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Equality of the critical inverse temperatures for the one- and two-sided Dyson models*

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Abstract

For the FK-Ising random-cluster representation of the Dyson long-range Ising model, we prove that the critical inverse temperatures on the half-line and on the full line coincide whenever the interaction exponent satisfies $1 < \alpha < 2$. The proof combines a finite-volume cluster estimate above the full-line critical point, a sprinkling comparison for random-cluster measures, and a block renormalization that reduces the half-line problem to the classical one-dimensional long-range site-bond percolation theorem of Newman and Schulman. The borderline case $\alpha = 2$ remains open for this argument.

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1 Introduction

The Dyson model is the one-dimensional Ising model with pair interaction $J(ij) = |i-j|^{-\alpha}$. For $1 < \alpha \leq 2$ the interaction is summable, but the model remains genuinely long range: the full-line model has a phase transition at low temperature, and the usual one-dimensional nearest-neighbour intuition does not apply.

There are two natural geometries. One may put the model on the full line $\mathbb{Z}$, or on the half-line $\mathbb{N}$. In the FK-Ising random-cluster representation, deleting the negative half-line can only make long connections harder. Consequently the half-line critical inverse temperature is at least as large as the full-line one. The purpose of this note is to prove the opposite inequality for $1 < \alpha < 2$, and hence equality of the two critical points.

One motivation comes from the transfer-operator approach to one-sided Dyson potentials. Johansson, Öberg and Pollicott [5] proved existence of continuous eigenfunctions in the subcritical regime $\beta < \beta_c^{\mathbb{Z}}$ (when $\alpha > 3/2$). Since a priori one only knows $\beta_c^{\mathbb{Z}} \leq \beta_c^{\mathbb{N}}$, the equality proved here shows, for $1 < \alpha < 2$, that this full-line threshold is the natural one-sided threshold as well. Previously, Johansson, Öberg and Pollicott [4] proved the quantitative comparison $\beta_c^{\mathbb{N}} \leq 8\beta_c^{\mathbb{Z}}$.

The proof is based on a renormalization idea from [1]. Fix $\gamma > \alpha/2$. Then above the full-line critical point, large finite intervals contain, with high probability, internal clusters of size at least N^γ. Domain monotonicity transfers this estimate to blocks of

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the half-line. After adding an independent long-range Bernoulli sprinkling, the large block clusters dominate a one-dimensional site-bond long-range percolation model. The theorem of Newman and Schulman for $1/r^\alpha$ percolation with $\alpha < 2$ then gives percolation on the half-line once the block scale is chosen large enough. A final sprinkling comparison converts this percolation statement back to the FK-Ising model.

The argument uses the strict inequality $2\gamma - \alpha > 0$ with $\gamma < 1$, and therefore leaves the borderline case $\alpha = 2$ open.

2 The critical inverse temperatures are equal when $1 < \alpha < 2$

Throughout we write $\mathbb{N} = \{0, 1, 2, \dots\}$. For a finite graph G, let $\mathcal{C}(G) = \{C_1, C_2, \dots\}$ be the set of connected components, let $\hat{C}(G)$ denote a component of maximum size, and let $\omega(G) = |\mathcal{C}(G)|$.

We consider graphs whose vertex sets are integer intervals $V \subset \mathbb{Z}$; $V^{(2)}$ denotes the set of unordered pairs of distinct vertices in V. The independent Bernoulli graph model $\eta(G; p)$ is parameterized by an edge probability function $p : V^{(2)} \to [0, 1]$. For an interaction strength $\mathrm{J}(ij) \geq 0$, the corresponding Dyson edge probabilities are

$$p_\beta(ij) = 1 - \exp(-\beta \mathrm{J}(ij)), \qquad i, j \in \mathbb{Z}. \tag{2.1}$$

Here

$$\mathrm{J}(ij) = |i - j|^{-\alpha}, \qquad i, j \in \mathbb{Z}, \tag{2.2}$$

where $\alpha > 1$ and $\beta \geq 0$. We write $p_\beta|_V$ for the restriction of p_β to $V^{(2)}$.

