0 + ie^{-\Phi} \\ G &= F_3 - \tau H_3 \end{aligned} \tag{4.42}$$

the action is

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{|g|} \left(R - \frac{\partial_M \tau \partial^M \bar{\tau}}{2(\operatorname{Im}\tau)^2} - \frac{G_3 \cdot \bar{G}_3}{12 \operatorname{Im}\tau} - \frac{|\tilde{F}_5|^2}{4 \cdot 5!} \right) \\ + S_{\text{CS}} + S_{\text{loc}}, \end{aligned} \tag{4.43}$$

where at every explicit or implicit appearance of a metric, the Einstein frame metric is understood (even if we suppressed the superscript E). The term S_{loc} is the action for localized sources like D-branes or O-planes. Our objective is to find solutions to the equations of motion. The solutions we find will all be Minkowski in four dimensions.

We start by making some ansätze that respect four-dimensional Poincaré invariance. As an ansatz for the Einstein frame metric g_{MN} we take (4.39), with $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}(x)$. Four-dimensional Poincaré invariance restricts $\tau = \tau(y)$ to be a function only of y and forces $G_{NMP} = G_{mnp}(y)$ to have legs only in the internal directions. The five-form field strength can actually have a non-compact part if it is proportional to the four-dimensional volume form:

$$\tilde{F}_5 = (1 + *)[d\alpha \wedge d\mathrm{Vol}_4]. \tag{4.44}$$

The factor $(1 + *)$ makes the ansatz self-dual.

Computing the energy-momentum tensor for these ansätze, and plugging the result into (4.41) produces

$$\tilde{\nabla}^2 e^{4A} = e^{2A}\frac{|G_3|^2}{12\mathrm{Im}\,\tau} + e^{-6A}\left(|\partial\alpha|^2 + |\partial e^{4A}|^2\right) + \kappa_{10}^2 e^{2A}\hat{T}^{\mathrm{loc}} \tag{4.45}$$

In the absolute squares in this equation contraction with the six-dimensional metric $e^{2A}\tilde{g}^{mn}$ is understood. $\tilde{\nabla}$ is with respect to $\tilde{g}_{mn}$. If we had omitted the energy-momentum tensor $\hat{T}^{\mathrm{loc}}$ originating from localized sources, this would have been an instance of the no-go theorem of Ref. [MN01] since the other terms on the right-hand side of (4.45) are manifestly positive, while the left hand side integrates to zero.

Now we used Einstein's equations, but there are of course more equations to fulfill. In particular we have the Bianchi identity for $\tilde{F}_5$. Since $F_5 = dC_4$ is sourced by D3 branes and O3-planes this equation reads

$$d\tilde{F}_5 = H_3 \wedge F_3 + (2\pi)^4(\alpha')^2\rho_3^{\mathrm{loc}}. \tag{4.46}$$

ρ_3^{loc} is the density of D3-charge from D3-branes and O3-planes. Inserting the GKP ansatz for $\tilde{F}_5$, H_3 and F_3 and subtracting the result from (4.45) gives

$$\begin{aligned}\tilde{\nabla}^2(e^{4A} - \alpha) = e^{2A}\frac{|iG_3 - *_6 G_3|^2}{6\mathrm{Im}\,\tau} + e^{-6A}|\partial(e^{4A} - \alpha)|^2 \\ + \kappa_{10}^2 e^{2A}[\frac{1}{2}\hat{T}^{\mathrm{loc}} - \mu_3\rho_3^{\mathrm{loc}}].\end{aligned} \tag{4.47}$$

GKP noted that something special happens if we consider only localized sources that fulfill

$$\frac{1}{2}\hat{T}^{\mathrm{loc}} - \mu_3\rho_3^{\mathrm{loc}} \geq 0. \tag{4.48}$$

If this inequality holds then all terms on the right hand side of (4.47) are non-negative. The left hand side still integrates to zero though. Thus, Minkowski compactifications with objects satisfying (4.48) are allowed only if all terms on the right hand side vanish! This gives two simple conditions that the fields must fulfill. The five form is directly related to the warping, $\alpha = e^{4A}$ and the three-form field strength G_3 is *imaginary self dual* (ISD): $*_6 G_3 = iG_3$. Furthermore, the sources must actually saturate the inequality (4.48).

There are a lot of relevant sources that saturate the inequality. Space-filling D3-branes and O3-planes are two examples, as are D7-branes or O7-planes wrapped on internal four-cycles. The ISD condition $*_6 G_3 = iG_3$ is what fixes the complex structure moduli and the value of the axio-dilaton τ. We return to a more detailed analysis of this fixing in the next subsection.

GKP go on to show that these conditions imply all the other equations of motion except for two equations. These two equations relate τ and the Ricci tensor $\tilde{R}_{mn}$ of $\tilde{g}_{mn}$ to the stress tensor of D7-branes. In the case of no D7 branes these equations are simply

$$\begin{aligned} \tilde{R}_{mn} &= 0 \\ \partial_m \tau &= 0. \end{aligned} \tag{4.49}$$

To complete the construction there are now only two conditions left, coming from the Bianchi identity (4.46) for $\tilde{F}_5$. First, since $\alpha = e^{4A}$ it gives an equation for the warping in terms of the D3-charges $F_3 \wedge H_3$ and ρ_3:

$$-\tilde{\nabla}^2 e^{-4A} = (2\pi)^2 \alpha'^2 \rho_3 + \frac{G_{mnp} G^{\tilde{m}\tilde{n}\tilde{p}}}{12 \mathrm{Im}\, \tau} \tag{4.50}$$

where we use $\tilde{g}^{mn}$ to raise the indices of G_3.

Second, integrating (4.46) over the internal manifold gives the condition that the total D3-brane charge on the compact space vanishes. The charge density ρ_3 comes from both D3-branes and O3-planes. Remembering that the latter carry $-1/4$ of D3-charge we get

$$N_{D3} - \frac{1}{4} N_{O3} + \frac{1}{(2\pi)^4 \alpha'^2} \int H_3 \wedge F_3 = 0. \tag{4.51}$$

This equation is a condition on three integers: the number of D3-branes N_{D3}, the number of O3-planes N_{O3} and the D3-charge induced from fluxes. This term is integral because of flux quantization.

Thus, for local sources saturating (4.48) we can construct solutions as follows. We choose a number of D3-branes, O3-planes and flux quanta satisfying (4.51) on a Calabi–Yau topology. The quantity $\alpha = e^{4A}$ is then determined by the $\tilde{F}_5$ Bianchi identity. This yields the five-form flux and the warping. Moreover $\tilde{g}_{mn}$ is Ricci flat, and thus a Calabi–Yau metric. The corresponding complex structure moduli and constant axio-dilaton τ are fixed by the condition that G_3 is ISD.

We see that the degeneracy of solutions to the equations of motion is partially lifted when fluxes are included. The complex structure and the axio-dilaton are not moduli anymore. In the next subsection we analyze the same effect from the viewpoint of effective four-dimensional actions.

Before turning to this issue, let us note that the GKP solutions have a radial modulus. It is possible to rescale the internal metric $\tilde{g}_{mn} \to \lambda^2 \tilde{g}_{mn}$ and still obtain a solution. The warping however scales non-trivially, as can be realized from (4.50). In this equation the $\tilde{\nabla}^2 e^{-4A}$ will scale as λ^{-2}, whereas the other two terms scale as λ^{-6}. This means that if we take the internal dimensions to be large relative to the string scale, the warping will be approximately constant. In this limit the internal geometry is effectively Ricci flat, and thus a Calabi–Yau metric. The whole machinery of Calabi–Yau geometry thus becomes applicable without modifications.

4.3.4 Effective supergravity actions

The ultimate goal of compactifying strings is of course to make contact with real-world phenomenology. To this end, the next step after finding the background is to derive the physics of low energy fluctuations around that background. The low energy effective field theories of superstring compactification are four-dimensional supergravity theories, whose actions are determined by a small set of functions. In this section we describe how the geometric data of a string compactification fit into this framework.

The field content of a supersymmetric theory splits up in *multiplets*, i.e. fields that transform into each other under supersymmetry transformations. In the superspace formalism the fields in a multiplet are components of the same superfield. Naturally any supersymmetry multiplet contains both bosons and fermions. Our focus will be completely on the bosons, so for the most part we simply ignore the fermionic field content. In string theory compactifications the multiplet structure depends on the cohomology structure of the Calabi–Yau manifold.

The restrictions that local supersymmetry puts on supergravity actions can often be formulated in terms of the geometry of field space. The scalar field space of $\mathcal{N} = 2$ supergravity is "quaternionic Kähler", and the corresponding space for $\mathcal{N} = 1$ supergravity is Kähler. The part of the $\mathcal{N} = 2$ field space we are interested in will be Kähler as well, so this is the only concept we need. A Kähler metric is specified by a real function called the *Kähler potential*. It determines the metric on field space, and thus the kinetic terms. E.g., for a set of complex scalar fields $(\phi^i, \phi^{\bar{\imath}})$ with Kähler potential $K(\phi, \bar{\phi})$ the metric $K_{i\bar{\jmath}}$ is

$$K_{i\bar{\jmath}} = \frac{\partial^2}{\partial \phi^i \partial \phi^{\bar{\jmath}}} K(\phi, \bar{\phi}), \tag{4.52}$$

yielding the kinetic term

$$S_{\text{kin.}} = -\frac{1}{2} \int \sqrt{|g|}\, K_{i\bar{\jmath}} \partial_\mu \phi^i \partial^\mu \phi^{\bar{\jmath}}. \tag{4.53}$$

Two remarks are in order. First, a metric coming from a Kähler potential is indeed a Kähler metric as this concept was defined in Section 4.2. The fact that partial derivatives commute makes it clear that $\partial_i K_{j\bar{k}} = \partial_j K_{i\bar{k}}$ which together with its complex conjugate is exactly the Kähler condition. Second, the metric does not determine K completely. There is a freedom of choice $K \to K + f(\phi) + \bar{f}(\bar{\phi})$ for any holomorphic f.

There are also other terms than the kinetic ones in supergravity actions. They are determined by *superpotentials* that are always holomorphic functions of the fields.

Performing a dimensional reduction of compactified type IIB supergravity leads to expressions for the Kähler and superpotentials of the resulting four-dimensional theory. This is hard work, and below we collect only the results. We are interested in two types of compactifications: fluxless Calabi–Yau compactifications leading to $\mathcal{N} = 2$ supergravity in four dimensions and flux compactifications with (at most) $\mathcal{N} = 1$ supersymmetry. Let us treat them in turn.

4.3.4.1 No flux: $\mathcal{N} = 2$

$\mathcal{N} = 2$ supergravity in four dimensions has four types of multiplets: the gravity multiplet, vector multiplets, hypermultiplets and the double tensor multiplet. The fields appearing in these multiplets come from a Kaluza–Klein reduction of ten-dimensional type IIB theory. We remember that the massless fields correspond to harmonic forms on the compact manifold. Denoting a basis of $H^3(\mathcal{M})$ by (α_I, β^J), $I, J = 0 \dots h^{2,1}$ a basis of $H^{1,1}(\mathcal{M})$ by ω_a $a = 1 \dots h^{1,1}$ and the corresponding basis of $H^{2,2}$ by $\tilde{\omega}_a$ we can make the expansions

$$\begin{aligned} &B_2(x,y) = B_2(x) + b^a(x)\omega_a, \qquad C_2(x,y) = C_2(x) + c^a(x)\omega_a, \\ &C_4(x,y) = D_2^a(x) \wedge \omega_a + V^I(x) \wedge \alpha_I - U_I(x) \wedge \beta^I + \rho^a(x)\tilde{\omega}_a \end{aligned} \tag{4.54}$$

The reader is allowed to forget about the gray terms: because of the self-duality of the five-form field strength, only half of the fields represent independent degrees of freedom.

Also the metric perturbations, i.e. the moduli, are parameterized by harmonic forms. The Kähler moduli correspond to deformations of the Kähler form J:

$$J = v^a(x)\omega_a. \tag{4.55}$$

The Kähler modulus v^a, corresponding to the $H^{1,1}(\mathcal{M})$ element ω_a transforms in the same multiplet as all other fields that are expansion coefficients of ω_a or its dual: b^a, c^a and ρ^a. These $h^{1,1}$ multiplets are called hypermultiplets.

The complex structure moduli $z^I(x) = X^I/X^0$ are associated to three-forms. However not all three-forms correspond to complex structure moduli: only the the $(2,1)$-forms do. As might be expected, the complex structure moduli form multiplets with the linear combinations of the $V^I(x)$ corresponding to $(2,1)$-forms. These multiplets are called vector multiplets, their namesake being the vector bosons. The vector boson corresponding to the $(3,0)$-form Ω transforms in a multiplet together with the four-dimensional graviton $g_{\mu\nu}$. This is the gravity multiplet and the vector boson is called the *graviphoton*. We denote it V^0 even if it is not proportional to α_0, but rather to Ω. Finally, there is a double tensor multiplet whose bosonic content is $B_2(x)$, $C_2(x)$ and τ. It can be dualized to a "universal" hypermultiplet.

Our main interest is in the complex structure, and thus in the vector multiplets. The constraints on the geometry of the vector multiplet moduli space were first studied from a four-dimensional perspective by B. de Wit, P.G Lauwers and A. Van Proeyen [dWLVP85]. A. Strominger then described this geometry that was named *special geometry* in a coordinate independent manner, and showed how it beautifully emerges Calabi–Yau geometry [Str90].

de Wit *et al.* showed that the action is locally defined by a single holomorphic function F. In terms of the homogeneous coordinates X^I on complex structure moduli space F has a very nice interpretation. It is nothing but the prepotential of the Calabi–Yau as defined in Section 4.2! The corresponding Kähler potential is

$$K = -\ln\left[-i\int \Omega \wedge \bar{\Omega}\right]. \tag{4.56}$$

Note that the rescaling $\Omega \to e^{-f}\Omega$ with f a holomorphic function on moduli space corresponds to a choice of Kähler gauge: $K \to K+f+\bar{f}$. By Eqs. (4.31) and (4.29) we can express the Kähler potential in terms of the prepotential:

$$K = -\ln\left[i(\bar{X}^I F_I - X^I \bar{F}_I)\right]. \tag{4.57}$$

We do not need to flesh out the full bosonic action in terms of these functions here. The important pieces of information are that the complex structure moduli form vector multiplets with the (2,1)-coefficients of C_4, that the left over (3,0)-coefficient is the graviphoton of the gravity multiplet, and that the prepotential F determines the action in terms of Calabi–Yau data.

4.3.4.2 Flux: $\mathcal{N}=1$

In the GKP setup, fluxes break supersymmetry at least down to $\mathcal{N}=1$. The relevant geometry for the entire field space of $\mathcal{N}=1$ supergravity is Kähler as explained by B. Zumino [Zum79]. In fact, assuming only terms with up to two derivatives, the entire action is specified by a single function. Symbolically, with ϕ denoting all scalars, the function is

$$G(\phi,\bar{\phi}) = K(\phi,\bar{\phi}) + \ln|W(\phi)|^2 \tag{4.58}$$

K is the Kähler potential determining the kinetic terms and W is the superpotential. The bosonic part of the Lagrangian is

$$\mathcal{L}_{\mathrm{bos}} = \sqrt{|g|}\left(-\frac{1}{2}R - G_{i\bar{\jmath}}\partial_\mu\phi^i\partial^\mu\phi^{\bar{\jmath}} - e^G[G_i\,G^{i\bar{\jmath}}\,G_{\bar{\jmath}} - 3]\right). \qquad (4.59)$$

Subscript i ($\bar{\imath}$) means differentiation with respect to ϕ^i ($\phi^{\bar{\imath}}$), and $G^{i\bar{\jmath}}$ is the matrix inverse of $G_{i\bar{\jmath}}$.

A quaternionic Kähler manifold is not Kähler, but a submanifold of it can be. Consistent with this, the $\mathcal{N} = 2$ spectrum is truncated when the theory only has $\mathcal{N} = 1$ supersymmetry. From the ten-dimensional point of view, the truncation of the field content is due to the orientifold projection. This projection also alters the multiplet structure. The graviphoton is projected out and thus leaves the gravity multiplet whose bosonic content is now just the graviton. The vector multiplets split up according to the parity under the orientifold projection I of the corresponding basis element of $H^{2,1}(\mathcal{M})$. We can decompose $H^{2,1}(\mathcal{M})$ into forms that are even $(+)$ or odd $(-)$ under I:

$$H^{2,1}(\mathcal{M}) = H^{2,1}_+(\mathcal{M}) \oplus H^{2,1}_-(\mathcal{M}). \qquad (4.60)$$

In the multiplets that correspond to forms of even parity, the complex structure modulus is projected out. This leaves $h^{2,1}_+$ vector multiplets (V^κ), $\kappa = 1, \ldots h^{2,1}_+$. The opposite happens for multiplets corresponding to forms having odd parity. Here, the vector boson is projected out, and the resulting $h^{2,1}_-$ so-called chiral multiplets contain the surviving complex structure moduli (z^k), $k = 1, \ldots h^{2,1}_-$. The hypermultiplets similarly split into chiral multiplets, while the double tensor multiplet loses the two two-forms B_2 and C_2. Any fluxes present must of course survive the orientifold projection, meaning that they must be harmonic elements of $H^3_-(\mathcal{M})$.

