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Spins and Giants

Fundamental Excitations in Weakly and Strongly Coupled ABJM Theory

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Abstract

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The discovery of integrability on both sides of the duality between planar N=4 super Yang-Mills theory and free type IIB string theory in $AdS_5 \times S^5$ has lead to great progress in our understanding of the AdS/CFT correspondence. Similar integrable structures also appear in the more recent three-dimensional superconformal N=6 Chern-Simons-matter theory constructed by Aharony, Bergman, Jafferis and Maldacena (ABJM), as well as in its gravity dual, type IIA string theory on $AdS_4 \times CP^3$. However, new interesting complications arise in the AdS_4/CFT_3 duality. In the conjectured all-loop Bethe equations by Gromov and Vieira the dispersion relation of the magnons has a non-trivial coupling dependence which is parametrized by a function that is only known to the leading order at weak and strong coupling. In the first part of this thesis I discuss our calculations of the next-to-leading correction to this function at weak coupling. We compute this function from four-loop Feynman diagrams in the $SU(2) \times SU(2)$ sector of the ABJM model. As a consistency check we have performed the calculation both in a component formalism and using superspace techniques. At strong coupling the fundamental excitations of the integrable model are the giant magnons. The topic of the second part of this thesis is the spectrum of these giant magnons in CP^3. Furthermore, I discuss our analyses of the finite-size corrections beyond the asymptotic Bethe ansatz. At weak coupling we have computed the leading four-loop wrapping diagrams in the ABJM model. At the strong coupling side of the duality I discuss our results for the exponentially suppressed finite-size corrections to the energy of giant magnons.

Keywords: String theory, AdS/CFT correspondence, Integrable field theory

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List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I J. A. Minahan, O. Ohlsson Sax and C. Sieg, *Magnon dispersion to four loops in the ABJM and ABJ models*, J. Phys. A43, 275402 (2010).[1]

II M. Leoni, A. Mauri, J. A. Minahan, O. Ohlsson Sax, A. Santambrogio, C. Sieg and G. Tartaglino-Mazzucchelli, *Superspace calculation of the four-loop spectrum in $\mathcal{N} = 6$ supersymmetric Chern-Simons theories*, JHEP 1012, 074 (2010).

III J. A. Minahan, O. Ohlsson Sax and C. Sieg, *A limit on the ABJ model*, `arXiv:1005.1786 [hep-th]`, (2010). To appear in the proceedings of *6th International Symposium on Quantum Theory and Symmetries (QTS6)*, Lexington, Kentucky, 20-25 Jul 2009.

IV T. Łukowski and O. Ohlsson Sax, *Finite size giant magnons in the $SU(2) \times SU(2)$ sector of $AdS_4 \times CP^3$*, JHEP 0812, 073 (2008).

V M. C. Abbott, I. Aniceto, and O. Ohlsson Sax, *Dyonic giant magnons in CP^3: Strings and curves at finite J*, Phys. Rev. D80, 026005 (2009).

The following papers by the author are referenced in the text but not reprinted in this thesis

[78] V. Giangreco Marotta Puletti, T. Klose and O. Ohlsson Sax, *Factorized world-sheet scattering in near-flat $AdS_5 \times S^5$*, Nucl. Phys. B792, 228 (2008).

[133] J. A. Minahan, and O. Ohlsson Sax, *Finite size effects for giant magnons on physical strings*, Nucl. Phys. B801, 97 (2008).

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[1] A Corrigendum for Paper I has been published in J. Phys. A44, 049801 (2011). This thesis contains the pre-print version `arXiv:0908.2463 [hep-th]` which incorporates the corrections.

[145] O. Ohlsson Sax, *Finite size giant magnons and interactions*, Acta Physica Polonica B 39, 1001 (2008).

[138] J. A. Minahan, O. Ohlsson Sax and C. Sieg, *Anomalous dimensions at four loops in $\mathcal{N} = 6$ superconformal Chern-Simons theories*, Nucl. Phys. B846, 542 (2011).

Contents

1. Introduction

There are many "revolutions" in theoretical physics.[1] Any text on the history of superstring theory[2] will mention two of these: The first superstring revolution was initiated in 1984 by the discovery of anomaly cancellation for type I superstrings. During the second superstring revolution, in the mid 1990s, it was realized that there are various dualities that relate the five different types of string theories[3] to each other, as well as to the newly conjectured eleven-dimensional M-theory. At first sight these theories look very different but they actually all describe different regimes of the same physical system. Furthermore, it was realized that superstring theories contain higher-dimensional non-perturbative objects – the D-branes and M-branes.

In the end of 1997, another revolution took place when Maldacena conjectured that the low-energy physics of a stack of d-dimensional D-branes or M-branes has two dual descriptions, either as a conformal quantum field theory (CFT) in d dimensions or as string theory or M-theory in a space-time containing $(d+1)$-dimensional anti-de Sitter space (AdS) [125]. The equivalence between these totally different physical systems is known as the AdS/CFT correspondence, and has been the subject of an enormous number of papers during the last decade.[4]

The canonical realization of the AdS/CFT correspondence is the duality between maximally supersymmetric four-dimensional Yang-Mills theory and string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$. This duality is referred to as the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence and gives two equivalent descriptions of the low-energy physics of a stack of N D3-branes. In more detail, the conformal field theory here

[1] At least according to the physics blogs.

[2] See for example the string theory text book by Becker, Becker, and Schwarz [35].

[3] The five different types of string theories are type IIA and type IIB superstrings, type I superstrings and two kinds of heterotic string theories with gauge group $\mathrm{SO}(32)$ and $\mathrm{E}_8 \times \mathrm{E}_8$, respectively.

[4] In fact Maldacena's original paper on AdS/CFT is now the secondly most cited high energy physics paper ever. At the time of writing the SPIRES database of high energy physics papers reports 7378 citations for Maldacena's paper. The only paper in the database with more citations is Steven Weinberg's 1967 paper on electro-weak interactions [168] with a citation count of 7387. Considering the number of citations these two papers have received in recent years it seems very likely that Maldacena's paper will pass the paper of Weinberg and conquer the top position in the near future.

is $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with gauge group $\mathrm{SU}(N)$ in four-dimensional Minkowski space, and the gravity dual describes type IIB strings propagating in a background of $\mathrm{AdS}_5 \times \mathrm{S}^5$.

During the last four years, *i.e.*, during my time as a PhD student, a number of papers have appeared on the online preprint archive that have sparked their own mini-revolutions. One such case was the construction by Jonathan Bagger, Neil Lambert and, independently, Andreas Gustavsson of a three-dimensional conformal field theory with $\mathcal{N} = 8$ supersymmetry, proposed as the world-volume action for two interacting M2-branes [25, 27, 26, 97]. Following these papers a vigorous search for a generalization to more than two branes took place.

M2 branes are three-dimensional objects embedded in an eleven-dimensional background. Hence the world-volume theory on a stack of such branes should be a three-dimensional gauge theory. In the low-energy limit, the theory should flow to a non-trivial fixed point. This theory will not be a Yang-Mills theory since the Yang-Mills coupling is dimensionful in three dimensions. A more promising start is Chern-Simons theory. The Chern-Simons coupling is automatically protected from any continuous quantum corrections due to the gauge symmetry, ensuring conformal invariance. However, we also expect the sought theory to be parity invariant, but the Chern-Simons term changes sign under a parity transformation.

The solution to this problem is hidden in the Bagger-Lambert-Gustavsson (BLG) construction. As first realized in [166, 33] we can obtain a parity invariant theory by considering a non-simple gauge group which is the product of two subgroups. The action contains two Chern-Simons terms, one for each factor in the gauge group, with a relative minus sign. Under a parity transformation the role of the two gauge fields will be exchanged. This allows us to write down a gauge invariant action that is invariant under both conformal transformations and parity. Such an action was constructed by Aharony, Bergman, Jafferis, and Maldacena (ABJM) [5]. This model is the primary subject of this thesis.

The ABJM model is a three-dimensional superconformal Chern-Simons-matter theory with gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ and Chern-Simons levels k and $-k$. In contrast to the BLG model, the ABJM model is not maximally supersymmetric, but preserves $\mathcal{N} = 6$ supersymmetries. In the special cases of $k = 1$ and $k = 2$ the supersymmetry is enhanced. In particular the theory obtained by also setting $N = 2$ is equivalent to the BLG model [5]. In this thesis I will mainly discuss the opposite limit where N and k are very large and their ratio $\bar{\lambda} = N/k$ finite. The parameter $\bar{\lambda}$ then plays the role of a coupling constant, and the field theory is weakly coupled when $\bar{\lambda}$ is small. Since the number of colors N is very large it can be analyzed perturbatively from planar Feynman diagrams.

According to the AdS/CFT correspondence at low energy there is an alternative description of a stack of M2-branes in terms of M-theory. If the branes live in an asymptotically flat eleven-dimensional space-time the corresponding background is $AdS_4 \times S^7$. However, the ABJM model describes M2-branes in a $\mathbb{Z}_k$ oribifold. On the gravity side we therefore need to consider the orbifold background $AdS_4 \times S^7/\mathbb{Z}_k$. For large values of k the action of the orbifold shrinks a circle in the seven-sphere, and the M-theory dual reduces to type IIA string theory in a ten-dimensional space-time $AdS_4 \times \mathbb{CP}^3$.

The AdS/CFT correspondence give us a relation between the parameters N and k of the gauge theory, and the string coupling constant and curvature of the background in the dual string theory. In the planar limit the string coupling goes to zero and the strings propagate freely in $AdS_4 \times \mathbb{CP}^3$. For large values of $\bar{\lambda}$ the background is weakly curved and the world-sheet sigma model of the string theory can be analyzed using perturbation theory.

Note that the perturbative regimes of the gauge theory and the dual string theory correspond the coupling $\bar{\lambda}$ being small and large, respectively. Hence the AdS_4/CFT_3 correspondence is a weak–strong coupling duality. From the perturbative string theory sigma-model we obtain strong coupling results in the gauge theory, and from the weakly coupled field theory we learn something about the spectrum of string theory in a strongly curved background. This is a general feature of AdS/CFT and means that the correspondence provides us with a very powerful as a calculational tool. On the other hand it makes it hard to prove the duality, since there is no region in the parameter space of $\bar{\lambda}$ where both the gauge theory and the sigma-model are weakly coupled.

AdS/CFT and integrability

The AdS/CFT correspondence gives us a dictionary relating various observables on the gauge theory and gravity sides [96, 169]. A central class of observables in a conformal field theory is the correlation functions of local gauge invariant operators. In the planar limit the most important correlation functions are the two-point functions. These are highly constrained by conformal symmetry and depend on the coupling constant of the theory only via the dimensions of the local operators. At the classical level the dimension of an operator is easily obtained from the field content of the operator. However, in an interacting quantum field theory this dimension in general receives quantum corrections. Moreover these corrections introduce mixing among operators with equal classical quantum numbers. A primary step in understanding the conformal field theory is to analyse the spectrum of local operators and their dimensions.

The local operators of the planar gauge theory are dual to freely propagating strings in the curved AdS background. The AdS/CFT dictionary tells us that the dimension of an operator is to be identified with the energy of the corresponding string state. Hence the spectral problem of the field theory is mapped to the problem of determining the energy spectrum of strings propagating in AdS.

In recent years a lot of progress have been made in understanding the spectrum of AdS_5/CFT_4 on both strong and weak coupling. The basis for this progress is the appearance of *integrable structures* at both sides of the duality.[5] In [134] it was discovered that the one-loop spectrum of $\mathcal{N}=4$ SYM can be obtained from a particular integrable spin-chain, which can be solved using the Bethe ansatz. At strong coupling classical string theory in $AdS_5 \times S^5$ was shown to contain an infinite number of higher conserved charges [41]. Over the years these results have been generalized and refined. In [36] an all-loop Bethe ansatz for AdS_5/CFT_4 was written down, yielding a set of equations from which exact result for the spectrum in principle can be obtained.

Also in the AdS_4/CFT_3 correspondence the solution to the spectral problem becomes feasible in the planar limit due to integrability. In weakly coupled ABJM an operator can be interpreted as a state in the Hilbert space of an integrable spin-chain, and the corresponding dimension is obtained using a Bethe ansatz. The sigma-model of the dual string theory has also been shown to be invariant under higher preserved charges. The integrability aspects of AdS_4/CFT_3 will be discussed in much more detail in the rest of this thesis.

The fundamental excitations of an integrable spin-chain above a ferromagnetic ground state are commonly referred to as *magnons*. The magnons hop from site to site in the spin-chain, carrying some specific momentum and interacting through local interactions. The higher preserved charges strongly simplifies the dynamics of the magnons. In particular they ensure that the scattering of three or more excitations factorizes into consecutive pair-wise scattering of the magnons. A fundamental ingredient in understanding an integrable system is the two-particle S-matrix. By imposing periodic boundary conditions on the spin-chain we can obtain a set of equations, the Bethe equations, that give us the valid momentum configurations for a given set of magnons. Furthermore, to obtain the energy of a given configuration we need to know the dispersion relation of the magnons.