For a finite interval $V \subset \mathbb{Z}$, the FK-Ising random-cluster measure (see [3]) with free boundary condition is

$$\nu_\beta^V(G) = \mathsf{FK}_2(G; p_\beta|_V) \propto 2^{\omega(G)} \prod_{ij \in G} \left(e^{\beta \mathrm{J}(ij)} - 1\right).$$

For $t \geq 0$, we denote by η_t^V the independent Bernoulli graph measure on V with edge probabilities $1 - e^{-t\mathrm{J}(ij)}$. For $V = \mathbb{N}$ and $V = \mathbb{Z}$, the free-boundary measures ν_β^V are obtained as weak limits along finite interval exhaustions. We call $\nu_\beta^{\mathbb{Z}}$ the two-sided model and $\nu_\beta^{\mathbb{N}}$ the one-sided model.

Let Perc be the event that the graph contains an infinite connected component. For $V = \mathbb{N}$ or $\mathbb{Z}$, define

$$\beta_c^V = \sup\{\beta \geq 0 : \nu_\beta^V(\mathsf{Perc}) = 0\}.$$

This zero/non-zero threshold is the only property of the critical point used below.

Theorem 2.1. *We have $\beta_c^{\mathbb{N}} = \beta_c^{\mathbb{Z}}$ when $1 < \alpha < 2$.*

Problem 2.2. Prove, or disprove, that the same equality $\beta_c^{\mathbb{N}} = \beta_c^{\mathbb{Z}}$ holds in the borderline Dyson case $\alpha = 2$. We expect this to be true; the case is important and is not reached by the argument below, which uses $2\gamma - \alpha > 0$ with $\gamma < 1$.

2.1 Proof of Theorem 2.1

We use stochastic domination for graph measures ordered by inclusion. Thus $\nu' \prec \nu$ means that there is a coupling (H, G) with marginals ν' and ν such that $H \subset G$ almost surely. By Strassen's theorem [8], this is equivalent on finite edge sets to the inequality $\int f \, d\nu' \leq \int f \, d\nu$ for every bounded increasing function f. In infinite volume we use the corresponding cylinder-event formulation and pass to limits along finite exhaustions.

We shall use two standard domination facts. First, suppose that $V \subset W$ and condition on all edges in $W^{(2)} \setminus V^{(2)}$. By the domain Markov property, the conditional law of the edges in V is the FK-Ising measure on V with the boundary condition induced by the

exterior connections. Every such boundary condition dominates the free one. Hence, by positive association, if A is any positive-probability event determined by edges of $W^{(2)} \setminus V^{(2)}$, then

$$\nu_\beta^V \prec \mathcal{L}_{\nu_\beta^W}\big(G[V] \mid A\big), \tag{2.3}$$

where the conditional law on the right is viewed as a law on graphs on V. Indeed, this law is a mixture of the conditional measures just described. In particular, taking A to be the whole sample space shows that the free-boundary measure increases stochastically when the interval is enlarged. Second, for every interval V we have the sprinkling relation

$$\nu_\beta^V(G) \prec (\nu_\beta^V \otimes \eta_\delta^V)(G \cup H) \prec \nu_{\beta+2\delta}^V(\tilde{G}), \tag{2.4}$$

where $(G, H) \sim \nu_\beta^V \otimes \eta_\delta^V$, i.e. G and H are independent edge configurations on V, sampled respectively by ν_β^V and η_δ^V. We give the finite-volume proof of (2.4) in Appendix A; the infinite-volume statement follows by weak limits.

By stochastic dominance (2.3), we have $\beta_c^{\mathbb{N}} \geq \beta_c^{\mathbb{Z}}$. It remains to prove the reverse inequality. Fix $\beta_0 > \beta_c^{\mathbb{Z}}$. Our goal is to prove that $\nu_{\beta_0}^{\mathbb{N}}(\mathsf{Perc}) = 1$.

To this end choose $\delta > 0$ so small that $\beta := \beta_0 - 2\delta > \beta_c^{\mathbb{Z}}$, and let $G \sim \nu_\beta^{\mathbb{N}}$ and $H \sim \eta_\delta^{\mathbb{N}}$ be independent. By the sprinkling relation (2.4), it is enough to show that percolation occurs in $G \cup H$ with positive probability. Monotonicity in β then implies $\beta_0 \geq \beta_c^{\mathbb{N}}$.