The presence of fluxes $G_3 = F_3 - \tau H_3$ leads to a non-zero superpotential W, which in the action (4.59) leads to a potential for the complex structure moduli z^k. In the limit of large volume and hence constant warping this potential was derived by GKP in Ref. [GKP02]. It has the form proposed by S. Gukov, C. Vafa and E. Witten (GVW) in the context of domain walls [GVW00]. It is quite simple and reads

$$W(z) = \int G_3 \wedge \Omega(z). \qquad (4.61)$$

In the same limit of large warping, the Kähler potential is identical to the one in $\mathcal{N} = 2$ compactifications. Including complex structure moduli, the axio-dilaton τ and the overall volume of the Calabi–Yau parameterized by a field ρ it reads

$$K = -\ln\left[-i\int\Omega\wedge\bar{\Omega}\right] - \ln\left[-i(\tau-\bar{\tau})\right] - 3\ln\left[-i(\rho-\bar{\rho})\right]. \qquad (4.62)$$

Inserted in the Lagrangian (4.59) this W and K give rise to a potential for the scalar fields:

$$V(z,\bar{z},\tau,\bar{\tau},\rho,\bar{\rho}) = e^K\big(K^{i\bar{\jmath}}D_iWD_{\bar{\jmath}}\bar{W} + K^{\tau\bar{\tau}}D_\tau WD_{\bar{\tau}}\bar{W} \\ + K^{\rho\bar{\rho}}D_\rho WD_{\bar{\rho}}\bar{W} - 3|W|^2\big). \tag{4.63}$$

Here the capital D's denote Kähler covariant derivatives. For an arbitrary field index A we define

$$D_AW = (\partial_A + \partial_AK)W. \tag{4.64}$$

The potential V sets the dynamics of the scalar fields. Minimizing it fixes the complex structure moduli and the axio-dilaton, and the matrix of second derivatives at the minimum determines the corresponding masses. In Chapters 5 and 6 we analyze the moduli dynamics, and V plays a prominent role.

To prepare for these chapters, let us analyze some of the properties of V. First, because of $\partial_\rho W = 0$ and the special dependence of K on ρ, the last two terms actually cancel:

$$K^{\rho\bar{\rho}}D_\rho WD_{\bar{\rho}}\bar{W} = 3|W|^2, \tag{4.65}$$

leaving

$$V = e^K\big(K^{i\bar{\jmath}}D_iWD_{\bar{\jmath}}\bar{W} + K^{\tau\bar{\tau}}D_\tau WD_{\bar{\tau}}\bar{W}\big). \tag{4.66}$$

The potential therefore is positive semidefinite, and is zero if and only if

$$D_iW = 0 \qquad \text{and} \qquad D_\tau W = 0. \tag{4.67}$$

These conditions are equivalent to

$$\int \partial_i\Omega \wedge G_3 = 0 \qquad \text{and} \qquad \int \Omega \wedge \bar{G}_3 = 0. \tag{4.68}$$

We see that $[G_3] \in H^{2,1} \oplus H^{0,3}$. Because of the geometric facts that the Hodge star has $+i$ $(-i)$ eigenspaces $H^{2,1} \oplus H^{0,3}$ $(H^{1,2} \oplus H^{3,0})$ this is exactly the condition that G_3 is ISD. Thus, the GKP solutions minimize the effective potential V to a global minimum $V = 0$. A background value of V corresponds to a cosmological constant in the effective action, so the GKP solutions are indeed Minkowski.

The $\mathcal{N} = 1$ condition for a supersymmetric background is that $D_AW = 0$ for all fields A. This is generally not the case for the GKP solutions because of (4.65). Only if

$$W = \int G_3 \wedge \Omega = 0, \tag{4.69}$$

implying the vanishing of the $(0,3)$-piece of G_3 is the solution supersymmetric.

Note that the radial modulus ρ of the GKP solutions is evident from this formulation. If $V = 0$ for a given value of the complex structure, then $\partial_\rho V = 0$ at this point since the only dependence on ρ comes from the prefactor e^K. This behavior is called *no-scale* behavior and the corresponding models are no-scale models.

Let us end this section by a few comments regarding corrections. Incorporating the effects of warping is quite complicated, but significant progress has been achieved recently. S. Giddings and O. DeWolfe [DG03] demonstrated that the superpotential is unaltered while the Kähler potential changes when warping is taken into account, and Giddings and A. Maharana [GM06] analyzed the effective field theory of perturbations around a warped background on the level of field equations. More recently G. Shiu, G. Torroba, B. Underwood and M.R. Douglas presented an elegant analysis [STUD08] resulting in an effective action including also KK modes that become light at large warping.

Apart from warping, the above analysis gets corrections from the α' and string loop expansions and from non-perturbative effects. The latter can result in a superpotential W depending also on ρ. This was used by S. Kachru, R. Kallosh, A. Linde and S. Trivedi to construct models where also Kähler moduli are stabilized. This is known as the KKLT scenario.

The α' and loop corrections also spoil the no-scale structure and can be used to find completely stable vacua with large compact volumes. These constructions were pioneered by V. Balasubramanian, P. Berglund, J. Conlon and F. Quevedo [CQ04].

Here the introductory part of this thesis ends. It is my hope that the reader now will be prepared for the following three chapters treating more recent developments, and in particular the questions that led to the papers I to VII.

Part II:

Developments

5. Black holes in type IIB

Me no wanna go in there
Kalle Baah

Black holes are at the core of studies of quantum gravity. They are predicted by Einstein's theory of general relativity. In fact, the solution discovered by K. Schwarzschild [Sch16] was the first non-trivial exact solution found. Black holes turn up-side-down our everyday experience almost as violently as quantum mechanics. For instance, it is impossible to get out of a black hole; the only way back goes backwards in time. Furthermore, even if an observer falling into a black hole certainly will pass the horizon, from the outside it is actually impossible to see something fall in! By now there is convincing evidence that these objects actually exist in our universe. We should therefore take physical questions about them very seriously.

Black holes become even more interesting when they are treated quantum mechanically. In 1975 S. Hawking showed by a semiclassical analysis that black holes radiate through quantum effects [Haw75]. Eventually a black hole will evaporate through this radiation. In Hawking's analysis the radiation does not contain any information. One way to understand this is that the radiation is produced outside the black hole horizon, by the gravitational field. Uniqueness theorems for black holes (so-called no-hair theorems) however state that from the outside a black hole is completely specified by a small set of conserved charges. The outside cannot contain any information. This poses a problem for a quantum mechanical description of a black hole. Quantum mechanics is *unitary* and unitarity implies that information cannot be destroyed. The process of some stuff collapsing to a black hole and then being radiated as Hawking radiation, thus seems to be incompatible with quantum mechanics.

Even before Hawking's discovery, it was known that black holes have a whole set of thermodynamical laws that follow from general relativity. In particular J. Bekenstein explained that they have a macroscopic entropy [Bek73]. In Einstein gravity the entropy is $S = A/4$ where A is the horizon area in Planck units. This relation receives corrections for theories with higher derivative terms in the action. In ordinary quantum mechanics, the entropy of a system is the logarithm of the number of possible microstates. The entropy of black holes cries out for such a microscopic explanation. The unitarity and microstate problems are perfect challenges for any theory of quantum gravity.

String theory has made impressive progress in understanding these questions. One major development followed in the wake of the AdS/CFT correspondence discovered by J. Maldacena [Mal98]. While still formally a conjecture, the evidence for the duality is by now overwhelming. In general, AdS/CFT relates string theory on an anti de Sitter background to a conformal field theory living on the conformal boundary of the AdS spacetime. The original example constructed by Maldacena was the duality between type IIB strings on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions.

The relevance of AdS/CFT for black holes comes from the fact that putting a black hole in the AdS background translates to letting the CFT have a finite temperature. Because the thermodynamics of conformal field theory certainly derives from a unitary system, this gives convincing evidence that string theory describes black holes without sacrificing unitarity.

Another equally impressive result is the derivation of the macroscopic black hole entropy from microscopic principles. Compactifying string theory results, as we saw in Chapter 4, in lower-dimensional supergravity theories. These generally have black hole solutions with computable macroscopic entropy. If the black holes are charged under the gauge fields of the supergravity, it means that they microscopically must consists of D-branes. In 1996 A. Strominger and C. Vafa [SV96] performed such a microscopic construction and computed the asymptotic degeneracy of states, finding agreement with the Bekenstein–Hawking entropy. The way they counted the states was by reducing the brane effective theory to a two-dimensional CFT and then applying the Cardy formula we met in Chapter 2.

In this thesis we consider a type of black holes that feature in Calabi–Yau compactifications of type IIB string theory. They consist of D3 branes wrapped around internal three-cycles. Being charged under the five-form field strength F_5, these black holes generate five form flux piercing three-cycles of the Calabi–Yau, and having two legs in the non-compact directions. Thus they appear as point particles in four dimensions charged under the various vector fields of $\mathcal{N} = 2$ supergravity.

Having flux through internal three-cycles affects the complex structure moduli much in the same way as the three-form flux did in Chapter 4. This interplay between black holes and the complex structure moduli is the interest of two of the papers on which this thesis is based. In particular, paper I deals with quantum corrections to the entropy of these black holes. Paper II estimates the relative strengths of the effect that black holes contra three-form fluxes have on the complex structure.

5.1 D3-brane black holes

In this section we describe how charged black holes in type IIB compactifications can be described as D3-branes. We follow more or less the charming treatment provided by F. Denef in Ref. [Den00]. Consider a flux-less Calabi–Yau compactification of type IIB string theory resulting in a low energy effective $\mathcal{N} = 2$ supergravity with massless vector and hypermultiplets. The gauge fields in the vector multiplets V^I originate from the four-form C_4 of the ten-dimensional theory, and are thus sourced by D3-branes.

A black hole is obtained by wrapping a D3-brane around a three-cycle in the Calabi–Yau. The brane is localized to a point in non-compact spacetime. Let us choose to wrap the brane q_I times around the cycle A^I and $-p^J$ times around the cycle B_J. This gives rise to a self dual five-form flux

$$\tilde{F}_5 = F_5 = (1 + *)\mathcal{F}. \tag{5.1}$$

Defining the three-form $\Gamma = q_I\beta^I + p^I\alpha_I$, a suitable $\mathcal{F}$ is

$$\mathcal{F} = \frac{1}{2\mu_3} d\mathrm{Vol}_{S^2} \wedge \Gamma. \tag{5.2}$$

Here the S^2 is a spacelike sphere in the non-compact directions centered at the D3-brane, and $d\mathrm{Vol}_{S^2} = \sin\theta d\theta d\phi$. To see that this field corresponds to the correct charge, we perform the integrations

$$\begin{aligned} \int_{S^2\times B^I} F_5 &= \int_{S^2\times B^I} \mathcal{F} = \frac{2\pi}{\mu_3}\int_{\mathcal{M}} \alpha_I \wedge \Gamma = (4\pi^2\alpha')^2 q^I \\ \int_{S^2\times A^I} F_5 &= \int_{S^2\times A^I} \mathcal{F} = -\frac{2\pi}{\mu_3}\int_{\mathcal{M}} \beta^I \wedge \Gamma = (4\pi^2\alpha')^2 p^I. \end{aligned} \tag{5.3}$$

Note that the manifold $S^2 \times A^I$ is “linked” with a D-brane wrapping the B_I-cycle somewhere inside the S^2. The same goes for $A^I \leftrightarrow B_I$, but then the orientation of the linking is reversed. This is why it makes sense to interpret the second equation in (5.3) as the presence of a brane wrapping the B^I cycle $-p^I$ times.

Thus, the brane surrounds itself with a five-form field strength wrapped on the three-cycles α_I and β^I. From a four-dimensional point of view, the object is a black hole charged under the fields V^I of Eq. (4.54). It will generate a potential for the complex structure moduli analogously to the presence of three-form fluxes. An important difference in this case is that the potential will depend on the distance to the black hole.

Let us investigate this potential. Performing a dimensional reduction with this brane and corresponding flux requires some care because of the self-duality of $\tilde{F}_5$. Writing the black hole metric in the form

$$ds^2 = -e^{2U(r)}dt^2 + R^2(r)d\mathrm{Vol}_{S^2} + f(r)dr^2 \tag{5.4}$$

where r is a radial coordinate from the black hole, and following the lead of Denef[1] the effective action becomes

$$S_{\text{eff.}} = \frac{1}{16\pi} \int d\text{Vol}_4 \left(\mathcal{R}_4 - 2K_{i\bar{\jmath}} \partial_\mu z^i \partial^\mu z^{\bar{\jmath}} - \frac{1}{R(r)^4} \langle \Gamma, \hat{\Gamma} \rangle \right). \tag{5.5}$$

Here $K_{i\bar{\jmath}}$ is the Kähler metric on moduli space as defined in (4.56) and $\langle \Gamma, \hat{\Gamma} \rangle$ is the intersection product (4.26). $\hat{\Gamma}$ is the six-dimensional Hodge dual of Γ.

The term with $\langle \Gamma, \hat{\Gamma} \rangle$ acts as a potential for the complex structure moduli. By expanding Γ in a harmonic basis $\{\Omega, D_i\Omega, \bar{\Omega}, D_{\bar{\imath}}\bar{\Omega}\}$ one sees that it has the standard form of an $\mathcal{N} = 2$ scalar potential coming from a holomorphic superpotential. This superpotential has the GVW form

$$W_{\text{bh}}(z) = \int_{\mathcal{M}} \Gamma \wedge \Omega, \tag{5.6}$$

and the potential expressed in this quantity is

$$V(z) \equiv \frac{1}{2} \langle \Gamma, \hat{\Gamma} \rangle = e^K \left(K^{i\bar{\jmath}} D_i W_{\text{bh}} D_{\bar{\jmath}} \bar{W}_{\text{bh}} + |W_{\text{bh}}|^2 \right), \tag{5.7}$$

which is the usual $\mathcal{N} = 2$ result. In paper II we consider a D3-brane black hole in a flux compactification. The potential for the complex structure moduli then also has a contribution from the fluxes of the form (4.66). We compare the effects of these potentials and find that the flux potential dominates except in fine-tuned cases. The last section in this chapter makes a few comments on this paper.

In place of W_{bh} it is convenient and customary to use a rescaled quantity $Z = e^{K/2} W_{\text{bh}}$ called the central charge[2] of the black hole. The central charge is not holomorphic, but its absolute value is independent of Kähler rescalings $\Omega \to e^{-f}\Omega$.

Let us now consider the special case of supersymmetric black holes. In $\mathcal{N} = 2$ supergravity there are black hole solutions preserving half of the supersymmetry. These states are called BPS states since they saturate the so-called BPS bound $M \geq |Z|$. In the context of magnetic monopoles, this inequality was discovered by E.B. Bogomolny [Bog76], and states saturating it were studied by M.K. Prasad and C.M. Sommerfield [PS75].

BPS black holes have a metric of a more restrictive form than (5.4):

$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} [r^2 d\text{Vol}_{S^2} + dr^2]. \tag{5.8}$$

[1] Who in his turn follows the lead of M. Henneaux in collaborations with C. Teitelboim [HT88] and X. Bekaert [BH99].

[2] The reason for this nomenclature is that theories with extended supersymmetry have nontrivial anti-commutation relations between the generators Q_α^A, $A = 1, \ldots, \mathcal{N}$ of the different supersymmetries: $\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB}$. The matrix Z^{AB} commutes with all other generators, and is thus a central term. Since Z^{AB} is antisymmetric it has only one component in $\mathcal{N} = 2$.

Plugging this metric into (5.5), assuming $z^i = z^i(r)$ and integrating over angular coordinates produces the following effective action per unit time

$$S_{\text{eff}} = -\frac{1}{2}\int_0^\infty d\tau \left(\dot{U}^2 + g_{i\bar{\jmath}}\dot{z}^i\dot{z}^{\bar{\jmath}} + e^{2U}V(z)\right) \tag{5.9}$$

where we ignored boundary terms and introduced $\tau = 1/r$. A dot denotes differentiation with respect to τ. The action (5.9) determines U and the moduli fields z^i as functions of the radial coordinate. Since we are considering an $\mathcal{N} = 2$ compactification without flux, there is no potential for the moduli far away from the black hole. At $\tau = 0$ they can take any values, $z^i(0)$, but closer to the black hole the z^i start experiencing the potential V.

For extremal black holes there is a remarkable result known as the *attractor mechanism*. It states that the moduli fields are fixed to take values determined by the charges q_I, p^I at the horizon, and that these values are independent of $z^i(0)$. Indeed, as we describe briefly below, the equations of motion derived from (5.9) describe an attractor flow converging to an attractor point.