Both the two-particle S-matrix and magnon dispersion relation for the integrable models of AdS_5/CFT_4 and AdS_4/CFT_3 are determined by the symmetries of the models up to an undetermined function of the coupling con-

[5] There is no definite definition of integrability in a quantum mechanical system. In this thesis I will refer to a system as integrable if there exists a tower of higher momentum dependent charges which generate additional symmetries of the system. These charges allow us to obtain the (asymptotic) spectrum using a set of Bethe ansatz equations [46].

stants. In AdS_5/CFT_4 this function has turned out to be linear. However, in AdS_4/CFT_3 it is much more complicated. One of the main subjects of this thesis is the origin and form of this function $h^2(\bar{\lambda})$.

At strong coupling the dual string states of the scalar spin-chain excitations are the giant magnons. The second part of this thesis discusses the spectrum of giant magnons in $\mathbb{CP}^3$.

* * *

In writing this thesis my goal has been to give a stand-alone review of how integrability helps us solve the spectral problem of the planar ABJM model at weak and strong coupling. The results discussed closely parallells the previous developments in the AdS_5/CFT_4 correspondence. Since I have chosen to present these results as they appear in the ABJM model, rather than by comparison with $\mathcal{N} = 4$ super Yang-Mills or string theory in $AdS_5 \times S^5$, many references to these earlier works are not included. For a general introduction to integrability in AdS_5/CFT_4 see the recent review in [39].

I will end this introduction with an outline of the rest of the thesis. The ABJM model and the more general ABJ model are introduced in chapter 2. In chapter 3 I discuss how perturbation theory leads us to an integrable spin-chain describing the spectrum of operators. Chapter 4 contains a description of the calculation of the four-loop magnon dispersion relation presented in Paper I and Paper II.

The focus of chapter 5 and chapter 6 is on the strong coupling side of the AdS_4/CFT_3 duality. The type IIA string background is described in detail and I show how the classical integrability of the string action leads to an algebraic curve description of the string spectrum. Furthermore I discuss the spectrum of giant magnons in $\mathbb{CP}^3$ and their finite-size corrections, summarizing the results of Paper IV and Paper V.

The last chapter of the thesis summarizes the results of the previous chapters and contains a general discussion of the ABJ(M) magnon dispersion relation. In particular the all-loop expressions for $h(\bar{\lambda})$ proposed in Paper II and Paper III are discussed.

2. The ABJM and ABJ models

In this chapter the ABJM and ABJ models are introduced. Since the two models are very closely related, with ABJ being a generalization of ABJM to a more general gauge group, I will discuss them both at the same time and refer to them collectively as the ABJ(M) model.

2.1 The field content

The ABJ(M) model is a three-dimensional Chern-Simons matter theory, which is invariant under $\mathcal{N} = 6$ superconformal symmetry.[1] The gauge group of the model is the non-semisimple group $\mathrm{U}(N) \times \mathrm{U}(M)$, where $M = N$ in the case of ABJM.

The gauge sector of ABJ(M) consists of two Chern-Simons terms at opposite Chern-Simons levels k and $-k$. The two gauge fields, A^μ and $\hat{A}^\mu$, transform as connections under the $\mathrm{U}(N)$ and $\mathrm{U}(M)$ subgroups of the gauge group, respectively. In the matter sector we have four complex scalars Y^A as well as four Dirac fermions ψ_A, with $A = 1, \dots, 4$. The matter fields transform in the bifundamental representation $(\mathbf{N}, \bar{\mathbf{M}})$ and their conjugates in the anti-bifundamental representation $(\bar{\mathbf{N}}, \mathbf{M})$. The gauge structure of ABJ(M) is visualized in the quiver diagram in figure 2.1.

The ABJ(M) model is invariant under a global $\mathrm{SO}(6)_R \cong \mathrm{SU}(4)_R$ symmetry, referred to as R-symmetry. It takes the form of a flavor symmetry, under which the scalars Y^A and fermions ψ_A transform as $\mathbf{4}$ and $\bar{\mathbf{4}}$, respectively. For a further discussion of the symmetries of ABJ(M), and the full representation theory of the fields, see section 2.3 and section 2.4.

2.2 The action

There are various forms for the action of ABJ(M). Here I will write it down in terms of $\mathcal{N} = 2$ superspace, as well as in components. The full action first

[1] The ABJM and ABJ models are special cases of $\mathcal{N} = 4$ superconformal Chern-Simons theories constructed in [75, 74, 105].

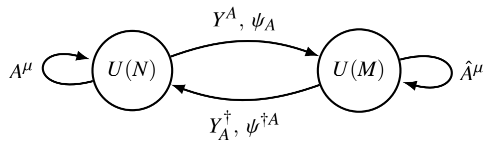


Figure 2.1: Quiver diagram of the ABJ(M) model. The arrows indicate the gauge group representations of the various fields, pointing from a fundamental to an anti-fundamental representation.

appeared in [42]. There are also known expressions for the action in $\mathcal{N} = 1$ [129], $\mathcal{N} = 3$ [54] and $\mathcal{N} = 6$ [56] superspace.

In the $\mathcal{N} = 2$ superspace form of the action, two of the supersymmetries and an $\mathrm{SU}(2) \times \mathrm{SU}(\dot{2})$ subgroup of the $\mathrm{SU}(4)_R$ flavor symmetry are manifest. The two gauge fields sit in vector superfields $\mathcal{V}$ and $\hat{\mathcal{V}}$, and the bifundamental matter is organized into four chiral superfields $\mathcal{Z}^A$ and $\mathcal{W}_{\dot{A}}$ which transform as two doublets under the global $\mathrm{SU}(2)$ and $\mathrm{SU}(\dot{2})$, respectively. The action is given by[2] [47, 48, 122]

$$\mathcal{S} = \frac{k}{4\pi}\left(\mathcal{S}_{\mathrm{CS}} + \mathcal{S}_{\mathrm{mat}} + \mathcal{S}_{\mathrm{pot}}\right) , \tag{2.1}$$

$$\begin{aligned}\mathcal{S}_{\mathrm{CS}} = \int d^3x d^4\theta \int_0^1 ds\, \mathrm{tr}\Big[&\mathcal{V}\bar{D}^\alpha(e^{-s\mathcal{V}}D_\alpha e^{s\mathcal{V}}) \\ &- \hat{\mathcal{V}}\bar{D}^\alpha(e^{-s\hat{\mathcal{V}}}D_\alpha e^{s\hat{\mathcal{V}}})\Big] ,\end{aligned} \tag{2.2}$$

$$\mathcal{S}_{\mathrm{mat}} = \int d^3x d^4\theta\, \mathrm{tr}\left(\bar{\mathcal{Z}}_A e^{\mathcal{V}} \mathcal{Z}^A e^{-\hat{\mathcal{V}}} + \bar{\mathcal{W}}^{\dot{A}} e^{\hat{\mathcal{V}}} \mathcal{W}_{\dot{A}} e^{-\mathcal{V}}\right) , \tag{2.3}$$

$$\mathcal{S}_{\mathrm{pot}} = \int d^3x d^2\theta W(\mathcal{Z},\mathcal{W}) + \frac{k}{4\pi}\int d^3x d^2\bar{\theta} \bar{W}(\bar{\mathcal{Z}},\bar{\mathcal{W}}) . \tag{2.4}$$

The superpotential $W(\mathcal{Z},\mathcal{W})$ is given by

$$W(\mathcal{Z},\mathcal{W}) = \frac{i}{2}\epsilon_{AB}\epsilon^{\dot{A}\dot{B}}\, \mathrm{tr}\, \mathcal{Z}^A\mathcal{W}_{\dot{A}}\mathcal{Z}^B\mathcal{W}_{\dot{B}} , \tag{2.5}$$

$$\bar{W}(\bar{\mathcal{Z}},\bar{\mathcal{W}}) = \frac{i}{2}\epsilon^{AB}\epsilon_{\dot{A}\dot{B}}\, \mathrm{tr}\, \bar{\mathcal{Z}}_A\bar{\mathcal{W}}^{\dot{A}}\bar{\mathcal{Z}}_B\bar{\mathcal{W}}^{\dot{B}} . \tag{2.6}$$

In the component form of the action, the $\mathrm{SU}(4)_R$ flavor symmetry is manifest. By expanding the superfields and integrating out the auxiliary fields in the $\mathcal{N} = 2$ action, we get, after a redefinition of the matter fields, the component action

$$\mathcal{S} = \frac{k}{4\pi}\int d^3x \left(\mathcal{L}_k - V_b - V_f\right) , \tag{2.7}$$

2 The superspace conventions follow those of [77]. Due to differences in conventions, the action here differs slightly from the one in [42].

where the kinetic term and potentials are given by

$$\mathcal{L}_k = \mathrm{tr}\Bigg[\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho - \hat{A}_\mu\partial_\nu\hat{A}_\rho - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\right) \\ - D_\mu Y_A^\dagger D^\mu Y_A + \psi^{\dagger A} i\not{D}\psi_A\Bigg], \tag{2.8}$$

$$V_b = -\frac{1}{12}\mathrm{tr}\Big[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger \\ - 6Y^A Y_A^\dagger Y^B Y_C^\dagger Y^C Y_B^\dagger + 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger\Big], \tag{2.9}$$

$$V_f = \frac{i}{2}\mathrm{tr}\Big[\left(Y_A^\dagger Y^B \psi^{\dagger C}\psi_D - \psi_D\psi^{\dagger C} Y^B Y_A^\dagger\right)\left(\delta_B^A\delta_C^D - 2\delta_C^A\delta_B^D\right) \\ + \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D - \epsilon_{ABCD} Y^A \psi^{\dagger B} Y^C \psi^{\dagger D}\Big]. \tag{2.10}$$

The covariant derivatives act as

$$D_\mu Y^A = \partial_\mu Y^A + iA_\mu Y^A - iY^A\hat{A}_\mu, \quad D_\mu Y_A^\dagger = \partial_\mu Y_A^\dagger + i\hat{A}_\mu Y_A^\dagger - iY_A^\dagger A_\mu. \tag{2.11}$$

In addition to the Chern-Simons terms and the kinetic terms for the matter fields, the action contains cubic gauge-matter interactions, quadratic interactions between the scalars and fermions and sextic scalar interaction terms.

The ABJM model contains two parameters: the rank of the gauge group, N, and the Chern-Simons level k. In ABJ there is an additional parameter, namely the difference in rank between the two factors of the gauge group, $N-M$. Without loss of generality, we can assume that $N-M \geq 0$.

Note that the Chern-Simons level k appears as an overall factor of the action, see (2.1) or (2.7). Hence, it plays the role of a coupling constant $g_{CS}^2 \equiv 1/k$. As an alternative we rewrite the action so that the kinetic terms take a conventional form, without any coupling constant, by rescaling the gauge and matter fields by a factor $\sqrt{4\pi/k}$. This facilitates the counting of factors of g_{CS} in perturbation theory, since the propagators now are independent of the coupling, while the cubic, quartic and sextic interaction terms are proportional to g_{CS}, g_{CS}^2 and g_{CS}^4, respectively.

While the complete action has been given above in both $\mathcal{N}=2$ superspace and components, we need to complement it with a gauge fixing term in order to use it in perturbation theory. The details of the gauge fixing will not be discussed in this thesis. The reader is referred to Paper I and Paper II, where gauge fixed actions are introduced in components and superspace, respectively.

2.3 Gauge symmetry

As mentioned above, the gauge symmetry of ABJ(M) is $\mathrm{U}(N)\times\mathrm{U}(M)$, with $M=N$ in the ABJM model. Since the action for each gauge field consists of a Chern-Simons term, gauge invariance requires the coupling constant k to be integer valued [62, 63]. In this section the origin of this quantization is discussed.

Under a gauge transformation by $(U,\hat{U})\in\mathrm{U}(N)\times\mathrm{U}(M)$, the scalars and gauge fields transform as

$$\begin{aligned} Y^A &\to U Y^A \hat{U}^\dagger\,, & A_\mu &\to U A_\mu U^\dagger - iU\partial_\mu U^\dagger\,,\\ Y_A^\dagger &\to \hat{U} Y_A^\dagger U^\dagger\,, & \hat{A}_\mu &\to \hat{U}\hat{A}_\mu\hat{U}^\dagger - i\hat{U}\partial_\mu\hat{U}^\dagger\,. \end{aligned} \tag{2.12}$$

We will now consider the transformation of the Chern-Simons action for A_μ,

$$\mathcal{S}_{CS}^A = \frac{k}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right)\,, \tag{2.13}$$

and restrict to the case where U is in an $\mathrm{SU}(2)$ subgroup of $\mathrm{U}(N)$. Under the transformation in (2.12), $\mathcal{S}_{CS}^A$ transforms as

$$\begin{aligned} \delta\mathcal{S}_{CS}^A =& \frac{k}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(i\partial_\mu\left(A_\nu U^\dagger\partial_\rho U\right)\right)\\ &-\frac{1}{3}\frac{k}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(U^\dagger\partial_\mu U\,U^\dagger\partial_\nu U\,U^\dagger\partial_\rho U\right)\,. \end{aligned} \tag{2.14}$$

We will only consider gauge transformations that approach the identity at infinity,

$$U(x)\to 1\,, \qquad \text{as } x\to\infty\,. \tag{2.15}$$

Since (2.14) is independent of the space-time metric, we can perform a Wick rotation and evaluate it in Euclidean space. With the above condition on the gauge transformation, we can also consider the Chern-Simons theory on S^3 instead of in flat space. It is then clear that the first term in (2.14) vanishes.