We base the proof on the following lemma. For an integer interval I contained in the vertex set of G, let

$$\mathcal{C}_I(G) = \{C \cap I : C \in \mathcal{C}(G)\} \tag{2.5}$$

and let $\hat{C}_I(G)$ be an element in $\mathcal{C}_I(G)$ of maximum size. Thus paths are allowed to leave I if G has a larger vertex set. When only the induced graph on I is meant, we write $\hat{C}_I^{\mathrm{int}}(G) := \hat{C}(G[I])$, where as before $G[I]$ is the restriction of the graph G to the vertex set I.

Lemma 2.3. *Assume $1 < \alpha < 2$, $\beta > \beta_c^{\mathbb{Z}}$, and $1 > \gamma > \alpha/2$. For any $\epsilon > 0$ there exist arbitrarily large integers N with the following property: for any integer interval $I \subset \mathbb{N}$ of length N,*

$$\nu_\beta^I\big(\{G : |\hat{C}_I(G)| \geq N^\gamma\}\big) > 1 - \epsilon. \tag{2.6}$$

We postpone the proof of Lemma 2.3 to Section 2.2. The estimate (2.6) has the following immediate consequence, which is the form used in the proof of the theorem. By domain monotonicity (2.3), if $J \supset I$, then, for $G \sim \nu_\beta^J$, the same lower bound holds for the induced graph $G[I]$ even after conditioning on any positive-probability event $\mathcal{A}$ determined by edges outside $I^{(2)}$:

$$\nu_\beta^J\big(|\hat{C}_I^{\mathrm{int}}(G)| \geq N^\gamma \mid \mathcal{A}\big) > 1 - \epsilon.$$

Consequently, for a sequence of disjoint intervals, the indicators of the corresponding internal good-block events, by which we mean the events $|\hat{C}_I^{\mathrm{int}}(G)| \geq N^\gamma$, can be coupled from below by independent Bernoulli variables of parameter $1 - \epsilon$; this is the standard sequential coupling consequence of Strassen's theorem. The terminology is introduced formally in the proof of Lemma 2.3 below.

Given a finite or countable family $\mathcal{S} = \{S_j : j \in U\}$ of disjoint subsets of an integer interval J indexed by a set U, let $H \sim \eta_\delta^J$ and let $\tilde{H} \sim \mathcal{H}(\mathcal{S})$ denote the random graph on vertex set U where an edge $ij \in U^{(2)}$ is added precisely when H contains an edge connecting a vertex in S_i with a vertex in S_j. Then $\tilde{H}$ is a Bernoulli graph $\eta(\tilde{p})$ with edge

probabilities $\tilde{p}(ij)$, $i,j \in U$, given by

$$\begin{aligned}\tilde{p}(ij) &= 1 - \exp\left(-\delta \sum_{x\in S_i,\, y\in S_j} |x-y|^{-\alpha}\right) \\ &\geq 1 - \exp\left(-\delta\, |S_i||S_j|\, D_{ij}^{-\alpha}\right), \end{aligned} \tag{2.7}$$

where D_{ij} is an upper bound on $\operatorname{diam}(S_i \cup S_j)$ and $\operatorname{diam} A = \max\{|k-l| : k,l \in A\}$. In particular, when the union has finite diameter,

$$\tilde{p}(ij) \geq q := 1 - \exp\left(-\delta\, (\min_{j\in U} |S_j|)^2\, \operatorname{diam}\left(\bigcup_{j\in U} S_j\right)^{-\alpha}\right). \tag{2.8}$$

Assume the lemma to be true. Fix $\gamma \in (\alpha/2, 1)$. By the one-dimensional long-range site-bond percolation theorem of Newman and Schulman [7], there are numbers $\lambda_0 < 1$ and $B < \infty$ such that the independent site-bond model on $\mathbb{N}$ with site density $\lambda \geq \lambda_0$ and bond probabilities at least $1 - \exp(-B|i-j|^{-\alpha})$ percolates with positive probability. The half-line formulation follows from the usual two-sided statement by the same comparison argument that yields the factor-eight bound of [4]; the constant is immaterial here because the block parameter below can be made arbitrarily large. Choose $\epsilon > 0$ so small that $1 - \epsilon \geq \lambda_0$, and then choose from Lemma 2.3 an arbitrarily large N for which $\delta 2^{-\alpha} N^{2\gamma-\alpha} \geq B$.