Let us explain how this works. Dropping a boundary term, the action (5.9) can be recast into

$$S_{\text{eff}} = -\frac{1}{2}\int_0^\infty d\tau \left(\left[\dot{U} + e^U|Z|\right]^2 + \left\|\dot{z}^i + 2e^U K^{i\bar{\jmath}}\partial_{\bar{\jmath}}|Z|\right\|^2\right). \tag{5.10}$$

From this the equations of motion are obvious:

$$\begin{aligned} \dot{U} &= -e^U|Z| \\ \dot{z}^i &= -2e^U K^{i\bar{\jmath}}\partial_{\bar{\jmath}}|Z|. \end{aligned} \tag{5.11}$$

The second equation gives

$$\partial_\tau|Z| = \partial_i|Z|\dot{z}^i + \text{c.c.} = -4e^U \left\|\partial_i|Z|\right\|^2 \leq 0. \tag{5.12}$$

We see that the equations of motion imply that $|Z|$ decreases as we approach the horizon, to a minimal value $|Z_\infty|$ at $\tau = \infty$. For solutions where $Z_\infty \neq 0$ the geometry (5.8) describes an extremal supersymmetric black hole with horizon at $\tau = \infty$. Straightforward analysis of the equations close to $\tau = \infty$ reveals that the near horizon geometry is $\text{AdS}_2 \times S^2$ with horizon area $A = 4\pi|Z_\infty|$ and that $\partial_\tau|Z| = 0$ there. The latter implies

$$\partial_i|Z|\Big|_{\tau=\infty} = 0 \tag{5.13}$$

by (5.12). So the function $|Z(q_I, p^I, z^i)|$ is minimized with respect to z^i at the horizon. The particular values of z^i and of the entropy $S_{\text{BH}} = A/4 = \pi|Z_\infty|$ is therefore determined as a function of the integer charges q_I and p^I, but are unaffected by the background values of the moduli $z^i(0)$. This is the celebrated attractor behavior. It was discovered by S. Ferrara, R. Kallosh and

A. Strominger in 1995 [FKS95], and developed further by the same authors in Refs. [Str96, FK96a, FK96b].

It is possible to explicitly solve the minimization problem. Defining the quantity $C \equiv 2ie^{K/2}\bar{Z}$ the solution is

$$\begin{aligned} p^I &= \mathrm{Re}\,(CX^I) \\ q_I &= \mathrm{Re}\,(CF_I). \end{aligned} \tag{5.14}$$

These equations are known as the attractor equations.

5.2 Higher order corrections and the OSV conjecture

Let us now describe how some stringy corrections to the entropy of our $\mathcal{N} = 2$ black holes can be taken into account. This was developed in a series of papers by G.L. Cardoso, B. de Wit and T. Mohaupt [LCdWM99], [LCdWM00b], [LCdWM00c], [LCdWM00a]. The presentation here follows closely Ref. [Von05].

Considering string-loop diagrams results in corrections to the low energy $\mathcal{N} = 2$ effective action. Interactions between the graviton and graviphoton field strength $T = dV^0$ for instance give higher curvature terms. The only non-zero string theory g-loop amplitudes between these fields involve $2g - 2$ graviphotons and 2 gravitons. To reproduce these interactions in the effective theory one needs to include terms that are quadratic in the Ricci scalar in the action. The correct terms are neatly summarized in superspace formalism. If W denotes the superfield of the gravity multiplet the corrections are given by a function $F(X^I, W^2)$:

$$S_{\text{corr.}} \sim \int dx^4 d\theta^4 F(X^I, W^2). \tag{5.15}$$

The function F is the string loop corrected holomorphic prepotential. It is given as an expansion in W^2:

$$F(X^I, W^2) = \sum_{g=0}^{\infty} F_g(X^I) W^{2g}. \tag{5.16}$$

The tree-level contribution $F_0(X^I)$ is identical to the prepotential of the Calabi–Yau which we denoted $F(X^I)$. We keep this potentially confusing double meaning of F, but always write out the second argument in the corrected quantity.

To determine the thermodynamics of a black hole in higher curvature gravity, one has to use the formalism developed by R. Wald [Wal93]. Cardoso *et* *al.* perform this analysis. They find that the value of W^2 is fixed at the horizon

$$W^2 = -64\bar{Z}^2 e^{-K} = 256/C^2, \tag{5.17}$$

and that the entropy is given by

$$S_{\mathrm{BH}}(p,q) = \pi[|Z| - 256 F_{\hat{A}}(X^I, W^2)]. \tag{5.18}$$

Here we used subscript $\hat{A}$ to denote differentiation of $F(X^I, W^2)$ with respect to the second argument. On the left-hand side X^I and W^2 should be evaluated at their attractor values. The attraction point itself is also affected by the corrections. To take this into account, all one has to do is to use the corrected prepotential in the new attractor equations:

$$\begin{aligned} p^I &= \mathrm{Re}\,(CX^I) \\ q_I &= \mathrm{Re}\left[CF_I\left(X^J, \frac{256}{C^2}\right)\right]. \end{aligned} \tag{5.19}$$

So if we can compute the corrections $F_g(X^I)$ we can compute string loop corrections to the entropy of a black hole! The terms $F_g(X^I)$ have indeed been calculated. In two impressive works, one by I. Antoniadis *et al.* [AGNT94] and one by Bershadsky *et al.* [BCOV94] it is shown that the $F_g(X^I)$ is the genus g contribution to the free energy of the B-model *topological string* propagating on the Calabi–Yau.

Topological string theory can be considered as a toy-model for the full string theory. It comes in two variants, the A- and B-models, related to the type IIA and IIB superstring theories, respectively. The topological strings care less than usual about the metric of the space on which they propagate. On a Calabi–Yau, the A-model only depends on the Kähler structure, and the B-model conversely only on the complex structure. In this sense they only depend on "half of" the metric. Topological string theory originated in a work by E. Witten [Wit90].

In 2004 H. Ooguri, A. Strominger and C. Vafa — henceforth OSV — beautifully combined the works of Cardoso *et al.* and Antoniadis *et al*. OSV advocate a very simple relation between the black hole entropy (5.18) and the free energy of the topological B-model. Let ϕ^I denote $\pi\mathrm{Im}\,(CX^I)$ and define the function

$$\mathcal{F}(\phi, p) = -\pi\mathrm{Im}\left[F\left(p^I + \frac{i}{\pi}\phi^I, 256\right)\right]. \tag{5.20}$$

Expressed in this function the entropy becomes

$$S_{\mathrm{BH}}(p,q) = \mathcal{F}(\phi,p) - \phi^I \frac{\partial}{\partial \phi^I}\mathcal{F}(\phi,p), \tag{5.21}$$

where ϕ^I should be substituted in favor of q_I through the q_I-attractor equations which in this language read

$$q_I = -\frac{\partial}{\partial \phi^I} \mathcal{F}(\phi, p). \tag{5.22}$$

The interpretation of Eqs. (5.21) and (5.22) is that S_{BH} is the Legendre transform of $\mathcal{F}$ and that ϕ^I are chemical potentials for q_I.

The main conjecture of OSV is the identification of the function $\mathcal{F}$ with the free energy $\mathcal{F}_{\mathrm{top}}$ of the topological B-model

$$\mathcal{F}_{\mathrm{top}}(t^A, g_{\mathrm{top}}) = \sum_{g=0}^{\infty} g_{\mathrm{top}}^{2g-2} F_g(t^A). \tag{5.23}$$

Here $t^A = X^A/X^0$ parameterize the complex structure. With an appropriate identification of the topological coupling constant g_{top}, and the t^A evaluated at the attractor point, the correspondence reads

$$\mathcal{F}(\phi, p) = \mathcal{F}_{\mathrm{top}}(t^A, g_{\mathrm{top}}) + \bar{\mathcal{F}}_{\mathrm{top}}(t^{\bar{A}}, \bar{g}_{\mathrm{top}}) \tag{5.24}$$

with

$$t^A = \frac{p^I + i\phi^I/\pi}{p^0 + i\phi^0/\pi} \qquad g_{\mathrm{top}} = \pm \frac{4\pi i}{p^0 + i\phi^0/\pi}. \tag{5.25}$$

Exponentiating yields the corresponding relation between partition functions

$$Z_{\mathrm{BH}}(\phi^I, p^I) = |Z_{\mathrm{top}}(t^A, g_{\mathrm{top}})|^2. \tag{5.26}$$

The fully corrected thermodynamics of the supersymmetric D3-brane black hole is consequently governed by a gas of topological strings propagating on the particular Calabi–Yau geometry realized at the horizon.

5.3 Matrix model description of black holes

Even if it is known in principle how to compute the topological amplitudes F_g for a given Calabi–Yau threefold, it is still very complicated. There are explicit results available though, e.g. in Ref. [HKQ09], M-X. Huang, A. Klemm and S. Quackenbush compute the topological partition function for the quintic threefold[3] to genus 51!

In Ref. [BCOV94] the behavior of topological strings close to singularities in moduli space known as *conifold singularities* is analyzed. It is found that it is related to simple toy models of string theory known as $c = 1$ strings. These

[3]The quintic Calabi–Yau can be described as the zero locus of degree five polynomials in the projective coordinates on $\mathbb{P}^4$. We will get to know this space more intimately in Chapter 6.

are string theories in two space-time dimensions that are free from the conformal anomaly, paying the price of sacrificing Lorentz invariance: the dilaton is linear in the spatial coordinate. The $c = 1$ string theories are equivalent to systems that are exactly soluble — *matrix models*[4].

D. Ghoshal and C. Vafa made this connection between topological strings and $c = 1$ strings much more explicit, identifying the free energies of the two systems [GV95]. Let us briefly describe both conifolds and matrix models. Conifolds are treated to greater extent in Chapter 6 and a good reference for matrix models is I. Klebanov's review, Ref. [Kle91].

A conifold singularity in a Calabi–Yau manifold can locally be described by a hypersurface in $\mathbb{C}^4$:

$$uv - st = 0. \tag{5.27}$$

Here u, v, s and t are complex coordinates on $\mathbb{C}^4$. The singularity is at $u = v = s = t = 0$, and is the tip of a cone with base $S^2 \times S^3$. These singularities occur in Calabi–Yau spaces. In such a case some region of the compact space can be mapped to the vicinity of the singular point in (5.27). Conifold singularities develop at special loci in the complex structure moduli space: when the moduli approach such a locus, some region of the manifold approaches the singular geometry (5.27). Close, but not at such a conifold locus the manifold is smooth, and the local geometry of the would-be singular region has the form

$$uv - st = \mu_0. \tag{5.28}$$

The deformation parameter μ_0 is a parameter on the moduli space[5]. The conifold locus is at $\mu_0 = 0$. As the parameter μ_0 goes to zero a three-sphere in the geometry shrinks to a point. A. Strominger showed in a beautiful paper [Str95] that type IIB string theory is smooth on such singular manifolds. The smoothening happens trough non-perturbative effects: D3-branes that wrap the vanishing cycle become massless and contribute to the dynamics.

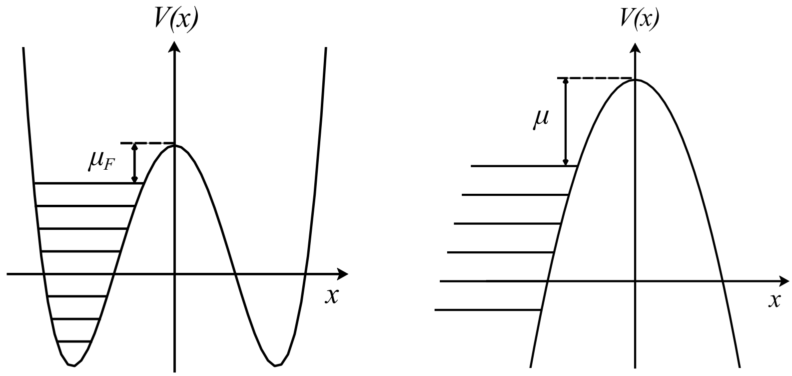
A matrix model is a simple version of non-abelian gauge theory. The basic degree of freedom is an $(N \times N)$-matrix X, and a typical action is

$$S = \int_0^{2\pi R} dt\, \beta \mathrm{Tr}\, \left(\dot{X}^2 - V(X) \right). \tag{5.29}$$

The function V acts as a potential, and the parameter β as the inverse of Planck's constant. We put the system at a finite temperature $T = 1/2\pi R$.

[4]Often the term matrix model is used for a zero-dimensional (i.e. timeless) model. In that language the models we consider are called matrix quantum mechanics.

[5]In paper II we denote the modulus μ which in the matrix model language introduced below is $\mu = \mu_0\beta$. On a non-compact space as (5.28) this is irrelevant, but in a compact Calabi–Yau, it is really μ_0 that corresponds to the deformation parameter.



(a) (b)

Figure 5.1: The potential for the $c = 1$ matrix model. To the left we see the full potential and to the right the effective potential after the double-scaling: a zoom-in on the top of the potential. The parameter μ is $\sim \beta(\mu_c - \mu_F)$.

By diagonalizing the matrix X the dynamics of the system can be reduced to that of the eigenvalues λ_i of X. Taking appropriate care of the volume measure in the path integral, one learns that the eigenvalues behave as non-interacting fermions living in the potential V! This is indeed a remarkable simplification; in fact it makes the theory so simple that it often is exactly soluble.

Matrix models are studied perturbatively by putting the N eigenvalues in the fermion ground state, corresponding to a Fermi level μ_F which is a function of N/β. The matrix model studied by Goshal and Vafa, simply called the $c = 1$ matrix model, has a potential with a maximum at $X = 0$: $V(X) \sim -X^2$. The appropriate limit for describing the conifold is when μ_F approaches a critical value μ_c where the Fermi surface reaches the top of the potential. At the same time one should send $\beta \to \infty$. More precisely the correct limit is taking $\mu_0 \sim \mu_c - \mu_F$ to zero, but keeping $\mu \equiv \mu_0\beta$ at a constant value. This *double-scaling limit* effectively zooms in on the top of the potential. An illustration of this is provided in Figure 5.1.

The free energy of the $c = 1$ matrix model can be explicitly computed:

$$\mathcal{F}_{c=1}(\mu, R) = -\frac{R}{2}\mu^2 \ln(\mu_0) - \frac{1}{24}\left(R + \frac{1}{R}\right)\ln(\mu_0) + \ldots \tag{5.30}$$

where the ellipsis denotes an expansion in powers of μ^{-2} all coefficients in which are known. Goshal and Vafa argued that at a the self-dual[6] radius $R = 1$ this exactly reproduces the B-model topological string free energy close to

[6]Note that the free energy is invariant under $R \to 1/R$ and $\mu \to R\mu$. This is indeed a symmetry of the whole theory.

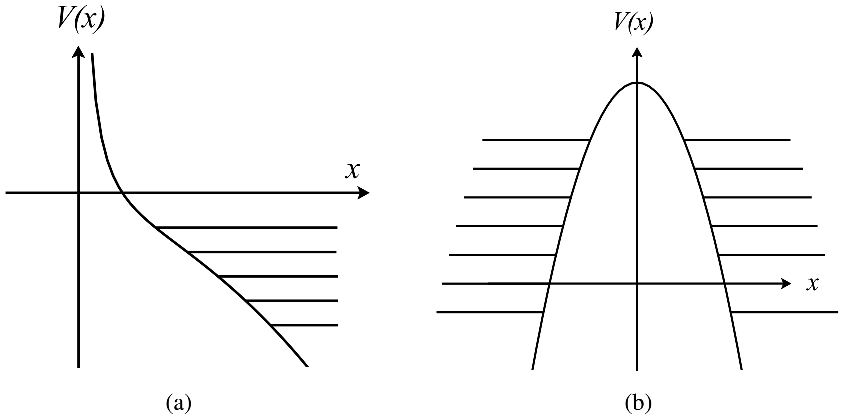


Figure 5.2: The potentials for the 0A and 0B matrix models. On the left we have 0A. The potential wall at $X = 0$ prevents tunneling. For 0B on the right the system is stable because both sides of the potential are filled.

a conifold point. In this correspondence the role of β is played by $1/g_{\text{top}}$. As explained by C. Vafa [Vaf95], the factor $-1/12$ in front of the genus one contribution is essential for the string theory resolution of the conifold through the Strominger D3-branes.

The OSV conjecture immediately made it possible to connect the thermodynamics of $\mathcal{N} = 2$ black holes to matrix models. Indeed, if $\mathcal{F}_{\text{BH}} = \mathcal{F}_{\text{top}} + \bar{\mathcal{F}}_{\text{top}}$ and $\mathcal{F}_{\text{top}} = \mathcal{F}_{c=1}$ then there really should exist such a correspondence! This is made explicit in the two papers Ref. [DOV04] by U.H. Danielsson, M.E. Olsson and M. Vonk, and in paper I of this thesis, which a careful reader at this point should be well prepared to read.

Let us make a few remarks on the results obtained. An important point is the appearance of two matrix models closely related, but not identical to the $c = 1$ matrix model. These are called the 0A and 0B models. They too are dual to two-dimensional string theories. One of their virtues is the non-perturbative stability. In the $c = 1$ matrix model only one side of the potential barrier is filled with fermions (see Figure 5.1). Allowing for tunneling to the other side, this configuration is unstable! The 0A and 0B models instead have potentials as depicted in Figure 5.2. In the 0A case the potential is

$$V(X) \sim -\frac{X^2}{2} + \frac{q - 1/4}{X^2}, \tag{5.31}$$

and in the 0B case both sides of the potential are filled with fermions. Both result in non-perturbatively stable theories.