The second term in (2.14) is more subtle. Let us rewrite it as $-2\pi k\,w(U)$, where

$$w(U) = \frac{1}{24\pi^2}\int d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(U^\dagger\partial_\mu U\,U^\dagger\partial_\nu U\,U^\dagger\partial_\rho U\right)\,. \tag{2.16}$$

I will now argue in a few steps, that $w(U)$ is an integer for any gauge transformation $U(x)$.[3]

(1) Let us consider two gauge transformations $U(x)$ and $U'(x)$ that only differ infinitesimally,

$$U'(x) - U(x) = iT(x)U(x)\,. \tag{2.17}$$

[3] The arguments given here closely follow Sidney Coleman's discussion of instantons in four-dimensional Yang-Mills theory [60].

The values of w for these two transformations differ by a total derivative

$$w(U') - w(U) = \frac{3i}{24\pi^2} \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{tr}\left(\partial_\mu U\, U^\dagger \partial_\nu U\, U^\dagger \partial_\rho T\right) \tag{2.18}$$

$$= \frac{3i}{24\pi^2} \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{tr}\, \partial_\rho \left(\partial_\mu U\, U^\dagger \partial_\nu U\, U^\dagger T\right) . \tag{2.19}$$

More generally, $w(U) = w(U')$ for any two gauge transformations $U(x)$ and $U'(x)$ that can be continuously deformed into each other. Hence $w(U)$ only depends on the *homotopy class* of $U(x)$.

(2) A simple set of mappings from S^3 to $\mathrm{SU}(2)$ are given by the functions

$$u_n(x) = [u(x)]^n\,, \qquad u(x) = z_0(x) + iz_i(x)\sigma^i\,. \tag{2.20}$$

Here z_i are embedding coordinates of $\mathrm{S}^3 \subset \mathbb{R}^4$, such that $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1$. The map u_n gives a n-fold cover of $\mathrm{SU}(2)$ and the integer n is known as the *wrapping number* of u_n.[4]

(3) Any map $U(x)$ from S^3 to $\mathrm{SU}(2)$ is homotopic to one of the maps u_n for some wrapping number n. Hence we can evaluate $w(U)$ by choosing the corresponding u_n as a representative for the homotopy class.

(4) To calculate $w(u_1)$ we use spherical coordinates

$$w(u_1) = \frac{1}{24\pi^2} \int_0^\pi d\theta \int_0^\pi d\phi \int_0^{2\pi} d\psi\, \sin^2\theta \sin\phi = 1\,. \tag{2.21}$$

Hence $w(u_1)$ gives the wrapping number of the map u_1.

(5) We now consider two consecutive gauge transformations by U_1 and U_2. As noted above (see point (1)), continuous deformations of U_i leave $w(U_i)$ invariant. In particular, we can deform U_1 and U_2 to be equal to the identity on the southern and northern hemisphere, respectively. We then get

$$w(U_2 U_1) = w(U_2) + w(U_1)\,. \tag{2.22}$$

Using this additive property of $w(U)$, we can calculate $w(u_n)$ for any $n \neq 1$,

$$w(u_n) = n\,. \tag{2.23}$$

This shows that $w(u_n)$ calculates the wrapping number of the maps u_n.

The above arguments show that $w(U)$ is an integer for any mapping $U(x)$ from S^3 to $\mathrm{SU}(2)$. Returning to the case of the Chern-Simons action (2.13), we have now seen that we can write the result of the gauge transformation (2.12) as

$$\delta \mathcal{S}_{CS}^A = -2\pi k\, w(U)\,, \tag{2.24}$$

[4] Since the group $\mathrm{SU}(2)$ as a manifold is isomorphic to S^3, the map u_w gives a w-fold mapping of S^3 to itself.

where $w(U)$ is an integer. Hence, the action is *not* invariant under generic gauge transformations. In particular, under a *large* gauge transformation, *i.e.*, a transformation that is not continuously connected to the identity, the action is shifted by a constant. However, the central object in a quantum field theory is not the action, but the *path-integral*

$$\int \mathcal{D}A \exp\left(i\mathcal{S}^{A}_{CS}\right). \tag{2.25}$$

Since the action sits in the exponential, a shift of the action by an integer multiple of 2π leaves the path-integral invariant. This leads us to the conclusion that gauge invariance requires the Chern-Simons coupling k to be an *integer* [63, 62]

$$k \in \mathbb{Z}. \tag{2.26}$$

In the above derivation of the quantization of the Chern-Simons coupling we restricted the gauge transformation to an $\mathrm{SU}(2)$ subgroup of $\mathrm{U}(N)$. However, the conclusion directly generalizes to the full gauge group. In particular the homotopy groups $\pi_3(\mathrm{U}(N))$, $\pi_3(\mathrm{U}(2))$ and $\pi_3(\mathrm{SU}(2))$ are all isomorphic to $\mathbb{Z}$ [141].

The requirement that k is integer valued is very important, since it protects the coupling from being renormalized – the quantized coupling cannot vary continuously with the energy scale.[5] The above argument only involved the action of the gauge fields, but the $\mathcal{N}=6$ supersymmetry of ABJ(M) forces all couplings in the action to be the same. Hence gauge symmetry and supersymmetry together guaranties that the coupling constants of the model do not receive any quantum corrections.[6]

2.4 Global symmetries

There are 12 supercharges that preserve the ABJ(M) action. In three dimensions these combine into six real spinors, resulting in a total of $\mathcal{N}=6$ supersymmetries. These spinors transform in the vector representation **6** of the R-symmetry group $\mathrm{SO}(6)_R \cong \mathrm{SU}(4)_R$. The supersymmetry relates the different interaction terms of the action, and hence also the β-functions of the corresponding coupling constants. Hence, the gauge symmetry and supersymmetry

[5] In general the Chern-Simons coupling can be shifted by an integer at one-loop. In pure Chern-Simons theory, this shift takes the form $k \to k + c_v$, where c_v is the quadratic Casimir in the adjoint of the gauge group [57, 14, 22, 128]. However, the analysis in [112] showed that for Chern-Simons theories with at least $\mathcal{N}=2$ supersymmetry, such a shift does not occur.

[6] As shown in [75], there is a large class of exactly marginal Chern-Simons-matter theories with $\mathcal{N}=3$ supersymmetry. At weak coupling, even $\mathcal{N}=2$ is enough to protect the Lagrangian from quantum corrections.

of ABJ(M) together ensure that the model is governed by a single coupling constant, and that this constant does not receive any quantum corrections.

In addition to the gauge symmetry and supersymmetry, the classical action of ABJ(M) is invariant under Lorentz transformations and space-time translations, which make up the Poincaré group, as well as scale transformations. These symmetries combine into the three-dimensional conformal group, which gives the full bosonic space-time symmetry of ABJ(M).[7]

For a generic model with classical conformal invariance, the scale invariance, and hence the conformal invariance, is broken at the quantum level through the renormalization of the coupling constant – the running coupling induces a scale in the model, *e.g.*, the energy at which the coupling is of order one. However, in some models the coupling constant is protected from quantum corrections and the quantum model is still conformally invariant. The most well known case of such a finite theory is the maximally supersymmetric Yang-Mills theory in four dimensions, where the $\mathcal{N}=4$ symmetry protects the coupling from renormalization [86, 126, 53]. As we noted above, the gauge symmetry and supersymmetry of ABJ(M) together ensures that the β-function vanishes to all orders in perturbation theory. Hence ABJ(M) gives another example of a finite theory where the conformal symmetry is exact also at the quantum level.[8]

The conformal symmetry, supersymmetry, and *R*-symmetry combine into a larger symmetry group – the supergroup $\mathrm{OSp}(6|4)$. The structure of the $\mathfrak{osp}(6|4)$ superalgebra will be discussed in more detail in section 2.5. There is also a "baryonic" $\mathrm{U}(1)_b$ symmetry [42] under which the bifundamental fields carry charge $+1$, the anti-bifundamentals -1, while the adjoint fields are uncharged. The representations of the various ABJ(M) fields under the gauge and global symmetries are collected in table 2.1.

In addition to the continuous local and global symmetries discussed so far, the ABJM model is invariant under a parity symmetry. Neither the Chern-Simons terms, nor the quartic potential is invariant under simple three-dimensional parity transformation. However, in the case of ABJM, where the two factors of the gauge group have the same rank, we can restore the symmetry by combining the parity transformation by an exchange of the two factors of the gauge group, *i.e.*, we swap A^μ and $\hat{A}^\mu$, as well as the bifundamental and anti-bifundamental matter. The ABJM action is invariant under this combined transformation, which in the literature is referred to simply as *parity*,

[7] The Poincaré group and scale transformations do not generate the conformal group on an algebraic level. However, almost all known scale and Poincaré invariant models are also conformally invariant [69].

[8] Other examples of finite field theories include classes of four-dimensional Yang-Mills theories with $\mathcal{N}=2$ [106] and $\mathcal{N}=1$ [148, 109, 114, 149] supersymmetry as well as $\mathcal{N}=3$ Chern-Simons-matter theories in three dimensions [75].

Table 2.1: Field content of the ABJ(M) model. This table shows the representations carried by the fields in the component action of ABJ(M) under the $\mathrm{U}(N) \times \mathrm{U}(M)$ gauge symmetry, the $\mathrm{SU}(4)_R$ R-symmetry, the Lorentz group $\mathrm{SO}(2,1)$ as well as the two $\mathrm{U}(1)$ charges – the dimension Δ and the "baryonic" charge.

	$\mathrm{U}(N)$	$\mathrm{U}(M)$	$\mathrm{SU}(4)_R$	$\mathrm{SO}(2,1)$	$\mathrm{U}(1)_\Delta$	$\mathrm{U}(1)_b$
Y^A	$\mathbf{N}$	$\bar{\mathbf{M}}$	$\mathbf{4}$	$\mathbf{1}$	$\frac{1}{2}$	1
$Y^\dagger_A$	$\bar{\mathbf{N}}$	$\mathbf{M}$	$\bar{\mathbf{4}}$	$\mathbf{1}$	$\frac{1}{2}$	-1
ψ_A	$\mathbf{N}$	$\bar{\mathbf{M}}$	$\bar{\mathbf{4}}$	$\mathbf{2}$	1	1
$\psi^{\dagger A}$	$\bar{\mathbf{N}}$	$\mathbf{M}$	$\mathbf{4}$	$\mathbf{2}$	1	-1
A^μ	adj	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	1	0
$\hat{A}^\mu$	$\mathbf{1}$	adj	$\mathbf{1}$	$\mathbf{3}$	1	0

even though it has a structure that more resembles a combination of parity and *charge conjugation*.

In the ABJ model the two subgroups of the gauge group do not have the same rank, so if we exchange them, we do not get back to the original model. Hence ABJ breaks the parity symmetry of ABJM. As we will see in chapter 5, this breaking of parity appears as a non-perturbative effect on the string theory side of the duality, and hence does not effect the perturbative string spectrum. See also chapter 4 for a discussion of this parity symmetry in relation to the spin-chain Hamiltonian.

2.5 The superconformal algebra $\mathfrak{osp}(6|4)$

The global symmetries of ABJ(M) combine to the superalgebra $\mathfrak{osp}(6|4)$.[9] The bosonic subalgebra of $\mathfrak{osp}(6|4)$ is $\mathfrak{so}(6)_R \times \mathfrak{sp}(4) \cong \mathfrak{su}(4)_R \times \mathfrak{so}(3,2)$. The three-dimensional conformal algebra, $\mathfrak{so}(3,2)$, has ten generators. Of these, six belong to the Poincaré algebra, which contains the Lorentz group $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ with generators[10] $\mathfrak{L}_\alpha^{\ \beta}$ ($\mathfrak{L}_\gamma^{\ \gamma} = 0$), as well as three space-time translations $\mathfrak{P}_{\alpha\beta} = \mathfrak{P}_{\beta\alpha}$. The additional generators are the dilatation, or scaling, operator $\mathfrak{D}$, as well as the generators of special conformal transforma-

[9] $\mathfrak{osp}(6|4)$ corresponds to $\mathfrak{D}(3,2)$ in Kac [110] classification of classical Lie superalgebras. See also [72]. The symmetry algebra of ABJ(M) corresponds to a real form of this algebra.