Consider the partition

$$\mathcal{I} = \{I_k = [kN, (k+1)N) : k \in \mathbb{N}\}$$

of $\mathbb{N}$ into blocks of length N. Here and below $G \sim \nu_\beta^{\mathbb{N}}$ and $H \sim \eta_\delta^{\mathbb{N}}$ are independent. Let $C_k = \hat{C}_{I_k}^{\mathrm{int}}(G)$ be a largest component of the induced graph $G[I_k]$, let $\tilde{H} = \mathcal{H}(\{C_i : i \in \mathbb{N}\})$, and put

$$\mathcal{G} = \{k \in \mathbb{N} : |C_k| \geq N^\gamma\}.$$

For $i \neq j$, the sets C_i and C_j are contained in an interval of length at most $(|i-j|+1)N \leq 2|i-j|N$. Hence, by (2.7), if $i,j \in \mathcal{G}$, then

$$\tilde{p}(ij) \geq 1 - e^{-\beta_N |i-j|^{-\alpha}}, \tag{2.9}$$

where

$$\beta_N \geq \delta\, 2^{-\alpha} N^{2\gamma-\alpha} \longrightarrow \infty \qquad \text{as } N \to \infty. \tag{2.10}$$

We now compare this coarse graph with a site-bond long-range percolation model. The conditional version of the lemma stated above implies that $\mathcal{G}$ stochastically dominates an independent Bernoulli subset of $\mathbb{N}$ with retention parameter $\lambda = 1 - \epsilon$. Conditional on the retained vertices, the edges of $\tilde{H}$ dominate independent long-range bonds with probabilities bounded from below by (2.9). By the choice of ϵ and N, this dominating site-bond model is supercritical. Hence $\tilde{H}$ percolates with positive probability.

Every infinite component of $\tilde{H}$ gives an infinite component of $G \cup H$. Therefore $G \cup H$ percolates with positive probability, and the sprinkling relation (2.4) implies percolation with positive probability under $\nu_{\beta_0}^{\mathbb{N}}$. Since $\beta_0 > \beta_c^{\mathbb{Z}}$ was arbitrary, this proves $\beta_c^{\mathbb{N}} \leq \beta_c^{\mathbb{Z}}$, completing the proof of Theorem 2.1. □

Remark 2.4. The argument extends without essential change to the random-cluster model with any fixed cluster weight $q \geq 1$. More precisely, let

$$\nu_{\beta,q}^V(G) \propto q^{\omega(G)} \prod_{ij\in G} \left(e^{\beta \mathrm{J}(ij)} - 1\right)$$

be the free q-random-cluster measure and let $\beta_c^{V,q}$ be its percolation threshold. Free-boundary domain monotonicity, the uniqueness-and-density input in Lemma 2.3, and the block renormalization are unchanged. The second sprinkling domination becomes

$$(\nu_{\beta,q}^V \otimes \eta_\delta^V)(G \cup H) \prec \nu_{\beta+q\delta,q}^V.$$

The corresponding calculation is given at the end of Appendix A. Thus the same proof yields $\beta_c^{\mathbb{N},q} = \beta_c^{\mathbb{Z},q}$ for $1 < \alpha < 2$. We have stated the main result for $q = 2$ in order to keep the focus on the Dyson Ising model.

2.2 Proof of Lemma 2.3

Write $\beta_\star$ and $\gamma_\star$ for the parameters called β and γ in the statement of the lemma. Choose

$$\alpha/2 < \gamma_\star < \gamma < 1$$

and choose β_1 with $\beta_c^{\mathbb{Z}} < \beta_1 < \beta_\star$. Next choose $\delta_* > 0$ so small that $\beta_1 + \delta_* < \beta_\star$, and set, for $n \geq 2$,

$$\delta_n = \delta_* 2^{-n}, \qquad \beta_n = \beta_1 + 2\sum_{m=2}^{n} \delta_m.$$

Then $\beta_n < \beta_\star$ for every n. The role of the small increments δ_n is to make the sprinkling used at the nth renormalization scale independent of the clusters constructed at the previous scale.

We say that an interval I of length $M = |I|$ is *good* in a graph G if

$$|\hat{C}_I(G)| \geq |I|^\gamma. \tag{2.11}$$

This is an increasing event. For an integer $L \geq 0$ and a half-open integer interval $I = [a, b)$, let $I \pm L = [a - L, b + L)$. We shall construct a fixed integer $L \geq 0$ and an increasing sequence $M_1, M_2, \ldots$ such that, for every interval I with $|I| = M_n$,

$$\nu_{\beta_n}^{I \pm L}\big(I \text{ is good in } G\big) > 1 - \epsilon_n, \tag{2.12}$$

where $\epsilon_n < \epsilon$.