In Ref. [DOV04] it is pointed out that the free energy of the 0A matrix model should correspond to the free energy of a black hole in a compactification with singular geometry. It is emphasized that the thermodynamics of the two systems are not dual in the naive way. In particular, the extremal black holes have zero temperature while the matrix model is at its self-dual temperature $T = T_s = 1/\pi$. Also other matrix model temperatures are studied and the corresponding geometries analyzed.

In paper I the proposed correspondence is slightly refined. We argue that the geometry naturally associated with the 0A matrix model is not a geometry leading to $\mathcal{F}_{0A} = \mathcal{F}_{\text{top}}$, but rather a geometry satisfying

$$\mathcal{F}_{0A} = \mathcal{F}_{\text{top}} + \bar{\mathcal{F}}_{\text{top}}, \tag{5.32}$$

something that also was hinted at in Ref. [DOV04]. Making this interpretation, the 0A theory at the self-dual radius corresponds to the conifold. We make a detailed match by the parameters in the matrix model and the $\mathcal{N} = 2$ black hole charges, and also discuss how, from a matrix model perspective, to incorporate large classical contribution to the black hole entropy coming from embedding the conifold in a compact Calabi–Yau.

Finally, we suggest how to make sense of the correspondence at multiples of the self-dual radius, in spite of the fact that the logarithmic genus one contribution does not come with the crucial factor $-1/12$. By rewriting the free energy in a way that mixes terms of different genera, it takes the form of the free energy of a black hole living in a compactification exhibiting multiple conifold singularities.

It is noteworthy that such simple systems as matrix models are able to compute highly nontrivial quantities in full-fledged string theory on Calabi–Yau manifolds. These toy-models are also well defined non-perturbatively and could potentially provide information about the non-perturbative nature of the topological string.

5.4 Black holes and flux compactifications

The second paper in this thesis, paper II, estimates the relative impact of a D3-brane black hole and a flux background on the complex structure moduli. The general argument is quite simple. It uses the result (5.5). The potential originating from the black hole comes with a factor $1/R^4$, where R is defined in (5.4). Denoting the integer flux quanta by Q and the black hole charges by q, this means that the black hole potential is suppressed by a factor

$$V_{\text{flux}} \sim \frac{1}{Q^2 q^2}. \tag{5.33}$$

at the horizon. To see this note that $V_{\text{flux}} \sim Q^2$, $V_{\text{BH}} \sim q^2$ and $R_{\text{hor.}} \sim q$. Since Q is typically of order 10 and q is very large for black holes of macroscopic size, there will be no effect from the black hole on moduli stabilization.

An end left loose in paper II is to incorporate the dependence of this reasoning on the size of the extra dimensions. Let us include this analysis here. Performing a dimensional reduction we see that the terms in the action come with the following factors:

$$\begin{array}{rcl} \frac{1}{2\kappa_{10}^2}\int dx^{10}\sqrt{g}\mathcal{R}_{10} & \sim & \frac{1}{2\kappa_4^2}\int d\mathrm{Vol}_4\mathcal{R}_4 \\ -\frac{1}{4\kappa_{10}^2}\int dx^{10}\sqrt{g}\frac{|\tilde{F}_5|^2}{2\cdot 5!} & \sim & \frac{1}{8\pi}\int d\mathrm{Vol}_4\frac{\langle\Gamma_{\mathrm{BH}},\hat{\Gamma}_{\mathrm{BH}}\rangle}{R^4} \\ -\frac{1}{4\kappa_{10}^2}\int dx^{10}\sqrt{g}\frac{|\tilde{F}_3|^2}{3!} & \sim & \frac{1}{8\pi}\int d\mathrm{Vol}_4\frac{g_s\langle\Gamma_{\mathrm{flux}},\hat{\Gamma}_{\mathrm{flux}}\rangle}{(4\pi\alpha')^2}. \end{array} \tag{5.34}$$

The four-dimensional Planck length κ_4 is

$$\kappa_4^2 = \frac{\kappa_{10}^2}{V_6} \tag{5.35}$$

where V_6 is the internal volume. The different Γ are integer cohomology forms: $\Gamma_{\mathrm{BH}} \sim q$ and $\Gamma_{\mathrm{flux}} \sim Q$. The horizon radius for the black hole will be of order $R_{\mathrm{hor.}} \sim q\kappa_4$ for an extremal black hole. For non-extremal black holes the radius will be larger, and the effect will be smaller.

Putting things together, using (3.52) we obtain that the black hole dominates at the horizon if the radius R_6 of the compact dimensions satisfies

$$R_6 > g_s^{1/3}Q^{2/3}R_{\mathrm{BH}}^{2/3}\kappa_4^{1/3}. \tag{5.36}$$

With our naive bound (4.6), and taking $\kappa_4 \sim 10^{-35}\mathrm{m}$, we see that we must require

$$R_{\mathrm{BH}} < \frac{10^{-9}\mathrm{m}}{Qg_s^2} \tag{5.37}$$

to see an effect. The interpretation of this is that a black hole can possibly affect the internal geometry of a flux compactification only at a late stage of Hawking evaporation. However, for such black holes the supergravity approximation we use breaks down.

6. Landscape topography

Over the hills and far away
Gary Moore

The GKP construction described in Chapter 4 comes a long way toward creating "semi-realistic" vacua of string theory. Stabilization of Kähler moduli is not taken into account, but as noted in the closing paragraphs of the same chapter, there are convincing models achieving also this. There are a lot of choices involved in these setups: the compact topology, the number of O3-planes and the number of flux quanta through each of the compact three-cycles are examples. There are also many other string theory models, based e.g. on type IIA or heterotic strings.

Since there are no known reasons why these vacuum solutions would be inconsistent, the emerging picture is that of a huge discrete set of possible low energy theories. This set is known as the string theory landscape, a term coined by L. Susskind [Sus03]. In principle, one could also speak of a landscape at the level of four-dimensional quantum field theories, but there are two crucial differences.

First and foremost the vacua in the string landscape have a consistent embedding in a full theory of quantum gravity. It is an interesting and challenging problem to understand which four-dimensional field theories are possible as effective descriptions of string theory. The set of those that cannot has been named "the swampland" by C. Vafa [Vaf05]. Some features distinguish the landscape from the swampland. As described by N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa [AHMNV07], one striking such feature is the weakness of gravity.

Second, string theory does not only provide a set of vacua — it also contains the dynamics for describing processes in which many vacua take part. There is an *effective potential* on the string theory landscape. A recent review making these aspects very clear is that of A.N. Schellekens [Sch08].

Assuming that we live in a string theory vacuum, i.e. at a critical point of this potential, several questions become pertinent. One is about the selection of the particular vacuum that describes our universe. Is it dynamically selected? Or is it a random process? Another issue concerns our immediate surroundings in the landscape. Since they are in principle dynamically accessible it is of great interest to understand if there are observable consequences that depend on them. For instance, the masses of the moduli fields depend on the second derivatives of the potential evaluated at the minimum, but there are also more elaborate effects.

Chain inflation is such an example. It was proposed by K. Freese and D. Spolyar [FS05] and analyzed in the landscape in collaboration with J.T Liu [FLS06]. More recently the idea was revisited by D. Chialva and U.H. Danielsson [CD08b, CD08a]. In this scenario inflation takes place in several different vacua between which the universe tunnels. If there are long series of connected vacua in the landscape, this scenario could have a natural embedding in string theory. Another consequence of long sequences of vacua was explained by H. Tye. Connected vacua can aid tunneling in the landscape by resonance effects [Tye06].

A very interesting, albeit speculative, idea is that our universe went through a phase where many vacua were accessed. This process may well leave an observable imprint on cosmological data, as was explored by H. Davoudiasl, S. Sarangi and G. Shiu [DSS07].

All these effects depend on the local topographic structure of the landscape, i.e., on the structure of the effective potential. It is therefore of considerable interest to study this structure. The two papers III and IV analyze the corner of the string theory landscape corresponding to some GKP models. We pick a particular Calabi–Yau manifold and perform both analytical and numerical investigations of the corresponding scalar potential (4.66) governing the dynamics of complex structure moduli and the dilaton. An important result is that we find that the long connected series of vacua relevant for chain inflation and resonance tunneling are a typical feature.

In both these studies the concept of *monodromies* of the internal space is of great importance. Let us therefore describe this in some detail in a simple example.

6.1 Monodromies

The best friend of many geometrical physicist is the two-dimensional torus T^2. Complex structure, Kähler structure, cohomology and monodromies are all abstract concepts that have simple, yet nontrivial realizations in T^2. This space is of immense help in trying to understand more complicated Calabi–Yau manifolds.

The torus also plays a very important part of its own in string theory. One-loop string amplitudes are amplitudes on the torus, and the $\mathrm{SL}(2,\mathbb{Z})$ self-duality of type IIB can be understood as the modular group of the torus in an elliptically fibered F-theory compactification. Furthermore, compactifications of string theory on flat or twisted tori are simple models with rich semi-realistic phenomenology.

Given these facts, it is very hard not to think as highly of tori as Homer Simpson does:

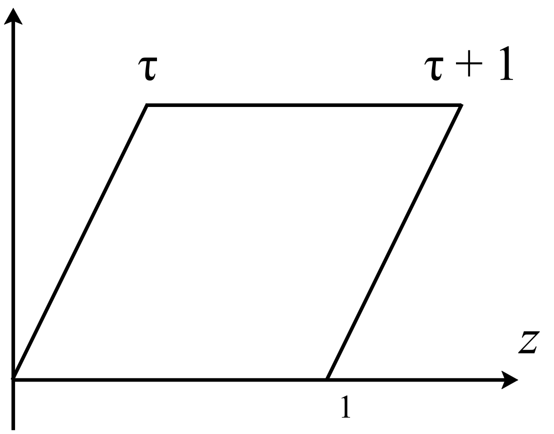


Figure 6.1: A torus. The best friend of many geometrical physicists. The torus is obtained from the complex plane by the identifications $z \sim z+m+n\tau$ with $m, n \in \mathbb{Z}$. It inherits its flat Kähler metric from that of the complex plane.

Doughnuts. Is there anything they can't do?

The complex geometry of the torus is described e.g. in Chapter 5 of Polchinski's book [Pola], but let us recall some facts.

To put a complex structure on T^2 we define it as the complex plane subject to the identification

$$z \sim z + m + n\tau \qquad m, n \in \mathbb{Z}. \tag{6.1}$$

Here $z = x + iy$ is the coordinate on $\mathbb{C}$. This defines a torus for any $\tau = \tau_1 + i\tau_2 \in \mathbb{C} \setminus \mathbb{R}$, which parameterizes the complex structure. As we shall see, not all τ define different structures. For instance $\tau \to \bar{\tau}$ clearly gives the same torus, so we restrict ourselves to $\tau_2 > 0$.

The fundamental domain of the identification (6.1) is depicted in Figure 6.1. The Ricci-flat (remember that in two dimensions this means flat) and Kähler metric corresponding to τ is inherited from the flat metric $ds^2 = (A/\tau_2)dzd\bar{z}$ on $\mathbb{C}$. With this perhaps strange-looking normalization of the metric, the parameter A is the area of the torus. Let us write this metric in standard periodic coordinates θ_1, θ_2, satisfying

$$\theta_i \in [0,1] \quad \text{and} \quad \begin{cases} (0,\theta_2) \sim (1,\theta_2) \\ (\theta_1, 0) \sim (\theta_1, 1) \end{cases} \tag{6.2}$$

The most natural ones, running along the sides of the torus in Figure 6.1 are

$$\theta_1 = x - \frac{\tau_1}{\tau_2} y \qquad \theta_2 = \frac{y}{\tau_2}. \tag{6.3}$$

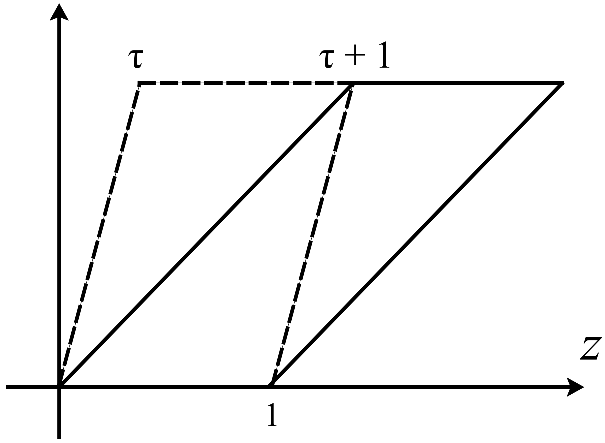


Figure 6.2: Two identical tori corresponding to different τ. The corresponding flat metrics and the coordinate transformation relating them are described in the text.

In these coordinates the metric reads

$$g_{ij} = \frac{A}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}_{ij} . \tag{6.4}$$

We see explicitly how different τ means different metrics on T^2. By our choice of normalization for the metric on $\mathbb{C}$ we factored out the overall volume A of the torus. This parameter corresponds to a Kähler modulus.

As mentioned, different τ can lead to the same complex structure. Indeed $\tau' = \tau + 1$ leads to exactly the same set of identifications as (6.1). Also $\tau' = -1/\tau$ does, provided that we scale the parameter on $\mathbb{C}$: $z \to \tau z$. Let us illustrate the former on the level of the metric — from (6.4) it looks as if they would be different. The two (identical) tori have fundamental domains as shown in Figure 6.2. Let us make a coordinate change to see that they are really the same. Define new coordinates $\tilde{\theta}_{1,2}$ on the first torus by

$$\tilde{\theta}_1 = \begin{cases} \theta_1 - \theta_2; & \theta_2 < \theta_1 \\ 1 + \theta_1 - \theta_2; & \theta_1 < \theta_2 \end{cases} \qquad \tilde{\theta}_2 = \theta_2. \tag{6.5}$$

These coordinates also satisfy (6.2). Expressed in these, the metric (6.4) is

$$g_{ij} = \frac{A}{\tau_2} \begin{pmatrix} 1 & \tau_1 + 1 \\ \tau_1 + 1 & |\tau + 1|^2 \end{pmatrix}_{ij} . \tag{6.6}$$

This is the metric corresponding to $\tau + 1$, so these two tori are really the same expressed in different coordinates.

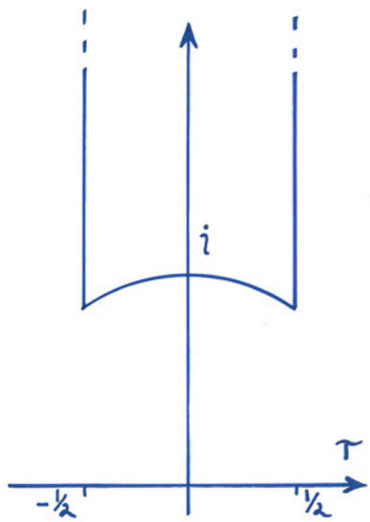


Figure 6.3: The famous picture of the torus moduli space $|\tau_1| \leq 1/2$ and $|\tau| \geq 1$. The two lines going off to infinity are identified, as are the circle arches on the left and right side of $\tau = i$.

The two transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$ can be performed any number of times and in any order. Together they generate the group $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$. A general transformation is of the form

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{6.7}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The factoring by $\mathbb{Z}_2$ is needed since changing the sign of a, b, c, d yields the same transformation. The true moduli space of the torus is therefore $H/\mathrm{PSL}(2,\mathbb{Z})$, where H is the upper half plane. The covering space H is called the *Teichmüller space* of T^2.

One can show that any point $\tau \in H$ can be mapped to the region $|\tau_1| \leq 1/2$ and $|\tau| \geq 1$. This map is unique except at the boundaries. With proper identifications at the boudaries, the strip in Figure 6.3 is a fundamental domain of the $\mathrm{PSL}(2,\mathbb{Z})$ action, and thus constitutes the moduli space of the torus.

Before analyzing the moduli space further, let us make a small digression on type IIB supergravity. Look back at the supergravity action (4.43), and the definition of the axio-dilaton τ (4.42). It is a straightforward exercise to show that the action is invariant under the combined transformation (6.7) and

$$G_3 \to \frac{G_3}{c\tau + d} \tag{6.8}$$

where τ now is the axio-dilaton. Type IIB supergravity thus has an $\mathrm{SL}(2,\mathbb{Z})$ symmetry. This is an instance of the S-duality of string theory, and config-

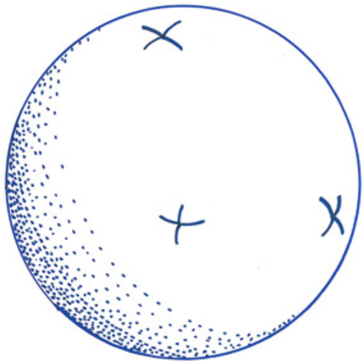


Figure 6.4: A sphere with three special points. They can be singular or removed. This space is a toy-model for the torus moduli space, or for the mirror quintic moduli space to be introduced below.

urations related by this symmetry should really be considered equivalent. If type IIB is viewed as a limit of a toroidal F-theory compactification the axio-dilaton τ really is the complex structure modulus of the torus fiber. Note that S-duality can interchange theories with large and small string coupling.