[10] Lowercase greek indices α, β, γ and δ take the values $+$ and $-$, while uppercase roman indices take values $I, J, K, L = 1, \ldots, 4$. Symmetrization and anti-symmetrization of indices is defined with a weighting factor, *e.g.*, $\mathfrak{P}_{\{\alpha\beta\}} = \frac{1}{2}(\mathfrak{P}_{\alpha\beta} + \mathfrak{P}_{\beta\alpha})$ and $\mathfrak{Q}_\alpha^{[IJ]} = \frac{1}{2}(\mathfrak{Q}_\alpha^{IJ} - \mathfrak{Q}_\alpha^{JI})$.

tions $\mathfrak{K}^{\alpha\beta}=\mathfrak{K}^{\beta\alpha}$. These ten generators satisfy the algebra [137, 175]

$$
\begin{aligned}
\left[\mathfrak{L}_\alpha{}^\beta,\mathfrak{P}_{\gamma\delta}\right] &= +2\delta^\beta_{\{\gamma}\mathfrak{P}_{\delta\}\alpha}-\delta^\beta_\alpha\mathfrak{P}_{\gamma\delta}\,, & \left[\mathfrak{D},\mathfrak{P}_{\alpha\beta}\right] &= +\mathfrak{P}_{\alpha\beta}\,,\\
\left[\mathfrak{L}_\alpha{}^\beta,\mathfrak{L}_\gamma{}^\delta\right] &= \delta^\beta_\gamma\mathfrak{L}_\alpha{}^\delta-\delta^\delta_\alpha\mathfrak{L}_\gamma{}^\beta\,, & \left[\mathfrak{D},\mathfrak{L}_\alpha{}^\beta\right] &= 0\,,\\
\left[\mathfrak{L}_\alpha{}^\beta,\mathfrak{K}^{\gamma\delta}\right] &= -2\delta_\alpha^{\{\gamma}\mathfrak{K}^{\delta\}\beta}+\delta^\beta_\alpha\mathfrak{K}^{\gamma\delta}\,, & \left[\mathfrak{D},\mathfrak{K}^{\alpha\beta}\right] &= -\mathfrak{K}^{\alpha\beta}\,,\\
&\left[\mathfrak{K}^{\alpha\beta},\mathfrak{P}_{\gamma\delta}\right] = 4\delta^{\{\alpha}_{\{\gamma}\mathfrak{L}_{\delta\}}{}^{\beta\}}+4\delta^{\{\alpha}_{\{\gamma}\delta^{\beta\}}_{\delta\}}\mathfrak{D}\,.
\end{aligned}
\tag{2.27}
$$

In $\mathfrak{osp}(6|4)$, this algebra is extended by twelve supercharges, which combine into six three-dimensional Dirac spinors $\mathfrak{Q}^{IJ}_\alpha=-\mathfrak{Q}^{JI}_\alpha$ making up the $\mathcal{N}=6$ superalgebra

$$
\left\{\mathfrak{Q}^{IJ}_\alpha,\mathfrak{Q}^{KL}_\beta\right\}=-\epsilon^{IJKL}\mathfrak{P}_{\alpha\beta}\,. \tag{2.28}
$$

Under Lorentz and scale transformations, the supercharges transform as

$$
\left[\mathfrak{L}_\alpha{}^\beta,\mathfrak{Q}^{IJ}_\gamma\right]=\delta^\beta_\gamma\mathfrak{Q}^{IJ}_\alpha-\tfrac{1}{2}\delta^\beta_\alpha\mathfrak{Q}^{IJ}_\gamma\,,\quad \left[\mathfrak{D},\mathfrak{Q}^{IJ}_\alpha\right]=+\tfrac{1}{2}\mathfrak{Q}^{IJ}_\alpha\,. \tag{2.29}
$$

There are twelve additional odd charges $\mathfrak{S}^\alpha_{IJ}=-\mathfrak{S}^\alpha_{JI}$, the *superconformal charges*, which are generated in the commutator between the supercharges $\mathfrak{Q}^{IJ}_\alpha$ and the special conformal generators $\mathfrak{K}^{\alpha\beta}$,

$$
\left[\mathfrak{K}^{\alpha\beta},\mathfrak{Q}^{IJ}_\gamma\right]=+\epsilon^{IJKL}\delta^{\{\alpha}_\gamma\mathfrak{S}^{\beta\}}_{KL}\,. \tag{2.30}
$$

These charges satisfy

$$
\begin{aligned}
\left[\mathfrak{L}_\alpha{}^\beta,\mathfrak{S}^\gamma_{IJ}\right] &= -\delta^\gamma_\alpha\mathfrak{S}^\beta_{IJ}+\tfrac{1}{2}\delta^\beta_\alpha\mathfrak{S}^\gamma_{IJ}\,, & \left[\mathfrak{D},\mathfrak{S}^\alpha_{IJ}\right] &= -\tfrac{1}{2}\mathfrak{S}^\alpha_{IJ}\,,\\
\left\{\mathfrak{S}^\alpha_{IJ},\mathfrak{S}^\beta_{KL}\right\} &= -\epsilon_{IJKL}\mathfrak{K}^{\alpha\beta}\,, & \left[\mathfrak{P}_{\alpha\beta},\mathfrak{S}^\gamma_{IJ}\right] &= -\epsilon_{IJKL}\delta^\gamma_{\{\alpha}\mathfrak{Q}^{KL}_{\beta\}}\,.
\end{aligned}
\tag{2.31}
$$

The final set of generators, the *R*-symmetry generators $\mathfrak{R}^I{}_J$ ($\mathfrak{R}^I{}_I=0$), appear in the anti-commutator

$$
\left\{\mathfrak{Q}^{IJ}_\alpha,\mathfrak{S}^\beta_{KL}\right\}=4\delta^\beta_\alpha\delta^{[I}_{[K}\mathfrak{R}^{J]}{}_{L]}-2\delta^I_{[K}\delta^J_{L]}\mathfrak{L}_\alpha{}^\beta-2\delta^I_{[K}\delta^J_{L]}\delta^\beta_\alpha\mathfrak{D}\,. \tag{2.32}
$$

The charges $\mathfrak{R}^I{}_J$ generate the $\mathfrak{su}(4)_R$ subalgebra of $\mathfrak{osp}(6|4)$, under which the odd charges $\mathfrak{Q}^{IJ}_\alpha$ and $\mathfrak{S}^\alpha_{IJ}$ transform in the six-dimensional vector representation,

$$
\begin{aligned}
\left[\mathfrak{R}^I{}_J,\mathfrak{R}^K{}_L\right] &= \delta^K_J\mathfrak{R}^I{}_L-\delta^I_L\mathfrak{R}^K{}_J\,,\\
\left[\mathfrak{R}^I{}_J,\mathfrak{Q}^{KL}_\alpha\right] &= +2\delta^{[L}_J\mathfrak{Q}^{K]I}_\alpha-\tfrac{1}{2}\delta^I_J\mathfrak{Q}^{KL}_\alpha\,,\\
\left[\mathfrak{R}^I{}_J,\mathfrak{S}^\alpha_{KL}\right] &= -2\delta^I_{[L}\mathfrak{S}^\alpha_{K]J}+\tfrac{1}{2}\delta^I_J\mathfrak{S}^\alpha_{KL}\,.
\end{aligned}
\tag{2.33}
$$

2.5.1 Representations of $\mathfrak{osp}(6|4)$: The field multiplets

Since the superalgebra $\mathfrak{osp}(6|4)$ is non-compact, all unitary representations are infinite dimensional.[11] The relevant representations are of highest weight type. The generators can be split into the raising operators

$$\mathfrak{L}_+{}^-\,,\quad \mathfrak{K}^{\alpha\beta}\,,\quad \mathfrak{R}^I{}_J\,,\, I<J\,,\quad \mathfrak{S}^{\alpha}_{IJ}\,, \tag{2.34}$$

the lowering operators

$$\mathfrak{L}_-{}^+\,,\quad \mathfrak{P}_{\alpha\beta}\,,\quad \mathfrak{R}^I{}_J\,,\, I>J\,,\quad \mathfrak{Q}^{IJ}_{\alpha}\,, \tag{2.35}$$

and the Cartan generators

$$\mathfrak{R}^I{}_I\,,\quad \mathfrak{L}_\alpha{}^\alpha\,,\quad \mathfrak{D}\,. \tag{2.36}$$

Note that there are no summations over the repeated indices in (2.36). The tracelessness of $\mathfrak{L}$ and $\mathfrak{R}$ reduces the number of Cartan elements to five.

Representations are labelled by the eigenvalues $R^I{}_I$, $L_\alpha{}^\alpha$ and Δ of the highest weight state under the Cartan generators $\mathfrak{R}^I{}_I$, $\mathfrak{L}_\alpha{}^\alpha$ and $\mathfrak{D}$, respectively, or by the Dynkin labels

$$\begin{gathered} p_1 = R^1{}_1 - R^2{}_2\,,\quad q = R^2{}_2 - R^3{}_3\,,\quad p_2 = R^3{}_3 - R^4{}_4\,,\\ s = \tfrac{1}{2}(L_+{}^+ - L_-{}^-)\,. \end{gathered} \tag{2.37}$$

The eigenvalue Δ is called the dimension of a state. The corresponding Dynkin diagram for $\mathfrak{osp}(6|4)$ is shown in figure 2.2.[12] It is convenient to combine the Cartan generators of $SU(4)_r$ into the charges

$$\begin{gathered} \mathfrak{J} = \frac{1}{2}\left(\mathfrak{R}^1{}_1 - \mathfrak{R}^4{}_4\right)\,,\quad \mathfrak{Q} = \frac{1}{2}\left(\mathfrak{R}^2{}_2 - \mathfrak{R}^3{}_3\right)\,,\\ \mathfrak{J}_3 = \mathfrak{R}^1{}_1 - \mathfrak{R}^2{}_2 - \mathfrak{R}^3{}_3 + \mathfrak{R}^4{}_4\,. \end{gathered} \tag{2.38}$$

The eigenvalues of these charges are related to the Dynkin labels p_1, q and p_2 by

$$J = \frac{p_1+q+p_2}{2}\,,\quad Q = \frac{q}{2}\,,\quad J_3 = p_1 - p_2\,. \tag{2.39}$$

The highest weight state of a superconformal algebra is known as a *primary* state. By acting on the primary with the translations $\mathfrak{P}_{\alpha\beta}$ or the supercharges $\mathfrak{Q}^{IJ}_\alpha$ we can construct states in the same multiplet with a higher dimension Δ. Such states are *descendants* of the primary state.

[11] Except for the trivial representation, which consists only of the identity.

[12] The Dynkin diagram of a superalgebra is not unique, but depends on the choice of simple roots.

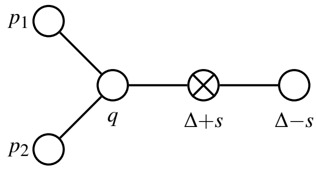


Figure 2.2: Dynkin diagram for $\mathfrak{osp}(6|4)$.

The states that makes up a representation can be grouped into submodules which are closed under the bosonic subgroup $\mathrm{SO}(3,2) \times \mathrm{SU}(4)_r$. The different submodules are related by the action of the supercharges $\mathfrak{Q}_\alpha^{IJ}$. A generic representation will consist of 2^{12} such submodules, since we can act with any combination of the 12 supercharges. Such a representation is called *typical*. For some particular representations the highest weight state is annihilated by one or more supercharge. In such *atypical* representation the number of submodules is reduced.

A particularly interesting case is a representation whose highest weight state is annihilated by half of the supercharges. Such a representation is known as a 1/2-BPS representation. As an example, let us consider a highest weight state annihilated by the six charges $\mathfrak{Q}_\alpha^{1K}$, $K = 2, 3, 4$. Being a state of highest weight it is also annihilated by the raising operators, and in particular by the superconformal charges $\mathfrak{S}_{IJ}^\alpha$. The algebra then implies that it is also killed by the anti-commutator

$$\left\{\mathfrak{Q}_\pm^{12}, \mathfrak{S}_{21}^\mp\right\} = \mathfrak{L}_\pm{}^\mp \,. \tag{2.40}$$

Hence this highest weight state is a Lorentz scalar. The action of the anti-commutator between $\mathfrak{Q}_+^{1K}$ and $\mathfrak{S}_{K1}^+$ on this state then reduces to

$$\left\{\mathfrak{Q}_+^{1K}, \mathfrak{S}_{K1}^+\right\} = \mathfrak{D} - \mathfrak{R}^1{}_1 - \mathfrak{R}^K{}_K\,, \quad K = 1,2,3\,. \tag{2.41}$$

For consistency of the algebra the Dynkin labels of the state has to be of the form

$$[p_1, q, p_2; \Delta, s] = [2J, 0, 0; J, 0]\,, \tag{2.42}$$

for some J. Equivalently, the R-charge J and the dimensionΔ satisfy the BPS condition[13]

$$\Delta = J\,. \tag{2.43}$$

Another set of important 1/2-BPS multiplets satisfying this condition have Dynkin labels $[0,0,2J;J,0]$, which corresponds to a highest weight state that is annihilated by $\mathfrak{Q}_\alpha^{12}$, $\mathfrak{Q}_\alpha^{13}$ and $\mathfrak{Q}_\alpha^{23}$.

[13] All unitary representations of $\mathrm{OSp}(6|4)$ satisfy the inequality [140]

$$\Delta \leq J\,.$$

A *BPS operator* has $\Delta = J$ and transform in an atypical representation.

By setting $J = 1/2$ we get the representations of the matter fields of ABJ(M). The bifundamental fields transform in the $[1,0,0;1/2,0]$ representation. The highest weight state is the scalar Y^1. By acting with the lowering operators we generate the module[14]

$$\mathcal{V} = \{D^n Y^I, D^n \psi_{I\alpha}\}, \tag{2.44}$$

consisting of the scalars Y^I and the fermions $\psi_{I\alpha}$ with any number of covariant derivatives D_μ acting on them. Similarly, the anti-bifundamental fields transform in $[0,0,1;1/2,0]$ representation and form the module

$$\bar{\mathcal{V}} = \{D^n Y_I^\dagger, D^n \psi_\alpha^{\dagger I}\}. \tag{2.45}$$

The content of the field multiplets $\mathcal{V}$ and $\bar{\mathcal{V}}$ is discussed in more detail in section 2.8, see in particular table 2.2.