This implies the lemma. Indeed, choose n arbitrarily large and put $N = M_n + 2L$, with n so large that

$$M_n^\gamma \geq (M_n + 2L)^{\gamma_\star} = N^{\gamma_\star}.$$

If $V \subset \mathbb{N}$ is any interval of length N, let I be the interval obtained from V by removing L sites from each end, so that $V = I \pm L$. By (2.12), under $\nu_{\beta_n}^V$ the interval V contains a cluster with at least $N^{\gamma_\star}$ vertices with probability at least $1 - \epsilon_n$. Since $\beta_n < \beta_\star$ and the event is increasing, monotonicity in β gives the same lower bound under $\nu_{\beta_\star}^V$. As the values $N = M_n + 2L$ are arbitrarily large, this proves (2.6).

Choose a large integer c_0 and, for $n \geq 2$, define

$$c_n = \max\{n^{2/(1-\gamma)}, c_0\}, \qquad d_n = c_n^{\gamma-1}. \tag{2.13}$$

We take c_0 large enough that $d_n \leq 1/2$ and $c_n^\gamma - 1 \geq c_n^\gamma/2$ for all $n \geq 2$. Put

$$\epsilon_n = (1 + 3d_n)\epsilon_{n-1}, \qquad n \geq 2.$$

Since $d_n = O(n^{-2})$, the product $\prod_{n\geq 2}(1 + 3d_n)$ is finite. Choose

$$\epsilon_1 = \epsilon \Big/ \prod_{k=2}^{\infty}(1 + 3d_k),$$

so that $\epsilon_n < \epsilon$ for every n.

We now choose the base scale M_1 and the fixed buffer L. For $G \sim \nu^{\mathbb{Z}}_{\beta_1}$ with $\beta_1 > \beta^{\mathbb{Z}}_c$, the infinite cluster is almost surely unique and has positive density $\theta > 0$; this is the standard uniqueness and density statement for translation-invariant finite-energy percolation measures [6]. By ergodicity, for all sufficiently large M, an interval of length M contains at least $(\theta/2)M$ vertices of the infinite cluster with probability arbitrarily close to one. Choose M_1 so large that

$$M_1^\gamma \le (\theta/2)M_1.$$

After setting recursively $M_n = M_{n-1}c_n$, we further choose M_1 large enough that, for every $n \ge 2$,

$$2\exp\left(\log c_n - \frac{\delta_n}{2} c_n^{\gamma-\alpha} M_{n-1}^{2\gamma-\alpha}\right) \le \epsilon_{n-1} d_n/2. \tag{2.14}$$

The additional requirement (2.14) is possible because $2\gamma - \alpha > 0$, the factor $M_{n-1}^{2\gamma-\alpha}$ grows super-exponentially in n, and $\delta_n = \delta_* 2^{-n}$ decays only exponentially.

For a fixed translate of an interval I of length M_1, the event that I contains at least M_1^γ vertices of the infinite cluster is the increasing limit, as $R \to \infty$, of the local event that at least M_1^γ vertices of I are connected to each other by paths contained in $I \pm R$. Hence we may choose R so that this local event has probability at least $1 - 2\epsilon_1/3$ under $\nu^{\mathbb{Z}}_{\beta_1}$. By weak convergence of free finite-volume measures to $\nu^{\mathbb{Z}}_{\beta_1}$, choose $L \ge R$ so large that the same local event has probability at least $1 - \epsilon_1$ under $\nu^{I\pm L}_{\beta_1}$, uniformly in the translate of I. Thus (2.12) holds at scale $n = 1$.