Returning to our description of the torus moduli space, there are three points of special significance in this space. First, the point $\tau_2 \to \infty$ corresponds to a degenerate torus where the ratio of the two radii goes to zero. A bit more subtle are the points $\tau = i$ and $\tau = \pm 1/2 + i\sqrt{3}/2$. The first of these is a fixed point of $\tau \to -1/\tau$. It does not correspond to a singular torus, but the moduli space is singular here. The reason for this is that moving in different directions in Teichmüller space corresponds to moving in the same direction in moduli space. E.g., $\tau = i + \epsilon i$ and $\tau = i - \epsilon i$ correspond to the same torus. Therefore this is an orbifold singularity of the moduli space. A similar thing happens at the point $\tau = \pm 1/2 + i\sqrt{3}/2$.

A sphere with three singular points is therefore a good picture of the T^2 moduli space. One point, $\tau_2 = \infty$, should really be removed, the other two correspond to regular tori but the moduli space is singular. A sphere with three special points is shown in Figure 6.4.

Let us now investigate in some detail what happens if we continuously deform a torus, but end up at the same geometry that we started with, i.e. if we take the torus around a loop in moduli space. At first sight it does not seem that exciting, but if the loop encircles one of the singular points interesting things can happen. Such loops lift to open curves in Teichmüller space.

Let us denote the continuous deformation of a torus $\tau \to \tau + 1$ by t_∞. This transformation corresponds to encircling the point $\tau_2 = \infty$. To generate the full fundamental group of the three times punctured S^2, also the continuous encircling of $\tau = i$ denoted t_i and corresponding to $\tau \to -1/\tau$ is needed.

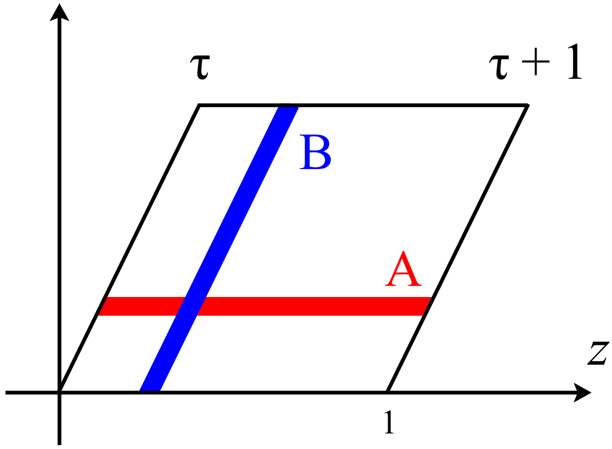


Figure 6.5: The torus with two of its one-cycles painted up. The A-cycle is red and the B-cycle is blue.

Suppose now that we wrap something on the cycles of the torus: rubber bands, fluxes, branes or something of the like. Or that we just paint them as in Figure 6.5. We choose two cycles, A and B, forming a basis of $H_1(T^2, \mathbb{Z})$. Now we perform the transformation t_∞. We continuously increase the real part τ_1 until we reach $\tau + 1$. At this stage the torus looks as the non-dashed part of Figure 6.6. We have now traversed the full loop and are back at the original torus. If we let our coordinate system vary continuously however, the metric has changed: τ became $\tau + 1$. Changing coordinates so that we get back our original coordinatization corresponds to the inverse of the reparameterization (6.5). This essentially corresponds to cutting out the rightmost triangle of Figure 6.6 and gluing it one unit to the left, producing the dashed torus.

The dashed torus has the original coordinatization, but something remarkable has occurred. The paint is now around different cycles! Indeed, the A-cycle is the same, but B has turned to $B + A$! There is apparently a nontrivial action of t_∞ on $H_1(T^2, \mathbb{Z})$. The action is linear, and in the $\{A, B\}$ basis the corresponding matrix is

$$t_\infty = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{6.9}$$

Such a transformation is called a *monodromy*.

Going around the point $\tau = i$ will also produce a monodromy, and one can show that the action is an interchange of the two cycles and a sign:

$$t_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.10}$$

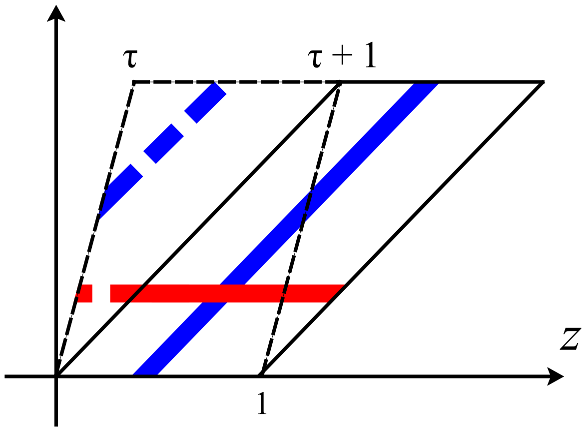


Figure 6.6: Under a continuous transformation $\tau \to \tau + 1$, the two painted cycles of Figure 6.5 end up as the solid cycles in the present figure. A torus with parameter $\tau + 1$ is however the same as a torus with parameter τ. Transforming back to the original coordinates corresponds to cutting out the triangle to the right and gluing it to the left (dashed). We see that the B-cycle has received an addition of the A-cycle. A monodromy!

The two matrices t_∞ and t_i generate the *monodromy group* of the torus. It is nothing but $\mathrm{SL}(2, \mathbb{Z})$.

One of the reasons for taking so much space to explain monodromies is that they often seem counter-intuitive at first encounter. It appears that it is possible to wrap a rubber band around one cycle and then *continuously* move it to a different cycle. This must be topologically impossible. Indeed it is impossible: there is no homotopy between homotopy inequivalent cycles. The magic trick is to realize that there is an ambiguity in the labeling of the cycles, and that if one changes also the space, then what was once an A-cycle might now look like a B-cycle!

Our interest in monodromies arises from the study of flux compactifications. Calabi–Yau spaces have monodromies just like the torus. Starting with flux through some of the cycles and performing a monodromy transformation, the geometry looks the same, but the fluxes have switched cycles. If the fluxes are just slightly changed by the monodromy, the resulting configuration is closely related to the original one, but different. And — more importantly — there is a continuous path between the configurations. In papers III and IV we explore several aspects of monodromies in flux compactifications, a distinguished feature being the discovery of long series of connected minima.

In these papers we deal not with tori but with Calabi–Yau three-folds. The monodromies are slightly more complicated, but the analogy is straightforward. In the next section we introduce the reader to two prominent members of the Calabi–Yau family: the quintic and its mirror.

6.2 The quintic and its mirror

Before describing the manifolds let us comment on the word "mirror". A wonderful and surprising feature of type II Calabi–Yau compactifications is that there are pairs of manifolds that result in the same effective theory. Type IIA compactified on a manifold $\mathcal{M}$ is identical to type IIB compactified on its mirror $\mathcal{W}$. In type IIA the relation between multiplets and cohomology is reversed as compared to type IIB: the vector multiplets correspond to $H^{1,1}$ and the hypermultiplets to $H^{2,1}$. This means that the Hodge diamonds (4.22) of mirror manifolds are each other's reflection in a diagonal line. For a long, thorough introduction to mirror symmetry, including prerequisites in mathematics and physics, the reader is referred to Ref. [H$^+$] by K. Hori *et al.*

One of the most popular ways to describe Calabi–Yau manifolds is as zero-sets of polynomials in (products of) complex projective spaces. Taking the polynomials to be holomorphic in the projective coordinates, the manifold inherits a complex structure from the projective space. Thus, the complex structure of the Calabi–Yau is parameterized by the coefficients of the polynomial.

The best known such manifold is the quintic[1] $\mathcal{W}_{(1,101)}$ in $\mathbb{P}^4$. Denoting the projective coordinates on $\mathbb{P}^4$ by $x_1, \ldots, x_5$ the quintic is given as the zero set of a general quintic polynomial $p(x)$:

$$p(x) = 0. \tag{6.11}$$

Note that for this zero set to make sense under rescalings $x_i \to \lambda x_i$ all monomials must be quintic.

Let us count the dimension of the complex structure moduli space. It is an amusing exercise to compute the number of degree d monomials in n variables. The result is

$$\binom{n+d-1}{n-1}. \tag{6.12}$$

Hence, there are 126 distinct quintic monomials in five variables, meaning that the polynomial $p(x)$ have 126 free complex coefficients. However, this is overcounting the degrees of freedom. Making a linear coordinate change of the z_i transmutes the monomials, and hence changes the coefficients. The group of such transformations is $\mathrm{GL}(5, \mathbb{C})$ having $5 \cdot 5 = 25$ complex dimensions. The dimension of the quintic complex structure moduli space is therefore $126 - 25 = 101$ as indicated by the notation $\mathcal{W}_{(1,101)}$.

Let us now study the mirror manifold $\mathcal{M}_{(101,1)}$. Its complex structure moduli space, which we denote $M_{(101,1)}$, is one-dimensional, facilitating its description. This is the manifold under study in papers III and IV. The mir-

[1] We let the subscript on Calabi–Yau manifolds denote their Hodge numbers in the order $(h^{1,1}, h^{2,1})$. The reader is warned, however. Specifying the Hodge numbers does *not* specify the manifold completely.

ror pair $\mathcal{M}_{(101,1)}$ and $\mathcal{W}_{(1,101)}$ was extensively and cleverly described in a beautiful paper by P. Candelas, X.C. De La Ossa, P.S. Green and L. Parkes [CDLOGP91].

The mirror quintic can in fact be obtained as a quotient of a special class of quintics, parameterized by the single complex parameter ψ:

$$\sum_{i=1}^{5} x_i^5 - 5\psi \prod_{i=1}^{5} x_i = 0. \tag{6.13}$$

The quotient is by a $\mathbb{Z}_5^3$ symmetry group leaving this class of hypersurfaces invariant. After taking the quotient some singularities require blow-up. The precise construction does not concern us.

The parameter ψ takes values in the complex numbers, but not all ψ correspond to different complex structures. Indeed if $\alpha^5 = 1$ then $\psi \sim \alpha\psi$, since the shift can be undone by a coordinate transformation $x_1 \to \alpha^{-1}x_1$. Consequently the moduli space is $\mathbb{C}/\mathbb{Z}_5$, having an orbifold singularity at the origin.

There are two more special points in the moduli space. One is at $\psi = \infty$ where the manifold is singular. This is referred to as the *large complex structure* limit. The other point is $\psi = 1$. Also here the manifold is singular, but in a much milder way. It is a conifold.

The topology of $M_{(101,1)}$ is thus similar to the one of the torus moduli space. It is a sphere with three special points (one singular and two removed). Precisely as the nontrivial fundamental group of the torus moduli space supports a monodromy group, so does the one of $M_{(101,1)}$. Let us describe it briefly, focusing on the transformation around the conifold point.

As mentioned in Chapter 5 a conifold singularity occurs when a three-cycle of topology S^3 in the manifold shrinks to zero volume. For the mirror quintic, this cycle is given explicitly in Ref. [CDLOGP91]. The situation is almost completely analogous to the torus, where the role of $\psi = 1$ is played by $\tau_2 = \infty$. At that point the A-cycle of Figure 6.5 shrinks to zero relative volume. Let us denote also the shrinking S^3 by A^2. It has a dual three-cycle B_2 intersecting it once. In fact the pair (A^2, B_2) can be chosen as one pair in a symplectic basis $\{A^I, B_I\}$ $I = 1, 2$ of $H_3(\mathcal{M}_{(101,1)}, \mathbb{Z})$. C.f. Subsection 4.2.3.

Completely analogously to the torus case, the cycle B_2 obtains an addition of A^2 as a loop encircling the conifold point $\psi = 1$ is traversed. The other basis cycles A^1, A^2 and B_1 are left unchanged by the transformation, which with the ordering[2] (A^2, B_1, A^1, B_2) is given by the matrix

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{6.14}$$

[2]This differs from Ref. [CDLOGP91]. It is chosen to concur with the papers III and IV.

The complete monodromy group of the mirror quintic is generated by this transformation together with any of the transformations corresponding to encircling $\psi = 0$ or $\psi = \infty$. To make the notation coincide with papers III and IV we denote these transformations T_∞ and T_0. The reason for this confusing naming is that papers III and IV use the variable $z = \psi^{-5}$, thus interchanging the meaning of 0 and ∞. In the conventions of papers III and IV T_0 is

$$T_0 = \begin{pmatrix} 1 & 1 & 3 & -5 \\ 0 & 1 & -5 & -8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.15}$$

At this point we would like to remind the reader of the period integrals (4.30). Recall that the corresponding prepotential (4.33) determines the low energy effective theory of type II strings compactified on the Calabi–Yau. Since the periods are integrals over a basis of three-cycles also they are subject to monodromies. Around the conifold point, e.g., we have

$$F_2 \to F_2 + X^2. \tag{6.16}$$

In particular this means that the prepotential F cannot be analytical at the conifold point! Instead it implies the presence of logarithmic branch cuts close to $X^2 = 0$:

$$F_2 = \frac{X^2}{2\pi i} \log\left(\frac{X^2}{X^1}\right) + \text{analytic}. \tag{6.17}$$

For the prepotential itself this translates to

$$F = \frac{(X^2)^2}{4\pi i} \log\left(\frac{X^2}{X^1}\right) + \text{analytic}. \tag{6.18}$$

An astute reader will recall that the genus zero term in the $c = 1$ matrix model free energy (5.30) has exactly this form with $\mu \sim X^2$. This is as it should, because this free energy is nothing but the free energy of topological strings on the conifold, the genus zero contribution of which is precisely the prepotential of the conifold.

Let us end this section by collecting our knowledge in a picture — Figure 6.7 — of the ψ-plane. A fundamental domain for $M_{(101,1)}$ is given by $\arg\psi \in [0, 2\pi/5)$, and $M_{(101,1)}$ itself can be thought of as the punctured sphere in Figure 6.4. The points $\psi = e^{2\pi i n/5}$ correspond to the conifold point, and logarithmic branch cuts extend from them to infinity. Since the ψ-plane is a five-fold cover of $M_{(101,1)}$ there is no trace of the singularity at $\psi = 0$: the monodromy T_∞, encircling $\psi = 0$, corresponds to $\psi \to e^{2\pi i/5}\psi$ when $|\psi| < 1$.

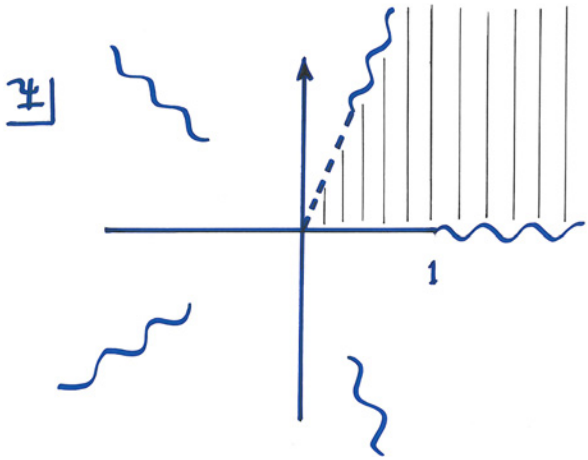


Figure 6.7: A picture of the ψ-plane. The fundamental domain $\arg\psi \in [0, 2\pi/5)$ has been striped, and logarithmic branch cuts emanating from the fifth roots of unity extend to infinity.

6.3 Series of minima and domain walls

At this point, the reader should be fairly fit to read paper III. There we study the scalar potential V given in (4.66). Minima to this effective potential correspond to string theory vacua. We use the mirror quintic as a technically simple model. Almost any Calabi–Yau manifold will, however display the same characteristic features, since these depend only on the presence of monodromies and minima of the potential.

Practically, we choose various sets of integer flux quanta and compute $V(\psi, \tau)$ numerically using expressions for the period integrals originally derived in Ref. [CDLOGP91]. Close to the conifold point, however, analytical treatment is feasible. A crucial point is that V is a multivalued function on the moduli space: it is subject to monodromies. This is clear since the superpotential $W = \int \Omega \wedge G_3$ is expressed in terms of the periods, and the periods transform. By the torus example it is also intuitively clear. Letting the paint on the tori in Figures 6.5 and 6.6 symbolize flux lines we see that traversing the moduli space loop $\tau \to \tau + 1$ *changes the flux quanta*. With new fluxes obviously the potential has changed.

Exploiting this structure of the scalar potential we find a number of interesting topographic traits. In particular, close to the singular points in moduli space the potential forms spiral staircases due to the logarithmic cuts. At the bottom of such a staircase, there may or may not be a minimum of the potential.