To construct more general representations we will consider tensor products of $\mathcal{V}$ and $\bar{\mathcal{V}}$. In particular, the tensor product of L copies of each of these representations, $(\mathcal{V} \otimes \bar{\mathcal{V}})^{\otimes L}$, contains a representation with a highest weight state of the form $(Y^1 Y_4^\dagger)^L$. Since both Y^1 and $Y_4^\dagger$ are annihilated by $\mathfrak{Q}_\alpha^{12}$ and $\mathfrak{Q}_\alpha^{13}$, so is $(Y^1 Y_4^\dagger)^L$. By the same argument as above, the charges Δ and J of this state satisfies (2.43). Since such a state is preserved by four of the twelve supercharges it is a 1/3-BPS state. I will refer to it as a *chiral primary*.

At quantum level the dilatation operator and the supercharges receive quantum corrections. In particular, the dimension of a generic state depends on the coupling constants. This deviation from the classical, or bare, dimension is known as the *anomalous dimension*. The generators of $\mathrm{OSp}(6|4)$ change the dimension of a state in half-integer steps. Hence all the states in an irreducible representation have the same anomalous dimension.

For a chiral primary state the classical dimension equals the R-charge J. This relation between Δ and J must be true also when interactions are turned on. The reason is that the chiral primary state must be annihilated by the same supercharges at all values of the coupling. If this was not the case, the representation would contain extra states as soon as the coupling was non-zero. But the number of states of a particular dimension is a finite integer that can not vary continuously with the coupling. Moreover, since the R-symmetry group $\mathrm{SU}(4)_r$ is compact, the charge J can not be continuously deformed. This means that the dimension of a chiral primary is protected and cannot receive any quantum corrections.

[14] The module $\mathcal{V}$ of the bifundamental fields should not be confused with the gauge vector superfield in the $\mathcal{N} = 2$ action in section 2.2.

2.6 The planar limit

The ABJM model contains two parameters – the Chern-Simons level k and the rank of the gauge group N – both taking integer values. The level k appears as an overall factor in the action. Hence $g_{CS}^2 \equiv \frac{1}{k}$ plays the role of a coupling constant in ABJM, and for $k \gg 1$ the theory is weakly coupled and can be treated using perturbation theory. As we will see below, the perturbative expansion simplifies considerably when we in addition to small coupling also have a large number of colors. We can then define the 't Hooft coupling constant

$$\lambda \equiv g_{CS}^2 N = \frac{N}{k}, \tag{2.46}$$

and take the planar limit, or 't Hooft limit, in which [165]

$$N \to \infty,\ k \to \infty, \quad \text{with } \lambda \text{ fixed.} \tag{2.47}$$

Since both N and k are positive integers, λ is a rational number. However, in the 't Hooft limit we can treat λ as a continuous parameter.

Since the gauge group of the ABJ model has two factors of different rank, ABJ contains an extra parameter compared to ABJM, $N-M$. This allows us to introduce two 't Hooft couplings

$$\lambda = \frac{N}{k}, \qquad \hat{\lambda} = \frac{M}{k}, \tag{2.48}$$

and a corresponding 't Hooft limit in which k, N and M are sent to infinity in such a way that λ and $\hat{\lambda}$ remain finite.

Instead of using λ and $\hat{\lambda}$, it is convenient to introduce the couplings

$$\bar{\lambda} = \sqrt{\lambda\hat{\lambda}}, \quad \text{and} \quad \sigma = \frac{\lambda - \hat{\lambda}}{\bar{\lambda}}. \tag{2.49}$$

Then $\bar{\lambda}$ controls the overall perturbative expansion, while σ describes the deviation away from the ABJM case of $\sigma = 0$.

In [4] it was argued that the ABJ model is unitary only if $|N-M| \leq k$. In terms of the 't Hooft couplings this means that $|\lambda - \hat{\lambda}| \leq 1$. Moreover, there is an equivalence between models with gauge groups $\mathrm{U}(N)_k \times \mathrm{U}(M)_{-k}$ and $\mathrm{U}(M)_k \times \mathrm{U}(2M-N+k)_{-k}$. In particular this means that the two cases $M = N$ and $M = N+k$ describe the same theory.

2.7 Physical observables

Conformal invariance puts stringent constraints on which physical observables are meaningful to consider. The basic observables are n-point functions of the operators in the theory. Since the ABJ(M) model is a gauge theory, only the correlation functions of *gauge invariant* operators are physical. There are several classes of gauge invariant operators, *e.g.*, Wilson loops, which have been studied in the context of ABJ(M) [100, 73, 113, 152, 67]. However, in this thesis, the focus is on *local* gauge invariant operators.

In the planar limit, the structure of the n-point correlation functions simplifies considerably. In fact, a quick counting reveals that the connected component of an n-point function is proportional to $1/N^{(n-2)}$. Hence, to the leading order in large N, any correlation function with $n > 2$ is dominated by *disconnected* diagrams, and factors into a sum of two-point functions,

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle = \sum_{\{k_1,\ldots,k_n\}} \langle \mathcal{O}_{k_1} \mathcal{O}_{k_2} \rangle \cdots \langle \mathcal{O}_{k_{n-1}} \mathcal{O}_{k_n} \rangle + \mathcal{O}(1/N) . \tag{2.50}$$

In particular, in the strict planar limit, any correlation function of an odd number of operators vanishes.

In a planar conformally invariant gauge theory, the study of two-point functions of gauge invariant local operators is central. In fact, conformal invariance gives strict restriction on the form of these correlation functions. It is possible to choose a basis in which the two-point functions are diagonal. We then get [79]

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta}} , \tag{2.51}$$

where Δ is the dimension of the involved operator. Hence, only the dimension of the operator depends on the dynamics of the theory and the central goal for understanding a large N conformal gauge theory is to determine the spectrum of local operators and their dimensions. This will be the focus of rest of this chapter, as well as the following two chapters.

2.8 Gauge invariant local operators and spin-chains

As discussed above, gauge invariant local operators are central objects in the study of the ABJ(M) model. To get a local operator we take any number of the various fields in the model, as well as their covariant derivatives, all evaluated at the same space-time point, and multiply them together. However, to ensure gauge invariance, we need to make sure all the gauge indices are fully contracted. This leads in general to *multi-trace* operators of the form

$$\mathrm{tr}\left(X_{i_1} X^\dagger_{i_2} X_{i_3} X^\dagger_{i_4} \cdots X_{i_n} X^\dagger_{i_n} \right) \cdots \mathrm{tr}\left(X_{j_1} X^\dagger_{j_2} \cdots X_{j_m} X^\dagger_{j_m} \right) \tag{2.52}$$

Here, the X_k are any bifundamental field, such as Y^1, ψ_3 or $D_0 Y^4$, while $X^\dagger_k$ transform in the anti-bifundamental. An important property of the sequence of fields within each trace factor is that it is *alternating*, *i.e.*, all the fields at odd positions are in the $(\mathbf{N},\bar{\mathbf{M}})$ representation of the gauge group, while the fields at even positions transform as $(\bar{\mathbf{N}},\mathbf{M})$.

In the planar limit, the most important class of gauge invariant operators are the *single trace* operators. This is due to the same effect that suppresses higher point functions – the leading contribution to the two-point function of a general multi-trace operator with itself factorizes into terms involving the individual trace factors.

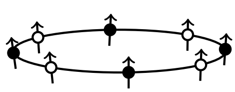
An operator of length L consists of L fields from each of the two field multiplets $\mathcal{V}$ and $\bar{\mathcal{V}}$, and belongs to one of the irreducible representations that appear in the L-fold tensor product between these multiplets, $(\mathcal{V}\otimes\bar{\mathcal{V}})^{\otimes L}$. Among all the states appearing in this product there is a unique state with the highest weight. This operator has Dynkin labels $[L,0,L;0;L]$, and hence satisfy the BPS condition $\Delta = J$. I will denote it by

$$\mathcal{O}^L_{gs} = \mathrm{tr}(Y^1 Y_4^\dagger)^L\,, \tag{2.53}$$

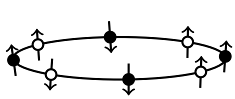
where the subscripts stands for “ground state”, as explained below. This operator is annihilated by the supercharges $\mathfrak{Q}^{12}_\pm$ and $\mathfrak{Q}^{13}_\pm$, and hence is a chiral primary.

More complicated operators can be built from $\mathcal{O}^L_{gs}$ by changing any of the scalars Y^1 ($Y_4^\dagger$) to any other field in the $\mathcal{V}$ ($\bar{\mathcal{V}}$) module. On the odd sites the resulting operator will have any of the 4_B+8_F fields Y^I, $\psi_{I\alpha}$, as well as any number of covariant derivatives $D_{\alpha\beta}$. Similarly, the even sites will contain $Y_I^\dagger$, $\psi^{\dagger I}_\alpha$, with covariant derivatives acting on them.

Pictorially, we can represent these operators as a *spin-chain* with $2L$ sites. At each site of the chain sits a spin transforming in the $\mathcal{V}$ or $\bar{\mathcal{V}}$ representations of $\mathrm{OSp}(6|4)$, with the two representations alternating between odd and even sites. The operator $\mathcal{O}^L_{gs}$ corresponds to the *ground state* of the spin-chain of length $2L$, with all spins “pointing up”:

$\mathrm{tr}\left(Y^1Y_4^\dagger Y^1Y_4^\dagger Y^1Y_4^\dagger Y^1Y_4^\dagger\right) \quad \leftrightarrow$ 

Other operators are represented by “flipping” one or more spins. This introduces an *impurity* or *excitation* in the chain:

$\mathrm{tr}\left(Y^1Y_4^\dagger Y^2Y_4^\dagger Y^1Y_3^\dagger Y^2Y_4^\dagger\right) \quad \leftrightarrow$ 

The spin-chain ground state (2.53) preserves an $\mathrm{SU}(2|2)\times\mathrm{U}(1)_{\text{extra}}$ subgroup of $\mathrm{OSp}(6|4)$ [76]. The bosonic part of this $\mathrm{SU}(2|2)$ symmetry is given by $\mathrm{SU}(2)_G\times\mathrm{SU}(2)_r\times\mathrm{U}(1)_E$, where $\mathrm{SU}(2)_r\cong\mathrm{SO}(2,1)$ is the Lorentz spin. The $\mathrm{SU}(4)_R$ R-symmetry group splits into $\mathrm{SU}(2)_{G'}\times\mathrm{SU}(2)_G\times\mathrm{U}(1)_{\text{extra}}$, but only the two latter factors are unbroken. Finally, $\mathrm{U}(1)_E$ is the spin-chain energy $E = \Delta - J$, which involves the scaling dimension Δ and the Cartan charge J of the broken $\mathrm{SU}(2)_{G'}$. The charges of the fields of the theory under these groups are given in table 2.2.

SPIN-CHAIN EXCITATIONS. As noted above, the ground state has $E = \Delta - J = 0$. Adding an excitation to the chain (by replacing one of the vacuum fields by some other field), will increase this classical energy by a half-integer. From table 2.2 we see that there are $4_B + 4_F$ excitation with $\delta E = 1/2$. The excitations on the odd ("A"-particles) and even ("B"-particles) sites are [76]

$$\text{“A”-particles:} \qquad (Y^2, Y^3 | \psi_{4+}, \psi_{4-}) \tag{2.54a}$$

$$\text{“B”-particles:} \qquad (Y_2^\dagger, Y_3^\dagger | \psi_+^1, \psi_-^1)\,. \tag{2.54b}$$

Under the $\mathrm{SU}(2|2) \times \mathrm{U}(1)_{\text{extra}}$ symmetry preserved by the spin-chain vacuum, these two sets of excitations transform as $(\mathbf{2}|\mathbf{2})_-$ and $(\mathbf{2}|\mathbf{2})_+$, respectively.

The other fields of the field multiplets do not correspond to elementary excitations of the spin-chain, but are composite excitations that mix with states involving more than one of the above excitations. For a discussion about these multi-excitations see, *e.g.*, [119].

SUBSECTORS. There are a number of closed sectors of the theory. To find a closed sector, we want to find a linear combination T of the conserved charges of the theory that is positive semi-definite on the field multiplets. The sector then consists of all fields for which this combined charge vanishes. Any combination of such fields will have a vanishing T, and since T is conserved, this will be true also when quantum corrections are included. For example, all fields have $T = \Delta - J \geq 0$, and only the fields Y^1 and $Y_4^\dagger$ have $\Delta = J$. This is the 1/3-BPS sector consisting only of the ground states. To get a more interesting sector, we can take $T = \Delta - J - Q$. The fields that have $T = 0$ are Y^1, Y^2, $Y_3^\dagger$ and $Y_4^\dagger$. This sector is usually called the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector, since it is made up of two sets of fields, on the odd and even sites, that transform as $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ under an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ subgroup of $\mathrm{SU}(4)_R$.[15]

Another interesting charge is $\Delta - L$, where L is half the spin-chain length. L is $1/2$ for the scalars and fermions, and 0 for the derivatives. The vanishing of this charge gives rise to the $\mathrm{SU}(4)$ sector consisting of operators built out of the scalars Y^I and $Y_I^\dagger$. However, L does not appear among the conserved charges in table 2.2, since it is not conserved beyond two-loop order in perturbation theory. As we will see in chapter 3, the $\mathrm{SU}(4)$ sector is a closed sector

[15] Note that the two $\mathrm{SU}(2)$ subgroups transforming the fields in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector are *not* $\mathrm{SU}(2)_{G'} \times \mathrm{SU}(2)_G$.