We prove (2.12) by induction. Assume it holds at scale $n-1$ and let $I = [a, a + M_n)$ be an interval of length M_n. Subdivide I into the c_n intervals

$$J_j = [a + jM_{n-1}, a + (j+1)M_{n-1}), \qquad j = 0, \dots, c_n - 1.$$

Let $G \sim \nu^{I\pm L}_{\beta_{n-1}}$. All probabilities and expectations up to the sprinkling step are with respect to this law; we denote them by P and E, respectively. For each subinterval J_j, set $\hat{C}_j = \hat{C}_{J_j}(G)$. By the induction hypothesis and domain monotonicity, the law induced by $\nu^{I\pm L}_{\beta_{n-1}}$ on the larger box $J_j \pm L$ stochastically dominates $\nu^{J_j\pm L}_{\beta_{n-1}}$. Hence

$$\mathsf{P}\big(|\hat{C}_j| \ge M_{n-1}^\gamma\big) > 1 - \epsilon_{n-1} \qquad \text{for every } j.$$

Let

$$\mathcal{J} = \{0 \le j < c_n : |\hat{C}_j| \ge M_{n-1}^\gamma\}, \qquad K = |\mathcal{J}|.$$

Then

$$\mathsf{E}(c_n - K) \le \epsilon_{n-1} c_n.$$

Since $c_n^\gamma = d_n c_n$, Markov's inequality gives

$$\mathsf{P}(K \ge c_n^\gamma) \ge 1 - \frac{\epsilon_{n-1}}{1 - d_n} \ge 1 - \epsilon_{n-1}(1 + 2d_n), \tag{2.15}$$

where the last inequality follows from the choice of c_0.

Now add an independent sprinkling $H_n \sim \eta^{I\pm L}_{\delta_n}$; from this point until the end of the induction step, probabilities are taken under the product law $\nu^{I\pm L}_{\beta_{n-1}} \otimes \eta^{I\pm L}_{\delta_n}$. Conditional on G and on the selected subinterval clusters $\mathcal{S} = \{\hat{C}_j : j \in \mathcal{J}\}$, the graph induced by the edges of H_n between the sets in $\mathcal{S}$ has independent edges. On the event $K \ge c_n^\gamma$, every selected set has size at least M_{n-1}^γ and is contained in I, whose diameter is at most $c_n M_{n-1}$. Therefore, for every pair of selected subintervals, by (2.8),

$$\tilde{p}(ij) \ge q_n := 1 - \exp\big(-\delta_n c_n^{-\alpha} M_{n-1}^{2\gamma-\alpha}\big).$$

Thus, conditional on G and K, the auxiliary graph dominates an Erdős–Rényi graph $\mathbb{G}(K, q_n)$.

We use the standard connectivity estimate for Erdős–Rényi graphs that, when Kq_n is large, the probability of disconnection is bounded, up to a factor 2, by the probability of having an isolated vertex; see, for example, Bollobás [2, Chapter 7]. Hence

$$\begin{aligned}\mathsf{P}(\tilde{H}_n \text{ disconnected} \mid K \geq c_n^\gamma) &\leq 2c_n \exp\left(-\delta_n(c_n^\gamma - 1)c_n^{-\alpha} M_{n-1}^{2\gamma-\alpha}\right) \\ &\leq 2\exp\left(\log c_n - \frac{\delta_n}{2} c_n^{\gamma-\alpha} M_{n-1}^{2\gamma-\alpha}\right) \leq \epsilon_{n-1} d_n/2,\end{aligned}$$

where the last inequality is (2.14).

If $K \geq c_n^\gamma$ and $\tilde{H}_n$ is connected, then the corresponding good subinterval clusters are all connected in $G \cup H_n$. Hence $|\hat{C}_I(G \cup H_n)| \geq K M_{n-1}^\gamma \geq M_n^\gamma$. Combining this with (2.15) and the preceding disconnection bound gives

$$(\nu_{\beta_{n-1}}^{I\pm L} \otimes \eta_{\delta_n}^{I\pm L})\big(I \text{ is good in } G \cup H_n\big) > 1 - \epsilon_{n-1}(1 + 3d_n) = 1 - \epsilon_n.$$

By the sprinkling relation (2.4), the law of $G \cup H_n$ is stochastically dominated by $\nu_{\beta_{n-1}+2\delta_n}^{I\pm L} = \nu_{\beta_n}^{I\pm L}$. Since goodness is increasing, (2.12) holds at scale n. The induction is complete. □

A Proof of the sprinkling relation

We prove (2.4) in finite volume; the infinite-volume statement follows by taking weak limits along finite intervals. The first domination in (2.4) is immediate from the coupling $(G, G \cup H)$. We prove the second one.