Furthermore, there are long series of minima, typically connected by conifold monodromies. The existence of such series could have been anticipated. Indeed, if the flux quanta on the conifold cycle are small, then the potential will not change much under the monodromy. If there is a minimum to begin with, it is likely to still be there after the monodromy. We also find seemingly infinite series that become periodic when the $\mathrm{SL}(2, \mathbb{Z})$ symmetry is taken into account. Despite some effort to find infinite series of connected minima, we do not achieve this. However we are able to relate their existence to open mathematical questions.

The possibility to connect configurations corresponding to different flux quanta by monodromies allows for the construction of domain walls. Consider e.g. two minima connected by a conifold monodromy. Letting the complex structure of the internal manifold encircle the conifold point when going from one side of the wall to the other, produces a flux changing domain wall. This object thus appears to be *charged* under the three-form fluxes. This is to say, a D5/NS5 brane wrapping the appropriate three-cycle acts as a domain wall relating the same two flux configurations! It is intriguing that there are two completely different ways of creating very related objects, and further studies of this correspondence would be interesting.

As mentioned in the beginning of this chapter, long series of vacua are very interesting from a cosmological point of view. The concept of chain inflation for instance, fits nicely in this framework as described by Chialva and Danielsson [CD08b, CD08a].

Another neat feature is that the potential between different minima is known and calculable. This makes it feasible to numerically search for regions suitable for slow-roll inflation, construct domain wall solutions and calculate tunneling amplitudes. An in-depth study of the flux potential regarding all these aspects was performed by M.C. Johnson and M. Larfors in Ref. [JL08b]. This work also established several novel results applicable to general multi-field tunneling, properly accounting for gravitational effects.

Extending the model to include Kähler moduli, and using topographic features of the potential, the same authors have also demonstrated that there are severe restrictions on the existence of tunneling instantons in the flux landscape [JL08a]. These results potentially have profound implications on the possibility to embed scenarios like eternal inflation into string theory.

6.4 Geometric transitions

To prepare for paper IV, we need to provide some more depth in our treatment of conifold singularities. In the 1980s it was realized, notably by P.S. Green and T. Hübsch, that the moduli spaces of many Calabi–Yau manifold of different topology are connected [GH88b, GH88a]. At the loci where different moduli spaces meet, the corresponding geometries are singular. The conifold singularity is exactly such a singularity, as we now describe. We follow closely the treatment in Ref. [CGH90] by P. Candelas, P.S. Green and T. Hübsch.

Consider the product $\mathbb{P}^4 \times \mathbb{P}^1$ of complex projective spaces. Let us denote projective coordinates on $\mathbb{P}^4$ by $[x_1 : x_2 : x_3 : x_4 : x_5]$ and on $\mathbb{P}^1$ by $[y_1 : y_2]$. A class of Calabi–Yau manifolds is described as the combined zero locus of polynomials

$$\begin{aligned} u(x)y_1 + t(x)y_2 &= 0 \\ s(x)y_1 + v(x)y_2 &= 0, \end{aligned} \tag{6.19}$$

where u and t are quartic and s and v are linear polynomials in the x_i. The manifold thus obtained has Hodge numbers $h^{1,1} = 2$ and $h^{2,1} = 86$ and we denote it $\mathcal{W}_{(2,86)}$.

If the $\mathbb{P}^1$ shrinks to zero size this manifold becomes singular. It is still a subset of $\mathbb{P}^4$ though, and it is in fact given by a quintic equation. To see this, note that since y_1 and y_2 can never vanish simultaneously we must have

$$p(x) \equiv u(x)v(x) - s(x)t(x) = 0. \tag{6.20}$$

Both uv and st constitute quintic polynomials. Hence this space is a singular version of the quintic three-fold! The singular locus is where p fails to be transverse: $dp = 0$, i.e. $u = v = s = t = 0$. An observant reader has already made the connection to (5.27) of the previous chapter. This is a conifold.

It is straightforward to make the quintic non-singular. We just add a suitable small quintic polynomial $\mu_0(x)$ to the left-hand side of (6.20):

$$u(x)v(x) - s(x)t(x) = \mu_0(x). \tag{6.21}$$

This blows up an S^3 in the quintic. There are thus two topologically distinct ways of making the conifold smooth. Inflating an S^3 is called *deforming* the conifold and blowing up a $\mathbb{P}^2 \approx S^2$ is called *resolving* it. This kind of transitions are called *geometric transitions* — this particular one relates the quintic $\mathcal{W}_{(1,101)}$ and the $\mathcal{W}_{(2,86)}$.

In paper IV we study the mirror of this process, relating the manifolds $\mathcal{M}_{(86,2)}$ and the mirror quintic $\mathcal{M}_{(101,1)}$ in much detail. The mirror quintic corresponds to the resolved geometry, whereas $\mathcal{M}_{(86,2)}$ corresponds to the deformed geometry. We find the six periods of the two-dimensional complex structure moduli space $M_{(86,2)}$ of $\mathcal{M}_{(86,2)}$ and find the corresponding monodromies.

Note that the conifold locus in $M_{(86,2)}$ exactly corresponds to the full complex structure moduli space $M_{(101,1)}$ of the mirror quintic. We provide this embedding of $M_{(101,1)}$ into $M_{(86,2)}$ explicitly, including monodromies. These constructions make heavy use of a Calabi–Yau geometry toolkit known as *toric geometry*, and are interesting in their own right.

However, we exploit this system to generalize the concept of monodromies to include paths that pass from one moduli space to another. The monodromy group of $\mathcal{M}_{(86,2)}$ is larger than that of the mirror quintic, so passing through the transition (deforming), performing a monodromy (revolving) and going back to the mirror quintic (resolving), thus allows for more general transformations than the monodromy group of the mirror quintic. Amusingly, this generalization is large enough to allow us to find infinite series of continuously connected minima.

Applying these transformations to configurations with fluxes introduces some issues only partially resolved in IV. If there is flux through the shrinking A-cycle in $\mathcal{M}_{(86,2)}$, this will persist as a flux through a chain on the mirror quintic, sourced by space-filling D5 branes wrapped on the blown-up S^2. The fate of flux through the dual B cycle is much more mysterious and is under current investigation.

7. Gravity in three dimensions

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Up until now, the developments described in this thesis have been within the rich and complicated structure of string theory. We described black holes in Calabi–Yau compactifications using topological strings, investigated the interplay between the potentials induced from fluxes and D3-brane black holes, and went on excursions in winding spirals in the string theory landscape.

In this chapter we switch gears a bit. Instead of using the full machinery, we try to isolate typical traits of quantum gravity and study them in as simple a setting as possible. Namely we study gravity in three dimensions. We do not reach as far as a quantization of the theory, but we attack questions that are very relevant for this quest at the classical level.

Let us stress that the objective is to quantize three-dimensional gravity *on its own*. We are not attempting to compactify seven of the dimensions of string theory to arrive at an effective theory containing 3D gravity. Such constructions are indeed possible, but all known models of this sort contain other fields than the metric itself. It is not at all clear whether our goal is possible to reach.

Actually both a consistent quantization of three-dimensional gravity and a no-go theorem would be very interesting results. The first would provide an excellent playground for asking questions about quantum gravity, and in particular about black holes. The second would be exciting news for string theory: as soon as gravity gets complicated enough, strings are needed to quantize it.

7.1 (2+1)-dimensional gravity as of 2007

In this section we review the developments in three-dimensional gravity up until E. Witten reignited the interest in the field with his talk at Strings 2007 and the corresponding paper [Wit07]. For the interested reader a nice review containing more information is the one by S. Carlip [Car05a].

Let us first make a general comment on gravitational actions in the presence of asymptotic boundaries. Gravity has an interesting feature when it comes to the relation between the action and the equations of motion. For most solutions the variation of the bulk action is *not zero* even if the fields satisfy the equations of motion. The reason for this is that the partial integration used in obtaining the equations produces boundary terms that do not vanish. To remedy this, boundary terms have to be added to the action. It is remarkable that to have a well-defined variational principle, e.g. for a black hole solution, puts restrictions on the form of the action infinitely far away. The boundary terms we encounter in paper V are the Gibbons–Hawking–York term proportional to the extrinsic curvature, and a boundary cosmological constant.

Let us now turn to the bulk action in three dimensions. We consider three natural terms containing just the metric and describe their corresponding dynamics.

7.1.1 Einstein–Hilbert term

The simplest term is the standard Einstein–Hilbert term

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int d^3x\,\sqrt{g}\,R, \tag{7.1}$$

yielding the equations of motion

$$R_{\mu\nu} = 0. \tag{7.2}$$

In three dimensions the Riemann curvature tensor has equally many independent components as the Ricci tensor. This means that Eq. (7.2) fixes the geometry completely. More specifically it implies

$$R^{\mu}{}_{\nu\kappa\lambda} = 0, \tag{7.3}$$

meaning that the geometry is flat. This sounds like a fairly boring theory, but there are some interesting results. S. Deser, R. Jackiw and G. 't Hooft studied the dynamics of point sources [DJtH84] and S. Carlip [Car93, Car05b] showed that on a manifold of nontrivial topology the gluing of flat Minkowski patches need not be unique. However, to be interesting as an analog system for four-dimensional gravity, this theory is too simple.

7.1.2 Cosmological constant

Surprisingly and interestingly, the inclusion of a cosmological constant Λ opens up a world of possibilities. The action is

$$S = S_{\mathrm{EH}} + \Lambda S_{\mathrm{cc}} \tag{7.4}$$

with

$$S_{\mathrm{cc}} = -\frac{2}{16\pi G}\int d^3x\,\sqrt{g}. \tag{7.5}$$

One of the earliest treatments of this model was done by S. Deser and R. Jackiw who explored the corresponding many-particle solutions [DJ84].

We will solely be concerned with $\Lambda < 0$, and therefore we choose the parametrization $\Lambda = -1/\ell^2$. The equations of motion read

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \left[\frac{R}{2} + \frac{1}{\ell^2}\right] g_{\mu\nu} = 0. \tag{7.6}$$

Note that we have included the cosmological constant term in defining the "cosmological" Einstein tensor $\mathcal{G}_{\mu\nu}$. Einstein's equations still determine the local geometry completely, but now imply (trace (7.6)) a constant scalar curvature $R = 6\Lambda$. The space is locally three-dimensional anti-de Sitter space, AdS_3. That the local geometry is fixed means that there are no propagating degrees of freedom, i.e., no gravitons.

The natural ground sate of the theory is global AdS_3. In papers V, VI and VII we meet this space in two different coordinate systems. In global coordinates the metric is

$$ds^2 = \ell^2\big(- \cosh^2\!\rho \, d\tau^2 + \sinh^2\!\rho \, d\phi^2 + d\rho^2\big). \tag{7.7}$$

Here $\rho \in [0, \infty)$ is a radial coordinate, $\phi \in [0, 2\pi)$ is the corresponding polar angle and $\tau \in \mathbb{R}$ is timelike. This coordinate system has the great virtue that it describes the entire AdS_3 space. The asymptotic boundary, corresponding to $\rho = \infty$ is parameterized by ϕ and τ. We often use light-cone coordinates $u, v = \tau \pm \phi$.

Another, often technically simpler coordinate system is the Poincaré patch

$$ds^2 = \ell^2 \left(\frac{dx^+ \, dx^- + dy^2}{y^2}\right) , \tag{7.8}$$

Here $y \in [0, \infty)$ and $x^{\pm} \in \mathbb{R}$ are light-cone coordinates. This patch covers half of global AdS, and the exact description of the boundary in these coordinates is very complicated[1].

AdS_3 has an $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ isometry group. The explicit generators can be found, e.g., in paper V.

7.1.2.1 The BTZ black hole

Given that any geometry solving the equations of motion is locally AdS, it came as a great surprise that there are black hole solutions. These were found by M. Bañados, C. Teitelboim and J. Zanelli in Ref. [BTZ92] and are called BTZ black holes after their discoverers.

BTZ black holes are very similar to ordinary four-dimensional Kerr black holes. The solutions are parameterized by two real parameters m and j denoting the mass and angular momentum, respectively. They are related to the

[1]For the complete map between local and global coordinates we refer to Ref. [BB07] by C.A. Bayona and N.R.F. Braga.

inner r_- and outer r_+ horizon radii as

$$m = \frac{r_+^2 + r_-^2}{8G\ell^2} \qquad j = \frac{r_+ r_-}{4G\ell}. \tag{7.9}$$

Defining the functions

$$L = \frac{(r_+ + r_-)^2}{16G\ell} \qquad \bar{L} = \frac{(r_+ - r_-)^2}{16G\ell} \tag{7.10}$$

the corresponding black hole metrics are

$$ds^2 = -4G\ell(Ldu^2 + \bar{L}dv^2) - (\ell^2 e^{2\tilde{\rho}} + 16G^2 L\bar{L}e^{-2\tilde{\rho}})dudv + \ell^2 d\tilde{\rho}^2. \tag{7.11}$$

The coordinate $\tilde{\rho}$ is shifted by a constant with respect to ρ:

$$\tilde{\rho} = \rho + \rho_0, \qquad e^{2\rho_0} = \frac{r_+^2 - r_-^2}{4\ell^2}. \tag{7.12}$$

If $|j| < \ell m$ (7.9) can be solved for real $r_\pm$, and (7.11) describes a black hole with two horizons. The Bekenstein–Hawking entropy is given by the area law

$$S_{\mathrm{BH}} = \frac{A}{4G} = \frac{2\pi r_+}{4G}. \tag{7.13}$$

Continuing formally to $m = -1/8G$ and taking $j = 0$ one obtains pure global AdS. Metrics corresponding to $-1/8G < m < 0$ are singular, and thus there is a mass gap in the spectrum of black hole states.

The geometry of the BTZ black holes has been studied in much detail by the BTZ trio in collaboration with M. Henneaux [BHTZ93]. Though not at all evident from the metric (7.11), the geometries are discrete quotients of global AdS_3. This is in accordance with the three-dimensional Einstein equations, requiring the space to be locally AdS. The singularity at $\rho = 0$ is a singularity in the causal structure rather than a curvature singularity.

The existence of the BTZ black holes makes a consistent quantization of three-dimensional gravity an extremely attractive scenario. If understood, three-dimensional quantum gravity would be an ideal gedanken laboratory for asking nontrivial questions about quantum black holes, and the answers are likely to be relevant to our four-dimensional world.

7.1.2.2 Asymptotic symmetries

As alluded to earlier, spacetimes with asymptotic boundaries require careful treatment in gravitational theories. For the action to have any extrema at all, boundary terms and boundary conditions must be specified. Another subtle feature that appears in gauge theories in general and in gravity in particular are the asymptotic symmetries.

It turns out that gauge transformations — i.e. diffeomorphisms — that do not vanish at the boundary should not be regarded as gauge transformations. Rather they are symmetries of the theory! One way to understand this is to consider the generator algebra in the canonical formalism. Here both symmetries and gauge transformations are generated by *constraints* that split up in two classes. First class constraints generate gauge transformations and second class constraints generate symmetries. The would-be-gauge transformations are generated by constraints that are first class in the bulk, but become second class at the boundary.

Three-dimensional gravity in asymptotically AdS spacetimes has a highly nontrivial structure in this regard. J.D. Brown and M. Henneaux [BH86] considered the canonical formulation of three-dimensional gravity, and found that imposing boundary conditions has interesting consequences. They considered metrics of the form

$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{AdS}} + h_{\mu\nu} \tag{7.14}$$

with h not necessarily small and imposed the *Brown–Henneaux* (BH) boundary conditions. In the Poincaré coordinates these read

$$\begin{pmatrix} h_{++} = \mathcal{O}(1) & h_{+-} = \mathcal{O}(1) & h_{+y} = \mathcal{O}(y) \\ & h_{--} = \mathcal{O}(1) & h_{-y} = \mathcal{O}(y) \\ & & h_{yy} = \mathcal{O}(1) \end{pmatrix}. \tag{7.15}$$

These conditions are preserved by a set of infinitesimal diffeomorphisms ζ^μ whose asymptotics fulfill

$$\begin{aligned} \zeta^+ &= \epsilon^+(x^+) - \frac{y^2}{2}\,\partial_-^2\epsilon^- + \mathcal{O}(y^4) \\ \zeta^- &= \epsilon^-(x^-) - \frac{y^2}{2}\,\partial_+^2\epsilon^+ + \mathcal{O}(y^4) \\ \zeta^y &= \frac{y}{2}\left(\partial_+\epsilon^+(x^+) + \partial_-\epsilon^-(x^-)\right) + \mathcal{O}(y^3). \end{aligned} \tag{7.16}$$

The leading order gauge-transformations parameterized by the two real one-variable functions $\epsilon^\pm$ do not vanish at the boundary. They generate symmetries rather than gauge transformations. The group of these transformations is known as the *asymptotic symmetry group* (ASG). Brown and Henneaux worked out the symmetry algebra and found that it is exactly the conformal algebra. The closed $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ sub-algebra is identified with the isometry algebra of pure AdS_3. Brown and Henneaux find that the Virasoro algebra has nonzero central charges:

$$c_L = c_R = \frac{3\ell}{2G}. \tag{7.17}$$

These discoveries were made several years before the AdS/CFT revolution, but in its light the appearance of the two-dimensional conformal group at the boundary of a three-dimensional gravitational theory is not surprising. Instead it is evidence for that any consistent quantization of three-dimensional gravity with negative cosmological constant is dual to a two-dimensional CFT.