Table 2.2: Classical charges of the fields of ABJ(M). For the spin-chain ground state (2.53), the $\mathrm{SU}(4)_R$ R-symmetry group is split into $\mathrm{SU}(2)_{G'} \times \mathrm{SU}(2)_G \times \mathrm{U}(1)_{\text{extra}}$, and the conformal group $\mathrm{SO}(3,2)$ is broken to $\mathrm{SU}(1,1)_r \times \mathrm{U}(1)_\Delta$. The remaining symmetry is given by $\mathrm{SU}(2|2) \times \mathrm{U}(1)_{\text{extra}} \supset \mathrm{SU}(2)_G \times \mathrm{SU}(1,1)_r \times \mathrm{U}(1)_E \times \mathrm{U}(1)_{\text{extra}}$ [76]. The spin-chain energy $E = \Delta - J$ is given by the dimension Δ and the eigenvalue of the Cartan generator of the broken $\mathrm{SU}(2)_{G'}$. The charges of the broken $\mathrm{SU}(4)_R$ are related to the Dynkin labels $[p_1, q, p_2]$ as $J = \frac{p_1+q+p_2}{2}$, $Q = \frac{q}{2}$ and $J_3 = p_1 - p_2$. The table is adapted from [119].

	$\mathrm{SU}(4)_R$ $[p_1,q,p_2]$	$\mathrm{SU}(2)_{G'}$ J	$\mathrm{SU}(2)_G$ Q	$\mathrm{U}(1)_{\text{extra}}$ J_3	$\mathrm{SU}(1,1)_r$ s	$\mathrm{U}(1)_\Delta$ Δ	$\mathrm{U}(1)_E$ $E=\Delta-J$
Y^1	$[1,0,0]$	$+1/2$	0	$+1$	0	$1/2$	0
Y^2	$[-1,1,0]$	0	$+1/2$	-1	0	$1/2$	$1/2$
Y^3	$[0,-1,1]$	0	$-1/2$	-1	0	$1/2$	$1/2$
Y^4	$[0,0,-1]$	$-1/2$	0	$+1$	0	$1/2$	1
$\psi_{1\pm}$	$[-1,0,0]$	$-1/2$	0	-1	$\pm1/2$	1	$3/2$
$\psi_{2\pm}$	$[1,-1,0]$	0	$-1/2$	$+1$	$\pm1/2$	1	1
$\psi_{3\pm}$	$[0,1,-1]$	0	$+1/2$	$+1$	$\pm1/2$	1	1
$\psi_{4\pm}$	$[0,0,1]$	$+1/2$	0	-1	$\pm1/2$	1	$1/2$
D_0	$[0,0,0]$	0	0	0	0	1	1
$D_{\pm\pm}$	$[0,0,0]$	0	0	0	±1	1	1
$Y_1^\dagger$	$[-1,0,0]$	$-1/2$	0	-1	0	$1/2$	1
$Y_2^\dagger$	$[1,-1,0]$	0	$-1/2$	$+1$	0	$1/2$	$1/2$
$Y_3^\dagger$	$[0,1,-1]$	0	$+1/2$	$+1$	0	$1/2$	$1/2$
$Y_4^\dagger$	$[0,0,1]$	$+1/2$	0	-1	0	$1/2$	0
$\psi_\pm^{\dagger 1}$	$[1,0,0]$	$+1/2$	0	$+1$	$\pm1/2$	1	$1/2$
$\psi_\pm^{\dagger 2}$	$[-1,1,0]$	0	$+1/2$	-1	$\pm1/2$	1	1
$\psi_\pm^{\dagger 3}$	$[0,-1,1]$	0	$-1/2$	-1	$\pm1/2$	1	1
$\psi_\pm^{\dagger 4}$	$[0,0,-1]$	$-1/2$	0	$+1$	$\pm1/2$	1	$3/2$

of the two-loop spin-chain Hamiltonian, but at higher orders the operators of this sector start to mix with operators outside the sector. For a more complete list of closed subsectors above the ground state considered here see the review article [119].

3. The integrable ABJ(M) spin-chain

3.1 Two-point functions

In this section we will consider the perturbative calculation of two-point functions, and the relation to the renormalization of the involved operators.

As discussed in chapter 2, in the planar limit, we are mainly interested in single trace operators. Let us start by considering the two-point function of the ground-state operator

$$\mathcal{O}_{gs}^{L} = C\,\mathrm{tr}(Y^1 Y_4^\dagger)^L\,, \tag{3.1}$$

where we have introduced a normalization contant C. The scalar propagator in position space is given by

$$\langle Y^I(x)^a{}_{\hat{a}} Y_J^\dagger(y)^{\hat{b}}{}_b\rangle = \frac{1}{k}\frac{\delta^I{}_J \delta^a{}_b \delta_{\hat{a}}{}^{\hat{a}}}{|x-y|}\,, \tag{3.2}$$

where the color indices have been written out explicitly ($a,b = 1,\ldots,N$ and $\hat{a},\hat{b} = 1,\ldots,M$). The tree-level diagrams that contribute to the correlation function

$$\langle \mathcal{O}_{gs}^{L}(x)\,\mathcal{O}_{gs}^{L\,\dagger}(y)\rangle\,, \tag{3.3}$$

are schematically given in figure 3.1. Here, the thick horizontal lines indicate the two operators, and the thin lines are the propagators between the individual fields inside the traces. The ordering of the propagators, and hence of the constituent fields, indicate the ordering of the gauge indices in the traces.

The first two diagrams in figure 3.1 are related by a relative shift of the fields by two sites in one operator compared to the other operator. Since the operator $\mathcal{O}_{gs}^{L}$ is invariant under this shift, we get L equivalent diagrams contributing to the two-point function. For each face of the diagram the two propagators on either side of the face gives a factor of $\delta^a{}_b\delta^b{}_a = N$ or $\delta_{\hat{a}}{}^{\hat{b}}\delta_{\hat{b}}{}^{\hat{a}} = M$, depending on whether the first field is in the bifundamental or anti-bifundamental representation. In total there are $L+L$ such pairs of propagators, so the total color factor for each diagram is $(NM)^L$.

In the third diagram in figure 3.1 the ordering of two of the propagators has been swapped. By carefully evaluating the traces over color indices resulting from this diagram, we see that this diagram is proportional to $(NM)^{L-1}$.

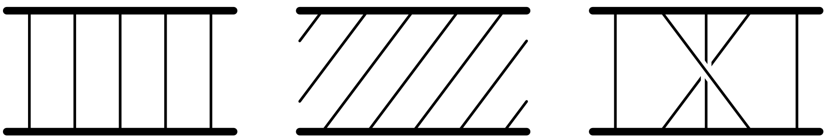


Figure 3.1: Tree-level diagrams contributing to the two-point function of two scalar operators $\mathcal{O}$ and $\mathcal{O}^\dagger$. Note that there are no interaction vertex in diagram (c), the three propagator simply cross at the midpoint. Diagram (a) and (b) give planar contributions while the diagram in (c) is non-planar and hence suppressed in the large N limit.

Hence diagram (c) is suppressed by a factor $1/(NM)$ compared to the first two diagrams in figure 3.1, and in the 't Hooft limit where $NM \gg 1$, we only need to consider the contribution of planar diagrams.[1]

The planar diagrams give a total contribution of

$$\langle \mathcal{O}_{gs}^L(x)\,\mathcal{O}_{gs}^{L\dagger}(y)\rangle = |C|^2 \frac{1}{k^{2L}} \frac{L(NM)^L}{|x-y|^{2L}} = |C|^2 \frac{L(\lambda\hat{\lambda})^L}{|x-y|^{2L}}\,. \tag{3.4}$$

By choosing the normalization constant to be $C = (\lambda\hat{\lambda})^{-L/2} L^{-1/2}$, we can write the above expression as

$$\langle \mathcal{O}_{gs}^L(x)\,\mathcal{O}_{gs}^{L\dagger}(y)\rangle = \frac{1}{|x-y|^{2\Delta_0}}\,, \tag{3.5}$$

where $\Delta_0 = L$ is the classical dimension of the operator $\mathcal{O}_{gs}^L$. This result agrees with the expected form in (2.51).

The above calculation can be straightforwardly generalized to the more general operators

$$\mathcal{O}^{I_1 I_2 \cdots I_L}_{\hat{I}_1 \hat{I}_2 \cdots \hat{I}_L} = C \,\mathrm{tr}\, Y^{I_1} Y^\dagger_{\hat{I}_1} Y^{I_2} Y^\dagger_{\hat{I}_2} \cdots Y^{I_L} Y^\dagger_{\hat{I}_L}\,. \tag{3.6}$$

By the same argument as above, the planar diagrams will dominate the correlation function

$$\langle \mathcal{O}^{I_1 I_2 \cdots I_L}_{\hat{I}_1 \hat{I}_2 \cdots \hat{I}_L}\, \mathcal{O}^{\dagger \hat{I}_1 \hat{I}_2 \cdots \hat{I}_L}_{I_1 I_2 \cdots I_L} \rangle\,. \tag{3.7}$$

In principle there could now be a contribution from diagrams involving self-contractions between, say, Y^{I_1} and $Y^\dagger_{\hat{I}_1}$. Such a diagram contains an ultraviolet divergence, and need to be regularized. However, if we use *dimensional regularization*, any such *tadpole* diagram vanishes identically. Hence, we still only

[1] The restriction to planar diagrams is only valid for operators with $L \ll N$. In general there are of the order of $L!$ Feynman diagrams in total. Out of these, L are planar. For L of the order of N, the suppression in $1/(NM)$ is cancelled by the huge number of contributing non-planar diagrams [139].

need to consider planar diagrams of the kind represented in figures 3.1 (a) and (b). These diagrams give the contribution

$$\langle \mathcal{O}^{I_1\cdots I_L}_{\hat{I}_1\cdots\hat{I}_L}\,\mathcal{O}^{\dagger\hat{J}_1\cdots\hat{J}_L}_{J_1\cdots J_L}\rangle = |C|^2\left(\delta^{I_1}_{J_1}\delta^{I_2}_{J_2}\cdots\delta^{I_L}_{J_L}\,\delta^{\hat{J}_1}_{\hat{I}_1}\delta^{\hat{J}_2}_{\hat{I}_2}\cdots\delta^{\hat{J}_L}_{\hat{I}_L}+\text{cycl.}\right)\frac{(\lambda\hat{\lambda})^L}{|x-y|^{2L}}\,, \quad (3.8)$$

where the flavor structure includes a sum over cyclic permutations of flavors of one of the operators. By the choice of the normalization factor C this can again be written in the form of (2.51).

Loop corrections. Consider now the structure of the loop corrections to the two-point function. By conformal invariance, it will still have the form [79]

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \frac{f(\bar{\lambda})}{|x-y|^{2\Delta(\bar{\lambda})}}\,. \quad (3.9)$$

The quantum corrections in the numerator can be absorbed by a $\bar{\lambda}$ dependent rescaling of the operators.[2] Hence, the physically interesting corrections appear in the dimension $\Delta(\bar{\lambda})$ of the operator. Expanding the above expression we then get

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \frac{1}{|x-y|^{2\Delta_0}}\left(1-\delta\Delta\,\bar{\lambda}^2\log|\mu x|+\cdots\right)\,, \quad (3.10)$$

where $\delta\Delta$ is the two-loop anomalous dimension. In order to make the argument of the logarithm dimensionless, we had to introduce the dimensionful constant μ. How does this constant arise in perturbation theory? Since the ABJ(M) model is scale invariant, it does not contain any dimensionful parameters. Hence, the only possibility is that μ is the renormalization scale, which appears in the perturbative calculation from the regularization of ultraviolet divergences.

In a generic field theory, such divergences can appear in any region of a Feynman diagram that contributes to the two-point function. This is not true in a conformal theory such as ABJ(M). In particular, a divergence at a generic point in space-time, not associated to the involved operators, is cancelled by a renormalization of the coupling constants in the theory. However, we know that due to gauge symmetry and supersymmetry, the coupling constant of ABJ(M) are inverse integers that do not renormalize. Hence, the only possible divergences appear in the ultraviolet regions close to the operators, and are associated with the renormalization of the operators themselves.

[2] To simplify the notation I will in this section only write out the coupling constant $\bar{\lambda}$. In the ABJ model, coupling dependent terms generally depends on the second coupling σ as well.

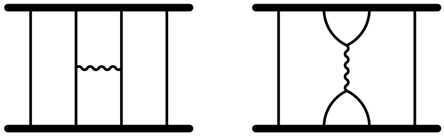


Figure 3.2: One-loop diagrams. These diagrams are not divergent, and hence do not contribute to the anomalous dimension of the involved operators.

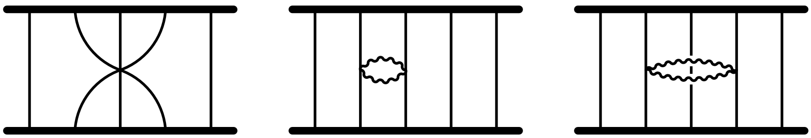


Figure 3.3: Two-loop diagrams. These diagrams are divergent, and hence contribute to the anomalous dimension of the involved operator. However, only first two diagrams are planar, so in the large N limit, we can ignore the third diagram.

In equation (3.10), the leading order correction to the dimension of the operator is proportional to $\bar{\lambda}^2$. This corresponds to a two-loop contribution in perturbation theory. There are one-loop diagrams that potentially could contribute to the two-point function. Two such diagrams are shown in figure 3.2. However, a simple power counting argument shows that in three dimensions only diagrams with an even number of loops can have an ultraviolet divergence, and hence contribute to the anomalous dimension. This means that the first correction to the dimension of an operator is a two loops.