Let $E = V^{(2)}$ and, for an edge $e = xy$, write

$$x_e(t) := e^{t\mathrm{J}(e)} - 1, \qquad r_e := e^{\delta \mathrm{J}(e)} - 1.$$

In finite volume the FK-Ising measure is

$$\nu_\beta^V(\omega) \propto 2^{k(\omega)} \prod_{e:\omega_e=1} x_e(\beta),$$

where $k(\omega)$ is the number of connected components of ω. Let λ be the law of $G \cup H$ when $(G, H) \sim \nu_\beta^V \otimes \eta_\delta^V$. For a fixed union configuration u, the event $G \cup H = u$ is the disjoint union, over $g \subset u$, of the events in which $G = g$ and $H \cup g = u$. Summing their probabilities gives, up to a normalising constant,

$$\lambda(u) \propto \prod_{e:u_e=1} r_e \sum_{g \subset u} 2^{k(g)} \prod_{e:g_e=1} \frac{x_e(\beta) e^{\delta \mathrm{J}(e)}}{r_e}.$$

Fix an edge $e = xy$ and a configuration u on $E \setminus \{e\}$. The ratio of the conditional weights of $u \cup \{e\}$ and u under λ is

$$R_e(u) = r_e\left(1 + \widetilde{x}_e \pi_u(x \leftrightarrow y) + \frac{\widetilde{x}_e}{2}\pi_u(x \not\leftrightarrow y)\right), \qquad \widetilde{x}_e := \frac{x_e(\beta) e^{\delta \mathrm{J}(e)}}{r_e},$$

where π_u is the FK-Ising measure on the subgraph determined by u with edge weights $\widetilde{x}_f$. Under $\nu_{\beta+2\delta}^V$, the corresponding conditional odds are $x_e(\beta + 2\delta)$ if x and y are connected in u, and $x_e(\beta + 2\delta)/2$ otherwise.

If x and y are connected in u, then

$$R_e(u) \leq r_e + x_e(\beta) e^{\delta \mathrm{J}(e)} = e^{(\beta+\delta)\mathrm{J}(e)} - 1 \leq e^{(\beta+2\delta)\mathrm{J}(e)} - 1.$$

If x and y are not connected in u, then $\pi_u(x \leftrightarrow y) = 0$ and

$$R_e(u) = r_e + \frac{1}{2}x_e(\beta)e^{\delta \mathrm{J}(e)} \le \frac{1}{2}x_e(\beta + 2\delta),$$

because, with $a = e^{\beta \mathrm{J}(e)}$ and $b = e^{\delta \mathrm{J}(e)}$,

$$(ab^2 - 1) - \{2(b-1) + (a-1)b\} = (b-1)(ab-1) \ge 0.$$

Thus every one-edge conditional probability under λ is bounded above by the corresponding one-edge conditional probability under $\nu^V_{\beta+2\delta}$. The FK-Ising single-edge conditional probabilities are increasing in the outside configuration. Together with the preceding estimates, this gives the standard single-edge comparison criterion for stochastic domination: for all configurations $u \le v$ on $E \setminus \{e\}$,

$$\lambda(\omega_e = 1 \mid \omega_{E\setminus\{e\}} = u) \le \nu^V_{\beta+2\delta}(\omega_e = 1 \mid \omega_{E\setminus\{e\}} = v).$$

Hence $\lambda \prec \nu^V_{\beta+2\delta}$. This proves the second domination in (2.4). □

For completeness, the same calculation also proves the q-random-cluster comparison used in Remark 2.4. Replace $2^{k(g)}$ by $q^{k(g)}$. Then

$$R_e(u) = r_e\left(1 + \widetilde{x}_e\pi_u(x \leftrightarrow y) + \frac{\widetilde{x}_e}{q}\pi_u(x \not\leftrightarrow y)\right).$$

If x and y are connected in u, then $R_e(u) \le x_e(\beta + \delta) \le x_e(\beta + q\delta)$. If they are not connected, then

$$R_e(u) = r_e + \frac{1}{q}x_e(\beta)e^{\delta \mathrm{J}(e)} \le \frac{1}{q}x_e(\beta + q\delta).$$

Indeed, with $a = e^{\beta \mathrm{J}(e)}$ and $b = e^{\delta \mathrm{J}(e)}$, the last inequality is equivalent to

$$ab^q - 1 - q(b-1) - (a-1)b \ge 0,$$

which follows from $a \ge 1$, $b \ge 1$, and $b^q - qb + q - 1 \ge 0$. The latter inequality is the tangent-line inequality for the convex function $b \mapsto b^q$ at $b = 1$.

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