Let us comment on the way these symmetries act on the black hole metrics (7.11). Using ϵ^-, it is possible to convert the constant L to an arbitrary function of u, and using ϵ^+ one can put $\bar{L}$ to an arbitrary function of v. So writing $L(u)$ and $\bar{L}(v)$ in (7.11), the metric still solves the equations of motion. Since the symmetries are not proper gauge transformations, these metrics represent different states. In a canonical treatment the L and $\bar{L}$ represent degrees of freedom that generate the conformal symmetry. Their Fourier components are the Virasoro generators of this theory.

As beautifully realized by A. Strominger [Str98] the stage is now set for a microscopic explanation for the entropy of the BTZ black holes. Namely, we have all the pieces needed to use the Cardy formula to count the states! The constant terms in L and $\bar{L}$ correspond as operators to L_0 and $\bar{L}_0$, and thus to the conformal weights $(h, \bar{h})$ of the black hole state. For a black hole with radii $r_\pm$

$$h = \frac{(r_+ + r_-)^2}{16G\ell} \qquad \bar{h} = \frac{(r_+ - r_-)^2}{16G\ell}. \tag{7.18}$$

As described in Chapter 2, the Cardy formula computes the logarithm of the density of states of a certain weight. Given the central charges $c_L = c_R = 3\ell/G$ we obtain

$$S_{\text{micro}} \sim 2\pi\sqrt{\frac{c_R h}{6}} + 2\pi\sqrt{\frac{c_L \bar{h}}{6}} = \frac{2\pi r_+}{4G} = \frac{A}{4G} = S_{\text{BH}}. \tag{7.19}$$

Note that the derivation uses no string theory, no extremality and no supersymmetry. This agreement between the microscopic and macroscopic entropies comprises striking evidence that gravity in three dimensions really is dual to a conformal field theory in two dimensions, and that the quantization of this field theory accounts for the black hole microstates.

7.1.3 Chern–Simons term

It is possible to deform pure gravity in a second way in three-dimensions. S. Deser, R. Jackiw and S. Templeton (DJT) showed that the inclusion of a gravitational Chern–Simons term has very interesting consequences [DJT82b, DJT82a]. The Chern–Simons term contains three derivatives of the metric:

$$S_{\text{CS}} = \frac{1}{32\pi G}\,\varepsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\nu\rho} + \frac{2}{3}\,\Gamma^\sigma{}_{\mu\tau}\Gamma^\tau{}_{\nu\rho}\right), \tag{7.20}$$

where $\Gamma^{\rho}{}_{\lambda\sigma}$ are the Christoffel symbols. Left on its own, this action would define a topological theory, but if combined with the Einstein–Hilbert term the resulting theory is anything but trivial. DJT studied a theory described by the action

$$S = -S_{\mathrm{EH}} + \frac{1}{\mu} S_{\mathrm{CS}}. \tag{7.21}$$

We shall comment on the sign in front of the Einstein–Hilbert term shortly. The equations of motion are

$$-\,G_{\mu\nu} + \frac{1}{\mu}\,C_{\mu\nu} = 0\,, \tag{7.22}$$

where

$$C_{\mu\nu} = \varepsilon_{\mu}{}^{\alpha\beta}\,\nabla_{\alpha}\left(R_{\beta\nu} - \frac{1}{4} g_{\beta\nu} R\right) \tag{7.23}$$

is known as the Cotton tensor. It is straightforward to check that $C_{\mu\nu}$ is traceless for any metric, and that it vanishes identically for Einstein metrics. Thus, including S_{CS} keeps all solutions to Einstein's equations (also if there is a cosmological term). There are however new solutions where both tensors $G_{\mu\nu}$ and $C_{\mu\nu}$ are nonzero but cancel.

DJT linearized the action and found that there is a propagating spin two degree of freedom with a Klein–Gordon action. The inclusion of the Chern–Simons term provides a graviton! This particle has mass proportional to μ, and mediates finite-range interactions. It is the presence of the graviton that forces us to choose the unconventional sign of the Einstein–Hilbert term; only with this choice the graviton has positive energy [DJT82b, DJT82a]. DJT named this theory *topologically massive gravity* (TMG).

Including all three terms yields cosmological topologically massive gravity (CTMG), first considered (in its supersymmetric version) by S. Deser [Des]. This theory exhibits both gravitons and black holes and constitutes the basic framework of papers V, VI and VII. Let us end this section by listing some of its properties.

First, note that the theory has two free dimensionless parameters ℓ/G and $\mu\ell$, so we are really dealing with a two-parameter family of theories. Second, the negative sign needed for positive graviton energy now becomes a serious issue[2]. The reason for this is that there are black holes in the theory, and their mass is bounded from below only if the "standard" positive sign is chosen! This signals an instability in the theory — an instability that is the object of study in papers V, VI and VII. In this thesis we define CTMG with the positive sign of the Einstein–Hilbert term, i.e., with negative energy gravitons

[2]This was actually not pointed out until Ref. [LSS08a].

but positive mass black holes:

$$S_{\mathrm{CTMG}} = S_{\mathrm{EH}} - \frac{1}{\ell^2} S_{\mathrm{cc}} + \frac{1}{\mu} S_{\mathrm{CS}}. \tag{7.24}$$

Third, since the Chern–Simons term treats left- and right-moving excitations differently, it shifts the central charges. P. Kraus and F. Larsen determined this shift by computing and comparing two anomalies: the non-diffeomorphism invariance in the bulk and the gravitational anomaly on the boundary [KL06]. Namely, the Chern–Simons term transforms under diffeomorphisms by a boundary term that exactly matches the corresponding anomaly in CFTs with unequal central charges. The result of the matching is

$$c_L = \frac{3\ell}{2G}\left(1 - \frac{1}{\mu\ell}\right) \qquad c_R = \frac{3\ell}{2G}\left(1 + \frac{1}{\mu\ell}\right). \tag{7.25}$$

Kraus and Larsen also determined the boundary stress tensor of the theory, as defined by J.D. Brown and J.W. York [BY93]. This object corresponds to the energy-momentum tensor of the dual CFT. It is obtained as follows. A three-dimensional metric satisfying the Einstein equations admits a *Fefferman–Graham* expansion [FG85]

$$ds^2 = \ell^2\, d\rho^2 + \left(e^{2\rho}\, \gamma_{ij}^{(0)} + \gamma_{ij}^{(2)} + \dots\right) dx^i dx^j\,. \tag{7.26}$$

Performing a variation of the metric $\gamma_{ij}^{(k)} \to \gamma_{ij}^{(k)} + \delta\gamma_{ij}^{(k)}$ changes the action by

$$\delta S_{\mathrm{CTMG}} = \frac{1}{2}\int_{\partial M} d^2x\, \sqrt{-\gamma^{(0)}}\, T^{ij}\, \delta\gamma_{ij}^{(0)}\,. \tag{7.27}$$

This is the defining equation for the boundary stress tensor T^{ij}. This tensor is a state dependent quantity and the spacelike boundary integrals of its components correspond to the mass M and angular momentum J of the state. For the black hole metrics in (7.11), Kraus's and Larsen's results are

$$\begin{aligned} M &= m + \frac{j}{\mu\ell^2} \\ J &= j + \frac{m}{\mu}. \end{aligned} \tag{7.28}$$

Interestingly, the Chern–Simons term shifts the mass and angular momentum of the spacetimes. Note that pure AdS_3 achieves a Casimir angular momentum:

$$J_{\mathrm{AdS}} = -\frac{1}{8G\mu}. \tag{7.29}$$

This concludes our summary of the status of three-dimensional gravity before Witten's 2007 paper. Let us now discuss some of the ideas presented there.

7.2 Witten and the Monster

In a beautiful and speculative paper [Wit07] Witten proposes that solving pure gravity with negative cosmological constant in three dimensions is equivalent to finding the dual CFT, and he sets out to find it. Let us follow in his footsteps. A first observation is that there is a theorem by A. B. Zamolodchikov [Zam86] stating that c_L and c_R of any continuous family of two-dimensional CFTs are constant. Since they in pure gravity are given by $c_L = c_R = 3G/2\ell$ this means that the theory makes sense at most at some discrete values of G/ℓ. To get a hint as to which are the correct values Witten uses that pure gravity is perturbatively dual to Chern–Simons theory. In fact, the action can be formulated as a sum of two Chern–Simons terms[3]. This correspondence was discovered by A. Achúcarro and P. Townsend [AT86] and expanded on by E. Witten [Wit88]. The central charges appear as the Chern–Simons levels, and those should be quantized. The suggested values are

$$c_L = 24k_L, \qquad c_R = 24k_R, \qquad k_{L/R} \in \mathbb{Z}. \tag{7.30}$$

Suggestively, as remarked in Chapter 2, these are precisely the values for which holomorphic factorization of the CFT is possible. Another circumstantial piece of evidence in this direction is the fact that the action is a sum of two Chern–Simons terms. Holomorphic factorization is one of the main assumptions in Ref. [Wit07].

With this assumption it suffices to study one of the holomorphic factors, described by $c = 24k$. The holomorphicity and the values of the central charges are already a lot of information, but they are not enough to uniquely determine the theory. At the lowest possible value of G/ℓ, i.e. at $c = 24$, A.N. Schellekens has argued [Sch93] that there are exactly 71 holomorphic CFTs. All save for one of these have some extended symmetry that would correspond to additional gauge fields in the bulk. The odd-one-out is a conformal field theory constructed by I.B. Frenkel, J. Lepowsky and A. Meurman [FLM84]. It has a large discrete symmetry group, in fact the largest of all sporadic finite groups. This is known as the Fischer–Griess monster group $\mathbb{M}$.

The states in the monster theory furnish representations of $\mathbb{M}$. The ground state is the AdS_3 vacuum with conformal weights $(0, 0)$. Conformal primaries of higher weights are identified with black hole states, i.e. the metrics (7.11) with constant L and $\bar{L}$. Counting these primaries reveals an agreement with the Bekenstein–Hawking entropy of the BTZ black holes. Descendants of the primaries correspond to metrics of the form (7.11) with $L(\bar{L})$ being functions of $u(v)$. We interpret these metrics as black holes surrounded by *boundary gravitons*. They are excitations that are pure gauge in the bulk but not on the

[3] While this statement remains true for CTMG the *dynamics* of CTMG is (contrary to pure gravity) *not* identical to that of Chern–Simons theory. For instance, in paper VI we use the Chern–Simons form of the action, but pay by introducing a Lagrange multiplier.

boundary. Descendants of the vacuum are just boundary gravitons without any black hole.

Witten goes on to make a proposal for a criterion that the CFTs dual to pure AdS gravity for $k > 1$ should fulfill. It states that there should be no primary fields of dimension lower than $k + 1$. Such CFTs are called *extremal* CFTs, and their partition functions are uniquely determined. The number of primaries again reproduces the Bekenstein–Hawking entropy.

The construction described above is a beautiful suggestion for a microscopic unitary theory describing black hole microstates. But it is not without problems. Firstly, there are *no known examples* of extremal CFTs at $k > 1$. This is despite considerable efforts to find them. Also, A. Maloney and E. Witten [MW07] actually computed all known contributions to the partition function of AdS gravity. The result does not factorize holomorphically, posing a serious problem for the scenario in [Wit07]. Possible resolutions suggested by Maloney and Witten are unknown states in the spectrum and the inclusion of complex geometries.

On the other hand, one can argue that the construction of Witten simply is too beautiful to be wrong. W. Li, W. Song and A. Strominger have not lost hope in this picture. We turn now to their proposal of *chiral gravity* in three-dimensions [LSS08a], and the wave of papers that followed in its wake.

7.3 Chiral gravity and log gravity

Li, Song and Strominger (LSS) aim in their January 2008 paper Ref. [LSS08a] to construct a gravitational dual of a holomorphic CFT. Their proposal is to consider CTMG with action (7.24) at a special tuning of one of the free parameters of the theory: $\mu\ell = 1$. At this point[4] (henceforth the *chiral* point) remarkable things happen.

First and foremost, as is evident from (7.25) the left-moving central charge vanishes! LSS point out that if the dual theory is unitary, then $c_L = 0$ implies that the left-moving sector is trivial and thus that the theory is purely right-moving. Such a theory is not only holomorphically factorizable — it is holomorphic. Unitarity thus implies Witten's all-important assumption that seems to fail for pure gravity. Second, the BTZ black holes all fulfill $M\ell = J$. This can be interpreted as the statement that all black holes are right-moving. The inequality $\ell m \geq |j|$ also implies that all black holes have non-negative mass $M \geq 0$.

Promising though these facts seem, a pertinent question arises at this point. Because of the sign in front of the Einstein–Hilbert term chosen in (7.24) the massive gravitons should have negative energy. Linearizing the equations of motion in global coordinates, LSS argue that at the chiral point the massive

[4]Since we only restrict one of two free parameters it is really a line, and with Witten's quantization argument it is an infinite discrete set of points on this line.

graviton ψ^M actually becomes identical to the left-moving boundary graviton ψ^L and that its energy vanishes. Together with the fact that $c_L = 0$ this indicates that the left moving sector is really pure gauge. Thus LSS conjecture the existence of a theory with only non-negative excitations at the chiral point. Since the technicalities of the LSS argument are reviewed in some detail in paper V we omit them here. Note that if everything works out well, chiral gravity is an ideal candidate for quantizing three-dimensional gravity. It is holomorphic, contains black holes and has a conformal symmetry at the boundary.

The claim that the massive gravitons disappear ignited a quite intense debate. The rest of this chapter is devoted to summarizing these discussions, in parts represented by the papers V, VI and VII. We do not provide a complete list of references; a reader interested in all aspects of the story is referred to the literature.

Starting off the discussions was a paper by S. Carlip, S. Deser, A. Waldron and D. Wise (CDWW) [CDWW09] (see also the follow-up paper Ref. [CDWW08]). Working in the Poincaré patch, CDWW linearize the action rather than the equations of motion, and find a propagating bulk degree of freedom — a massive graviton.

This apparent discrepancy with the LSS conjecture was the inspiration for papers V and VI, which the reader is now encouraged to turn her or his attention to. In the former paper we use the LSS setting and construct the missing graviton solutions, thus resolving the paradox. The main observation is that even if the massive graviton becomes identical to the left-mover as $\mu\ell \to 1$:

$$\psi^M(\mu\ell) \to \psi^L, \tag{7.31}$$

this does not imply that one linearly independent solution disappears. Instead a new branch appears:

$$\psi^{\mathrm{new}} = \lim_{\mu\ell\to 1} \frac{\psi^M(\mu\ell) - \psi^L}{\mu\ell - 1}\,. \tag{7.32}$$

The mode ψ^{new} also has an infinite tower of descendants. This mode has a number of interesting properties. In particular it grows linearly in τ and ρ. Since ρ is the logarithm of the proper radius we call this a *logarithmic mode*. Despite the divergence its energy is finite, negative and time-independent. Furthermore, the variational principle is well-defined, including the boundary issues mentioned in the beginning of this chapter. Because of the divergent behavior, this is highly non-trivial. In fact, the result depends crucially on the specific form of ψ^{new}. Only because of non-trivial cancellations are the boundary quantities finite — any logarithmic behavior in ψ^{new} would not be allowed.

We also make an important observation regarding the matrix representations of the operators L_0 and $\bar{L}_0$. The $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ subalgebra of the Virasoro algebra is realized as the isometry algebra of AdS_3, and thus it is straightforward to analyze its action on bulk modes. For the new mode we obtain representations identical to those in logarithmic CFT (LCFT). We discussed them in Subsection 2.2.4. ψ^{new} is identified as the logarithmic partner of ψ^L. This immediately shows that the theory cannot be unitary and that ψ^{new} cannot be decomposed in L_0 and $\bar{L}_0$ eigenstates. These facts lead us to propose that CTMG at the chiral point is holographically dual to a *logarithmic* CFT! As pointed out to us by M. Gaberdiel, a logarithmic CFT can never be chiral.

The mode ψ^{new} is compatible with spacetime being asymptotically AdS. However, the linear growth in ρ means that it does not obey the Brown–Henneaux boundary conditions. This fact leaves an escape route open for the chiral gravity conjecture of LSS: if the restriction to Brown–Henneaux boundary conditions is a consistent truncation, this subsector of the theory might be unitary and chiral. This would map holographically to a unitary truncation of an LCFT. In this way the logarithmic modes resurrect the graviton in the LSS setting, but also provide a candidate exclusion mechanism through the BH boundary conditions.

As a complement to the above perturbative analysis, paper VI counts the number of degrees of freedom at the non-perturbative level. We make a classical canonical analysis, counting the number of first and second class constraints. We find one degree of freedom per point — the graviton. This analysis, however, does not take any specific boundary conditions into account. Our results were confirmed as a special case of a more general analysis performed by S. Carlip [Car08].