Some of the diagrams that contributes to the two-loop correction of the two-point function between two scalar operators are shown in figure 3.3. The third of these diagram is non-planar, and hence suppressed in $1/(NM)$. The other two diagrams will give non-vanishing contributions.

The first diagram in figure 3.3 involves the sextic scalar interaction vertex of ABJ(M). This vertex has a non-trivial flavor structure. For example, this diagram will give a non-zero contribution to the two-point function between operators of the form

$$\langle \mathrm{tr}(\cdots Y^1 Y_4^\dagger Y^2 \cdots)\, \mathrm{tr}(\cdots Y^2 Y_4^\dagger Y^1 \cdots)^\dagger \rangle\,, \tag{3.11}$$

where the two $\mathrm{SU}(4)$ indices 1 and 2 have been exchanged in the second operator. The tree-level two-point function (3.8) is *diagonal*, in the sense that only the correlator between an operator $\mathcal{O}$ and its conjugate $\mathcal{O}^\dagger$ is non-vanishing. As the above example shows, turning on quantum corrections introduces *mixing* between different operators. Note that the superconformal symmetry of the theory ensures that only operators with the same bare dimension, spin and R-charge can mix. Still, for operators consisting of many fields, this mixing is in general highly non-trivial.

In the next section we will calculate the two-loop contribution of scalar operators in a general form, and discuss the mixing problem in more detail.

3.2 The dilatation operator of ABJ(M)

All physically relevant quantum corrections to two-point functions of local operators are captured by the coupling dependence of the scaling dimensions of the operators. The most straightforward way of calculating these corrections perturbatively is not by calculating the full two-point function, but by calculating how the operators are affected by renormalization.

To study the renormalization of a local operator $\mathcal{O}_a$, we need to extract any ultraviolet divergences appearing in the loop corrections to the operator. To compensate for these divergences we introduce counter terms in the action. The renormalized operator takes the form

$$\mathcal{O}_a^{(\mathrm{ren})} = \mathcal{Z}_a{}^b \mathcal{O}_b^{(bare)}, \tag{3.12}$$

where the matrix $\mathcal{Z}$ depends on the coupling constants and on the renormalization scale μ. The fact that $\mathcal{Z}$ is a matrix, and not scalar, indicates that different bare operators mix under renormalization. The renormalization factor has the perturbative expansion

$$\mathcal{Z} = 1 + \bar{\lambda}^2 \mathcal{Z}_2 + \bar{\lambda}^4 \mathcal{Z}_4 + \dots, \tag{3.13}$$

where, for instance, $\mathcal{Z}_2$ is given by the negative of the divergent part of the two-loop correction to the operator.

Once we know $\mathcal{Z}$, it is straightforward to extract the mixing matrix of anomalous dimensions of the involved operators. However, let us first take a step back and try to understand what we are actually calculating. As we saw in chapter 2, the scaling dimension of an operator is measured by the dilatation operator $\mathfrak{D}$. On an operator $\mathcal{O}$ with definite dimension Δ, $\mathfrak{D}$ acts as

$$\left[\mathfrak{D}, \mathcal{O}(x)\right] = \left(\Delta - x_\mu \frac{\partial}{\partial x^\mu}\right) \mathcal{O}(x). \tag{3.14}$$

In particular, if we evaluate the operator at $x = 0$ the eigenvalue of $\mathfrak{D}$ is Δ. The dimension can be split into two parts: the classical, or bare, dimension Δ_0, and the anomalous dimension $\delta\Delta$. To calculate the bare dimension of $\mathcal{O}$, we just need to sum up the dimensions of the fundamental fields that make up the operator. The bare dimension is measured by the operator $\mathfrak{D}^{(0)}$. The anomalous dimension, on the other hand, is a function of the coupling constants, and hence strongly depends on the dynamics of the theory. It is measured by the operator $\delta\mathfrak{D} = \mathfrak{D} - \mathfrak{D}^{(0)}$.

Once we turn on the interactions of the theory, most operators will *not* be eigenstates of the dilatation operator. As noted above, this is due to mixing between operators with equal quantum numbers under renormalization.

In a perturbative calculation, we can extract the anomalous dimension part of dilatation operator from renormalization factor $\mathcal{Z}$ by

$$\delta\mathfrak{D} = \frac{d\log\mathcal{Z}}{d\log\mu}\,, \tag{3.15}$$

where μ is the renormalization scale. As we will see below, this formula provides a very explicit expression for the action of $\delta\mathfrak{D}$ on local operators. The matrix elements of $\delta\mathfrak{D}$ in some basis of operators gives a matrix whose eigenvalues are the anomalous dimensions of the operators in the theory. In general, it is a non-trivial problem to diagonalize this matrix.

Expanding $\mathcal{Z}$ as in (3.13), we get

$$\log\mathcal{Z} = \bar{\lambda}^2\mathcal{Z}_2 + \bar{\lambda}^4\left(\mathcal{Z}_4 - \frac{1}{2}\mathcal{Z}_2^2\right) + \mathcal{O}(\bar{\lambda}^6)\,. \tag{3.16}$$

Since the ABJ(M) model is renormalizable and conformally invariant, anomalous dimensions should be independent of the renormalization scale μ. Hence $\log\mathcal{Z}$ will depend linearly on $\log\mu$.

In a planar gauge theory, we can restrict ourselves to diagrams that only involve a number of neighboring fields. It is easy to see that a diagram with k loops can involve at most $k+1$ external fields. Hence, in the spin-chain picture the resulting dilatation operator can be written as a sum of terms acting *locally* on consecutive spin-chain sites:

$$\mathfrak{D} = \sum_{k=0}^{\infty}\bar{\lambda}^k\sum_{l=0}^{2L}\mathfrak{D}_l^{(k)}\,, \tag{3.17}$$

where $\mathfrak{D}_l^{(k)}$ acts on the fields at the $k+1$ neighboring sites l to $l+k$.

To find the spectrum of anomalous dimensions we need to solve the eigenvalue equation

$$[\delta\mathfrak{D},\mathcal{O}] = \delta\Delta\,\mathcal{O}\,. \tag{3.18}$$

This equation has the form of a time-independent Schrödinger equation, where $\delta\mathfrak{D}$ plays the role of the Hamiltonian. Instead of $\delta\mathfrak{D}$, it is convenient to introduce the charge

$$\mathcal{H} = \mathfrak{D} - \mathfrak{J}\,, \tag{3.19}$$

where $\mathfrak{J}$ is given by the R-charge generators $\mathfrak{J} = \mathfrak{R}^1{}_1 - \mathfrak{R}^4{}_4$. Since the $\mathrm{SU}(4)_R$ R-symmetry group is compact, $\mathfrak{J}$ will not receive any quantum corrections.

Hence $\mathcal{H}$ is related to $\mathfrak{D}$ by a shift at the classical level, but their quantum corrections coincide:

$$\mathcal{H} = \sum_{k=0}^{\infty} \bar{\lambda}^k \mathcal{H}^{(k)} = \mathfrak{D}^{(0)} - \mathfrak{J} + \sum_{k=1}^{\infty} \bar{\lambda}^k \delta\mathfrak{D}^{(k)} . \tag{3.20}$$

It is natural to interpret $\mathcal{H}$ as the Hamiltonian of the spin-chain, and the corresponding eigenvalues $E = \Delta - J$ as the energy of a spin-chain state.

In the rest of this chapter, as well as in the next chapter, we will consider perturbative corrections to the dilatation operator. Since these coincide with the corrections to the spin-chain Hamiltonian we will not make a distinction between the two.

Dimensional reduction. As noted above, we can perturbatively compute the dilatation operator by considering the renormalization of local operators. This renormalization is associated to the appearance of ultraviolet divergences in the perturbative calculation. To extract useful information from divergent integrals, we need to regulate them in a sensible way. We will do this by using *dimensional reduction*. This procedure is similar to *dimensional regularization*, in that all integrals are performed by analytically continuing the number of space-time dimensions to $D = 3 - 2\epsilon$. Divergences then appear as poles in $1/\epsilon$.

In dimensional regularization [163] all the Feynman rules are treated in D dimensions. However, this procedure is known to break the Slavnov-Taylor identity of pure Chern-Simons theory at two loops [58]. Intuitively, we can understand this breaking from the difficulty of continuing the three-dimensional anti-symmetric tensor $\epsilon_{\mu\nu\rho}$ away from $D = 3$. As we have seen in chapter 2, the gauge invariance of the Chern-Simons terms strongly relies on the three-dimensional structure of the theory.

We can overcome this problem by not continuing $\epsilon_{\mu\nu\rho}$. Instead we will use the three-dimensional Feynman rules and perform all tensor algebra in $D = 3$. The analytic continuation to $3 - 2\epsilon$ dimensions is instead performed on the level of scalar integrals. In [58], it was shown that the Slavnov-Taylor identities are then satisfied.

Away from three dimensions, the coupling constants become dimensionful. We absorb this by a rescaling $\bar{\lambda} \to \mu^{2\epsilon}\bar{\lambda}$. The renormalization constant $\mathcal{Z}$ is then a function of the coupling $\mu^{2\epsilon}\bar{\lambda}$ and of ϵ (and of the coupling σ, but we suppress this dependence since σ remains dimensionless). The dilatation operator is given by

$$\delta\mathfrak{D} = \lim_{\epsilon\to 0} \frac{d \log \mathcal{Z}(\mu^{2\epsilon}\bar{\lambda}, \epsilon)}{d \log \mu} = \lim_{\epsilon\to 0} 2\epsilon \frac{d \log \mathcal{Z}(\bar{\lambda}, \epsilon)}{d \log \bar{\lambda}} . \tag{3.21}$$

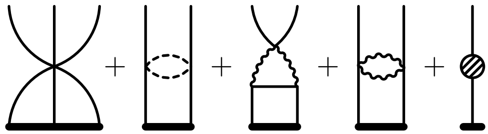


Figure 3.4: Feynman diagrams contributing to the two-loop dilatation operator of ABJ(M) in the $SU(4)$ sector. The shaded circle in the last diagram indicates a sum of scalar self-energy diagrams. These diagrams are given, *e.g.*, in [135, 30, 31]. The first three diagrams depend only on $\lambda\hat{\lambda}$, while the last two diagrams also give contributions depending on λ^2 and $\hat{\lambda}^2$. The diagrams also differ in the flavor structure – the second and third diagrams gives flavor traces, the last two diagrams act as the identity in flavor space, and the first diagram contains parts acting as the trace, the identity, and as a next-to-nearest neighbor permutation.

At loop order l, the last expression effectively extracts the coefficient of the $1/\epsilon$ pole multiplied by $2l$. For the $\epsilon \to 0$ limit to be convergent, $\mathcal{Z}$ cannot contain any higher poles in $1/\epsilon$. This requirement is equivalent to the statement above that $\mathcal{Z}$ is linear in $\log\mu$.

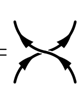
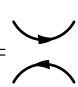
THE TWO-LOOP DILATATION OPERATOR. There are essentially four different Feynman diagrams that contribute to the two-loop dilatation operator in the $SU(4)$ sector of ABJ(M). These diagrams are shown in figure 3.4, and only involve a single operator, depicted by the thick horizontal line at the bottom. We also suppress any propagators that are not connected to an interaction vertex. The last diagram in figure 3.4, is not a single Feynman diagram. Instead the bubble indicates the scalar self-energy. The exact form of the diagrams contributing to the scalar self-energy can be found, *e.g.*, in Paper I.

Only the first three diagrams in figure 3.4 act non-trivially on the flavors of the operator. It will be enough to focus on these diagrams, since the rest of the dilatation operator can be reconstructed using supersymmetry.

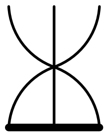
To capture the flavor structure of the diagrams we introduce two operators acting in flavor space: the permutation operator $P : \mathcal{V}\otimes\mathcal{V} \to \mathcal{V}\otimes\mathcal{V}$ (or $P : \bar{\mathcal{V}}\otimes\bar{\mathcal{V}} \to \bar{\mathcal{V}}\otimes\bar{\mathcal{V}}$) , and the trace $K : \mathcal{V}\otimes\bar{\mathcal{V}} \to \mathcal{V}\otimes\bar{\mathcal{V}}$ (or $K : \bar{\mathcal{V}}\otimes\mathcal{V} \to \bar{\mathcal{V}}\otimes\mathcal{V}$), which are defined as

$$P^{I'J'}_{I\,J} = \delta^{I'}_{J}\delta^{J'}_{I}\,, \qquad K^{I'I}_{J'J} = \delta^{I'}_{J'}\delta^{I}_{J}\,. \tag{3.22}$$

We can represent these operators graphically as

$P =$ , $\qquad K =$ .