Continuing the debate on bulk modes fulfilling the Brown–Henneaux conditions LSS dispute [LSS08b] the results of CDWW. They argue that the CDWW modes blow up at points on the boundary of AdS_3 not included in the set $y = 0$. CDWW reply, in arXiv version 2 of [CDWW09] that their set of modes is complete, and that they can form wave-packets of compact support. Furthermore, G. Giribet, M. Kleban and M. Porrati (GKP) consider a descendant of ψ^{new} [GKP08]. They show that by a suitable diffeomorphism it is actually possible to transform this mode into a form that respects the BH conditions. They conclude that chiral gravity is not unitary. At this point the status of the spectrum of linearized chiral gravity is unclear.

Meanwhile, A. Strominger presents a proof of the triviality of the left-moving sector in Ref. [Str08]. The Brown–York procedure is used to compute the stress tensor for a perturbation $h_{\mu\nu}$ precisely as was done by Ref. [KL06], resulting in (using Poincaré coordinates)

$$\begin{aligned} T_{++} &= \left(1 + \frac{1}{\mu\ell}\right) \frac{1}{8\pi G\ell} h_{++} \\ T_{--} &= \left(1 - \frac{1}{\mu\ell}\right) \frac{1}{8\pi G\ell} h_{--} \\ T_{+-} &= 0. \end{aligned} \tag{7.33}$$

The generators $Q(\zeta)$ for the ASG can now be obtained as boundary integrals of these quantities. For a transformation of the form (7.16)

$$\begin{aligned} Q(\zeta) = &\left(1+\frac{1}{\mu\ell}\right)\frac{1}{8\pi G\ell}\int_{\partial\Sigma} dx^+\, h_{++}\epsilon^+ \\ &+\left(1-\frac{1}{\mu\ell}\right)\frac{1}{8\pi G\ell}\int_{\partial\Sigma} dx^-\, h_{--}\epsilon^-. \end{aligned} \tag{7.34}$$

We see that all left-moving charges go to zero as $\mu\ell \to 1$.

Although to some extent implicit in paper V, we were triggered by Ref. [Str08] and private communications with LSS, to make the boundary conditions required for the logarithmic mode ψ^{new} explicit. This we do in paper VII. Guided by the results of paper V, we show that the appropriate procedure is to only logarithmically weaken the conditions on h_{--} and h_{-y}: $h_{--} = \mathcal{O}(\log y)$ and $h_{-y} = \mathcal{O}(y \log y)$. We demonstrate that the asymptotic symmetry group for these relaxed boundary conditions is still the conformal group, and that the symmetry generators are finite.

These results were confirmed and much generalized by M. Henneaux, C. Martinez and R. Troncoso [HMT09]. This paper also points out a subtlety overlooked in paper VII. (This is also noted in Ref. [MSS09].) The Brown–York formalism is not directly applicable when the boundary conditions are weaker than BH, so our result for the generators of the ASG misses a finite term. This term makes the left-moving charge non-zero, and thus shows explicitly the LCFT inspired conclusion of paper V: the theory with log boundary conditions is not chiral.

The paper Ref. [MSS09] by A. Maloney, W. Song and A. Strominger represents the latest development in the chiral gravity saga, and appeared just days before this thesis was sent to press. As already noted Maloney, Song and Strominger extend the Brown–York analysis of Ref. [Str08] so that it applies also to weaker boundary conditions. It is confirmed that BH boundary conditions imply that the left-moving charges vanish. The GKP modes have non-vanishing left-moving charges, and this is shown to result from a logarithmic behavior of the second order perturbation. Therefore the fully corrected solution corresponding to the GKP modes does not fulfill BH boundary conditions. MSS go even further. Using the complete set of modes found by CDWW, they argue that any linearized solution that is not a solution to the linearized Einstein equations must violate the boundary conditions! Since the linearized Einstein solutions have positive energy, these results are very encouraging for the prospects of chiral gravity. However, a non-perturbative positive energy theorem would be desirable. MSS determine the partition function of chiral gravity by somewhat non-rigorous methods, and find that it is exactly the partition function of the extremal CTF with the appropriate central charge.

With log boundary conditions the theory (denoted log gravity) is, as already mentioned, not chiral and the existence of the LCFT dual proposed in papers

V and VII seems plausible. MSS argue that there is a consistent truncation to chiral gravity from log gravity. The truncation is achieved by restricting to the sector with vanishing left-moving charges. We should note, however, that D. Grumiller and I. Sachs find that gravitational three-point correlators between boundary gravitons and the logarithmic modes do not vanish [GS09] which appears to be incompatible with such a truncation

The LCFT interpretation opens up the possibility of a very interesting scenario [MSS09]. One can imagine that no *unitary* extremal CFTs exist for $k > 1$, explaining why they have not been found, but that *logarithmic* extremal CFTs do exist. Log gravity could then be dual to such a theory. If the truncation to chiral gravity is consistent, it could then correspond to a unitary subsector of this LCFT that is not a local CFT on its own. It is amusing to note that when we made sense of two-dimensional gravity in Chapter 3 we encountered a similar situation: a non-unitary CFT at $c = 0$ was truncated to yield a unitary subsector, not itself a CFT. The existence of log extremal CFTs is a pivotal open question at this point.

Another alternative is that extremal unitary CFTs really exist and that these are dual to chiral gravity. Both alternatives would provide good hope to find consistent theories of quantum gravity in three-dimensions, something that would be a major breakthrough.

In addition to this there is a long list of other interesting open questions related to three-dimensional CTMG, and one can be certain that the future will hold exciting results, continued debates and many surprises.

Epilogue

What a long strange trip it's been
The Grateful Dead

Even if the land we finally sat foot on turned out to be the gangplank aboard another ship, we have now reached the end of this particular journey. Let us briefly contemplate on what we learned and what it was all good for.

The goal of the natural sciences is to understand the various features of the world around us, whether it is the formation of a drop of rain, the anisotropy of the cosmic microwave background, or the complexity structure of Minesweeper. This curiosity is part of what makes us human.

However, the topics treated in the present text seem to lie far from the real world. Indeed, who will ever use the landscape maps that we drew in Chapter 6 to find a safe way home? Who lives in the worlds corresponding to the minima we found? Will anyone ever visit a supersymmetric D3-brane black hole? Does anybody sleep calmer just because flux compactifications are stable in the presence of charged black holes? And who cares if the dual monsters of three-dimensional gravity have logarithmic tails?

Admittedly, it is unlikely (although not impossible!) that our present results will have direct impact on how we understand the world. But this is the path one has to take to understand the questions of study here. Even if a particular scenario is not realized in our universe, its description can trigger new thoughts, discover weak points and suggest future directions and new, more relevant questions. The entire field of quantum gravity is currently in the middle of such a process. We do not know what the correct theory is or if its description will have any practical consequences. We *do* know, however that there is a theory to be found and we have very good indications that understanding it will revolutionize the way we view our world.

The work presented in this thesis is part of this process. By viewing string theory from different angles we gained physical insight in models of great complexity. For instance, quite general features of the landscape topography were revealed. In all thinkable models where monodromies occur, the main reasoning of Chapter 6 will be applicable. Furthermore, in finding the lost log in three-dimensional gravity we took an important step toward a toy-model suitable for analyzing the microstates of reasonably realistic quantum black holes. These insights have inspired to new investigations, and led to novel points of view. They made us and other researchers stop worrying about some problems and allowed us to look ahead.

It is a good feeling to understand something. Be it small or large, be it simple or complex. This is what research as well as teaching is all about. At the end of the day, the world should seem bigger and more exciting, but at the same time safer. My time as a PhD student and the work in this thesis certainly achieved this for me personally. It is my hope that they have contributed in this direction also to my colleagues and my students. And, dear reader, to you.

Acknowledgments

...and nothing else matters.
James Hetfield

During my time as a PhD student I benefited immensely from a great number of amazing people. This is the place to thank some of them.

To start off, I would like to express my heartfelt thanks to **Ulf D**, my thesis advisor. It has been a privilege working with and learning from you. Your enthusiasm, knowledge and intuition have been indispensable during these years. Thanks for keeping our projects on track, taking so much time for discussions and answering questions, and for always being available at the department. Thanks also for sharing your pedagogical skills during the quantum mechanics courses.

Magdalena, my PhD student twin, roommate, teaching colleague and co-author of four papers — thanks are really not enough! Thanks anyway for our collaborations and our innumerable and invaluable discussions, not least during the writing of our theses. My PhD time would not have been anything near as good without you.

For good friendship and much fun I thank **Martin O**. Your teaching me how to beat you at hockey is warmly acknowledged: I learned some and then I won some. Thanks also for our collaborations, for our numerous discussions on physics and life, for good parties, and for making me nervous during the EPT. My deep gratitude goes also to **Marcel** who has been my teacher, and co-author of two papers. Thank you for all I learned, for all the fun discussions, for many laughs, for easy money and for rigorous, highly appreciated scrutiny of this entire manuscript. The apple-cheeked **Diego** I acknowledge for our collaboration, for salsa moves I never learn and for correlated shavings.

My friend, teacher and collaborator (and boss *in spe*) **Daniel** I thank for reigniting in times of trouble my physics inspiration by his contagious enthusiasm, honest curiosity, knowledge and creativity. I have you to thank for much of what I have learned and experienced during my last PhD year, and thereby also for the fact that I am continuing in physics. Thanks also for proofreading parts of the thesis. **Roman**, with whom I co-authored one paper I thank for his knowledge, skill, interest and encouragement during my stay at MIT and afterward.

I cordially wish to thank my assistant advisors **Joe** and **Ulf L** for continuous support and encouragement. Your help with physics, language and good advice can hardly be overestimated. Joe in particular suggested (and paid for!) my visit to MIT, which had a profound impact on my work and on my whole life. **Kostya**, **Lisa** and **Maxim**, I cannot thank you enough for your patient answers to my many questions. You are an invaluable asset to any student at the department! **Antti** is acknowledged for interesting discussions on knots and boiled eggs.

All post-docs, PhD and diploma students at the department over the years are thanked for the nice working atmosphere. In particular I thank my roommate **Olof**, for all nice discussions both in and out of office — I learned so much from you even if it should have been the other way around. **Sergey** is acknowledged for learning Swedish as quickly as I forget Russian, **Joel** for being my hockey opponent and for many stimulating discussions, **Malin** for perilous punch and perfect parties, **Valentina Giangreco Marotta Puletti** for slapping me when I misbehave, **Johan K** for putting up with my snoring, **Martin L** and **Shuang-Wei** for many nice lunches and **Kasper** for keeping me company at the department during late nights and weekends. The short-haired but dynamic duo **Jonas** and **Andreas** I thank for all the fun times and Friday fika, **Kristel** for inspiration and **Keizo** for philosophical insights, ice-breakers and semi-Japanese enzyme deficiency. I thank post-docs **Jian**, **Laura** and **Thomas v R** for discussions, laughs, mind openers and advice. Och, så klart, **Thomas K** — något av det bästa som hänt Uppsala sen Sundsvall brann.

Rolf is warmly thanked for teaching me how to teach and for unsuccessfully trying to keep my butt down during push-ups, and **Staffan** for stimulating and enjoyable discussions on sometimes unexpected topics. The indispensable administrative personnel at the department I thank for making everything work and run smoothly.

Albeit many years ago, I still want to acknowledge my colleagues at Quantum Chemistry, **Erik**, **David** and **Johan Å** — I am forever thankful for your introducing me to research in physics. I also want to take the opportunity to send a thought of gratitude to all my students. You put things into perspective and brighten my eight o'clock mornings.

Last, but not least I thank my dear, dear family and my dear, dear friends without whom truly nothing would matter.

Summary in Swedish

Kvantgravitation ur klassiska synvinklar

Den här avhandlingen behandlar ett antal relativt skilda aspekter av kvantgravitation. Att formulera en teori som beskriver fenomen där både kvantmekanik och gravitation är viktiga har blivit något av en helig graal för den teoretiska fysiken. En sådan teori behövs för att kunna beskriva universums födelse och död, samt svarta hål. Den klassiska teorin för gravitation är *allmän relativitetsteori* och den beskriver tyngdkraften som rumtidens krökning. En kvantmekanisk beskrivning av gravitation kommer därför sannolikt att i grunden förändra vår syn på tid och rum.

Den i särklass mest utvecklade kandidaten för kvantgravitation är strängteori — en teori som ursprungligen utvecklades som en möjlig beskrivning av den den starka kraften. Enligt denna består alla partiklar av vibrerande strängar: vibrationsmönstret avgör vilken partikel det är frågan om. En av partiklarna, *gravitonen*, förmedlar den gravitationella växelverkan. Att strängteori innehåller denna partikel möjliggör en störningsteoretisk beskrivning av kvantgravitation.

Strängteori har inre motsägelser om rumtiden inte har tio dimensioner. Vår värld har bara fyra, så för att kunna beskriva verklig fysik måste teorin anpassas på något sätt. Ett framgångsrikt sätt att göra detta på är så kallade *kompaktifieringar*. Man antar här att rumtiden har en icke-kompakt fyrdimensionell del och en kompakt sexdimensionell del med mycket liten storlek. Om man betraktar strängteori på en sådan bakgrund slipper man motsägelser, men resultatet blir ändå en effektiv fyrdimensionell teori som innehåller gravitation!

Kompaktifieringar är därför ett mycket aktivt forskningsområde inom strängteorin. Ett viktig resultat från dessa studier är att det tycks finnas ett mycket stort antal fyrdimensionella teorier som kan uppkomma från strängteori. Hur teorin ser ut beror till exempel på den kompakta geometrin. Mängden av dessa fyrdimensionella teorier kallas strängteorins *landskap*. I denna avhandling studeras kompaktifieringar och landskapet ur olika synvinklar.

Ett annat sätt att komma till insikter om kvantgravitation är att försöka hitta enkla modeller som ändå innehåller relevanta drag. För att åstadkomma sådana kan man till exempel reducera antalet dimensioner. Gravitation i två rumtidsdimensioner kan kvantiseras, och leder till många intressanta resultat. I tre dimensioner är problemet svårt, men mycket tyder på att lösningen kommer att vara extremt intressant. Till exempel innehåller tredimensionell gravitation svarta hål som är mycket lika våra fyrdimensionella. Gravitation i tre dimensioner studeras i avhandlingens sista kapitel.

Forskningen som presenteras i denna avhandling har publicerats i sju artiklar, och jag skulle vilja ägna några meningar åt var och en av dessa.

I artikel I studerar vi en särskilt enkel gräns av en strängteorikompaktifiering, och undersöker ett svart hål i denna kompaktifiering. Genom att kombinera två tidigare resultat visar vi att man kan förstå det svarta hålets termodynamik i termer av ett mycket enklare system: en *matrismodell*.

Artikel II studerar ett svart hål i en liknande kompaktifiering. Den kompakta geometrin stabiliseras i detta fall av magnetiska flöden. Det visar sig att inte bara flödena utan också det svarta hålet påverkar den kompakta geometrin. Vi försöker ta reda på vilken effekt som är starkast. Om det svarta hålet har en betydande effekt skulle man kunna tänka sig att världen ser väldigt annorlunda ut nära ett svart hål. Kanske kan det svarta hålet till och med göra kompaktifieringen instabil! Våra resultat visar att detta inte sker. Flödena har en starkare effekt, och kompaktifieringen är stabil.

I de två artiklarna III and IV studerar vi strängteorins landskap. Vi väljer en stor klass av modeller och visar att ett typiskt drag är långa serier av likartade fyrdimensionella teorier. Vi förklarar att, och hur, det är möjligt att röra sig mellan dessa teorier. Detta kan ha haft betydelse i det tidiga universum, kanske till och med på ett sätt som lämnat avtryck som fortfarande finns kvar. De två artiklarna använder ett tekniskt komplicerat verktyg för att beskriva den kompakta geometrin: *torisk geometri*.

De tre sista artiklarna V, VI and VII behandlar en speciell sorts gravitation i tre dimensioner — *topologiskt massiv gravitation*. Denna teori innehåller både gravitoner och svarta hål. Gravitonerna är, i motsats till vanliga gravitoner, massiva. Denna massa är en fri parameter i teorin: man kan välja vilket värde som helst. Ett problem som denna teori har är att gravitonerna har negativ energi, vilket skulle innebära att teorin är instabil. Om man väljer ett speciellt värde på gravitonmassan händer emellertid två mycket intressanta saker.

För det första verkar det som att en hel sektor av teorin blir trivial och försvinner. Detta skulle göra teorin mycket enklare att kvantisera. För det andra verkar det som att gravitonerna med negativ energi försvinner! Om detta är sant, eller om man kan utesluta gravitonerna från teorins spektrum utan att motsägelser uppstår, skulle detta vara en utmärkt utgångspunkt för en teori för kvantgravitation.

I artikel V visar vi att gravitonerna verkligen finns, också för det speciella värdet på massan, men att de har ett annat asymptotiskt beteende. Artikel VI utför en kanonisk analys av systemet och bestämmer antalet frihetsgrader genom att räkna första och andra klassens villkor. Resultatet är att det finns en frihetsgrad per punkt — gravitonen. Den sista artikeln VII visar att det nya asymptotiska beteendet inte utgör något problem i sig.

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