The first diagram involves the sextic scalar vertex. Using the momentum space Feynman rules from [138],



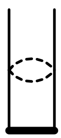
$$= \frac{(4\pi)^2}{2}\frac{N}{k}\frac{M}{k}\int \frac{d^dk\,d^dq}{q^2(q-k)^2(k-p)^2}\sum_{l=1}^{2L}\Xi_l \tag{3.23}$$

$$= -\frac{\bar{\lambda}^2}{8\epsilon}\sum_{l=1}^{2L}\Xi_l + \mathcal{O}(1)\,, \tag{3.24}$$

where Ξ_l is the flavor structure obtained from the sextic vertex

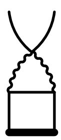
$$\Xi_l = 1 - 2P_{l,l+1} + K_{l,l+1}P_{l,l+2} + K_{l+1,l+2}P_{l,l+2} - \tfrac{1}{2}K_{l,l+1} - \tfrac{1}{2}K_{l+1,l+2}\,. \tag{3.25}$$

The second and third diagram in figure 3.4 are given by



$$= 2(4\pi)^2\frac{N}{k}\frac{M}{k}\int d^Dq\,d^dk\frac{k\cdot(q-k)}{k^2q^2(k-q)^2(q-p)^2}\sum_{l=1}^{2L}K_{l,l+1} \tag{3.26}$$

$$= -\frac{\bar{\lambda}^2}{4\epsilon}\sum_{l=1}^{2L}K_{l,l+1}\,, \tag{3.27}$$



$$= -(4\pi)^2\frac{M}{k}\frac{N}{k}\int d^Dk\,d^Dq\frac{q\cdot(k-q)\,(p-q)\cdot(k-q)}{k^2q^2(q-p)^2(k-q)^2}\sum_{l=1}^{2L}K_{l,l+1} \tag{3.28}$$

$$= \frac{\bar{\lambda}^2}{8\epsilon}\sum_{l=1}^{2L}K_{l,l+1}\,. \tag{3.29}$$

The total contribution to the dilatation operator is then

$$\delta\mathfrak{D} = \bar{\lambda}^2\sum_{l=1}^{2L}\left(\tfrac{1}{2} + c - P_{l,l+2} + \tfrac{1}{2}K_{l,l+1}P_{l,l+2} + \tfrac{1}{2}K_{l+1,l+2}P_{l,l+2}\right)\,. \tag{3.30}$$

Note that the terms that act only as a trace on two neighboring fields have completely cancelled. c is the contribution from the last two diagrams in figure 3.4. These were directly calculated in, *e.g.*, [30, 31], and also as part of the four-loop calculation in Paper I. However, to determine the two-loop dilatation operator, we only need to consider the action of the above operator on any chiral primary operator. Chiral primaries transform in a symmetric, traceless representation of SU(4). Hence, they are annihilated by K and have eigenvalue one under P. Acting with the above expression for $\delta\mathfrak{D}$ on a chiral primary operator $\mathcal{O}_L$ of length L then gives

$$\left[\delta\mathfrak{D}, \mathcal{O}_L\right] = 2L\bar{\lambda}^2\left(c - \frac{1}{2}\right)\mathcal{O}_L\,. \tag{3.31}$$

Since the dimension of $\mathcal{O}_L$ is protected to all orders in perturbation theory, supersymmetry requires $c = 1/2$. Our final expression for the two-loop dilatation operator of ABJ(M) is then

$$\delta\mathfrak{D} = \bar{\lambda}^2 \sum_{l=1}^{2L} \left(1 - P_{l,l+2} + \tfrac{1}{2}K_{l,l+1}P_{l,l+2} + \tfrac{1}{2}K_{l+1,l+2}P_{l,l+2}\right). \tag{3.32}$$

In the next section we will study this Hamiltonian in more detail.

3.3 Bethe equations for scalar sectors of ABJ(M)

THE $\mathrm{SU}(2)\times\mathrm{SU}(2)$ SECTOR. The spin-chain Hamiltonian (3.32) was first studied in [135]. Let us restrict to operators in the closed $\mathrm{SU}(2)\times\mathrm{SU}(2)$ subsector, where the operators are built up of the fields Y^1, Y^2, $Y_3^\dagger$ and $Y_4^\dagger$. It is clear that the trace operator K acts trivially on any operator in the sector. Hence the two-loop Hamiltonian takes the form

$$\mathcal{H}^{(2)}_{\mathrm{SU}(2)\times\mathrm{SU}(2)} = \sum_{l=1}^{2L}(1 - P_{l,l+2}). \tag{3.33}$$

Note that the permutation operator acts on either two odd or two even sites. Hence, the fields on the even and odd sites are completely decoupled. The structure of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ Hamiltonian is that of two independent Heisenberg $\mathrm{XXX}_{1/2}$ spin-chains of length L.

The Heisenberg spin-chain is one of the most well known integrable quantum models. For a simple introduction see, *e.g.*, the recent review by Minahan [139]. Here I will just recollect some basic results. The spectrum of the Heisenberg model is given by a set of Bethe roots, or rapidities, u_i, $i = 1,\ldots,K$ with $K \leq L/2$. The roots are solutions to the Bethe equations

$$\left(\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}}\right)^L = \prod_{j\neq i}^{K} \frac{u_i - u_j + i}{u_i - u_j - i}, \qquad i = 1,\ldots,K. \tag{3.34}$$

Given a solution to these equations, the corresponding momentum P and two-loop correction to the energy E can be calculated as

$$P = -i\sum_{i=1}^{K} \log\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}}, \qquad \delta\Delta = i\bar{\lambda}^2 \sum_{i=1}^{K}\left(\frac{1}{u_i + \frac{i}{4}} - \frac{1}{u_i - \frac{i}{4}}\right). \tag{3.35}$$

These expressions lead to the energy and momentum of a single magnon

$$p_i = -i\log\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}}, \qquad \epsilon(p_i) = 4\bar{\lambda}^2 \sin^2\frac{p_i}{2}. \tag{3.36}$$

These results can be directly imported to the ABJ(M) model. The spectrum of gauge invariant operators in the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ sector is described by two sets of Bethe roots, u_i and v_j, satisfying the equations

$$\begin{aligned}
\left(\frac{u_i+\frac{i}{2}}{u_i-\frac{i}{2}}\right)^L &= \prod_{j\neq i}^{K_u}\frac{u_i-u_j+i}{u_i-u_j-i}\,, \qquad i=1,\dots,K_u\,,\\
\left(\frac{v_i+\frac{i}{2}}{v_i-\frac{i}{2}}\right)^L &= \prod_{j\neq i}^{K_v}\frac{v_i-v_j+i}{v_i-v_j-i}\,, \qquad i=1,\dots,K_v\,.
\end{aligned} \tag{3.37}$$

A solution to these equations corresponds to an operator with anomalous dimension

$$\delta\Delta = i\bar{\lambda}^2\left[\sum_{i=1}^{K_u}\left(\frac{1}{u_i+\frac{i}{4}}-\frac{1}{u_i-\frac{i}{4}}\right)+\sum_{i=1}^{K_v}\left(\frac{1}{v_i+\frac{i}{4}}-\frac{1}{v_i-\frac{i}{4}}\right)\right]. \tag{3.38}$$

Not all solutions to (3.37) correspond to a gauge invariant operator. The reason is that the gauge invariant operators we consider are given by the trace of a product of fields. The trace is invariant under a shift by two sites. For a state of the spin-chain to actually correspond to a gauge invariant operator, it needs to possess this symmetry. This requires

$$e^{i(P_u+P_v)}=1\,, \qquad \text{or} \qquad P_u+P_v\in 2\pi\mathbb{Z}\,, \tag{3.39}$$

where

$$P_u=-i\sum_{i=1}^{K_u}\log\frac{u_i+\frac{i}{2}}{u_i-\frac{i}{2}}\,, \qquad P_v=-i\sum_{i=1}^{K_v}\log\frac{v_i+\frac{i}{2}}{v_i-\frac{i}{2}}\,, \tag{3.40}$$

are the total momenta of the two $\mathrm{SU}(2)$ sectors.

THE $\mathrm{SU}(4)$ SECTOR. The Hamiltonian in the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ sector turns out to be integrable. However, this result is fairly trivial, since any $\mathrm{SU}(2)$ spin-chain with spin-1/2 sites and only nearest-neighbor interactions is integrable.[3] Let us therefore instead study the full $\mathrm{SU}(4)$ Hamiltonian. There are several ways to try and answer the question of whether a given spin-chain Hamiltonian is integrable. In [139, 30] the authors directly construct the corresponding R-matrix. As a complement to these derivations I will now check that the Hamiltonian gives the same spectrum as the Bethe equations of an integrable alternating spin-chain with $\mathfrak{su}(4)$ symmetry.

[3] A nearest-neighbor Hamiltonian for a spin-chain with sites in the fundamental representation of $\mathrm{SU}(N)$ is given by an identity term and a permutation. By shifting the energy of the ferromagnetic ground state, we can always write it in the conventional form for the Heisenberg spin-chain: $1-P$.

From the construction in [144], it is possible to write down a set of Bethe equations for any spin-chain with a symmetry algebra given by a simple Lie-algebra, and where the sites of the chain transform in fundamental representations, or symmetric tensor products of such representations. The only ingredients that go into the construction are the Cartan matrix of the algebra, and the weights of the representations. For a spin-chain with $\mathfrak{su}(4)$ symmetry and with alternative sites transforming in the fundamental and antifundamental representations, this construction gives the Bethe equations

$$\left(\frac{u_i+\frac{i}{2}}{u_i-\frac{i}{2}}\right)^L=\prod_{j\neq i}^{K_u}\frac{u_i-u_j+i}{u_i-u_j-i}\prod_{j=1}^{K_r}\frac{u_i-r_j-\frac{i}{2}}{u_i-r_j+\frac{i}{2}}\,, \tag{3.41a}$$

$$1=\prod_{j\neq i}^{K_r}\frac{r_i-r_j+i}{r_i-r_j-i}\prod_{j=1}^{K_u}\frac{r_i-u_j-\frac{i}{2}}{r_i-u_j+\frac{i}{2}}\prod_{j=1}^{K_v}\frac{r_i-v_j-\frac{i}{2}}{r_i-v_j+\frac{i}{2}}\,, \tag{3.41b}$$

$$\left(\frac{v_i+\frac{i}{2}}{v_i-\frac{i}{2}}\right)^L=\prod_{j\neq i}^{K_v}\frac{v_i-v_j+i}{v_i-v_j-i}\prod_{j=1}^{K_r}\frac{v_i-r_j-\frac{i}{2}}{v_i-r_j+\frac{i}{2}}\,. \tag{3.41c}$$

As for the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ case, the energy of a state is related to the Bethe roots as in (3.38). A state with excitation numbers K_u, K_r and K_v describe a state in the $\mathfrak{su}(4)$ representation with Dynkin labels $[L-2K_u+K_r, K_u+K_v-2K_r, L-2K_v+K_r]$.

We will now reconstruct a Hamiltonian that produces the spectrum we get from these equations. The Hamiltonian should be acting on three consecutive spins. The tensor product of three alternating spins, where the first spin is at an odd site, decomposes into the representations

$$\mathbf{4}\otimes\bar{\mathbf{4}}\otimes\mathbf{4}=\mathbf{4}\oplus\mathbf{4}\oplus\mathbf{20}\oplus\mathbf{36}\,. \tag{3.42}$$

Starting with an even site, the set of representations appearing in the tensor product is the conjugate of the above. A general $\mathrm{SU}(4)$ invariant Hamiltonian acting on the alternating spin-chain can be written as

$$\begin{aligned}\mathcal{H}=&\bar{\lambda}^2\sum_{l\text{ even}}\left(c_0\mathcal{P}_l^{(\mathbf{36})}+c_1\mathcal{P}_l^{(\mathbf{20})}+c_2\mathcal{P}_l^{(\mathbf{4})}+c_3\mathcal{P}_l^{(\mathbf{4}')}\right)\\&+\bar{\lambda}^2\sum_{l\text{ odd}}\left(\bar{c}_0\mathcal{P}_l^{(\bar{\mathbf{36}})}+\bar{c}_1\mathcal{P}_l^{(\bar{\mathbf{20}})}+\bar{c}_2\mathcal{P}_l^{(\bar{\mathbf{4}})}+\bar{c}_3\mathcal{P}_l^{(\bar{\mathbf{4}}')}\right)\,,\end{aligned} \tag{3.43}$$

where $\mathcal{P}_l^{(R)}$ projects the sites l, $l+1$ and $l+2$ onto the representation R. The prime in the last projector is included to distinguish the two projectors onto the two four-dimensional representations. For simplicity I will assume that the Hamiltonian is parity invariant, and hence that $\bar{c}_i=c_i$.[4] The projectors can be

[4] The same argument goes through without this assumptions, though we have to consider all spin-chain states of length four, not only those corresponding to physical operators.

Table 3.1: The anomalous dimensions of scalar operators with $L = 2$, calculated from the $\mathrm{SU}(4)$ Bethe equations (3.41). These results were obtained in [135].

$\mathrm{SU}(4)$ representation	**84**	**20′**	**15**	**15**	**1**	**1**
Anomalous dimension	$0\bar{\lambda}^2$	$8\bar{\lambda}^2$	$6\bar{\lambda}^2$	$6\bar{\lambda}^2$	$2\bar{\lambda}^2$	$10\bar{\lambda}^2$

expressed in terms of the operators P and K as

$$\begin{aligned}
\mathcal{P}_l^{(\mathbf{36})} &= \mathcal{P}_l^{(\bar{\mathbf{36}})} = \tfrac{1}{2}\left(1 - \tfrac{1}{5}\left(K_{l,l+1} + K_{l+1,l+2}\right)\right)\left(1 + P_{l,l+1}\right) , \\
\mathcal{P}_l^{(\mathbf{20})} &= \mathcal{P}_l^{(\bar{\mathbf{20}})} = \tfrac{1}{2}\left(1 - \tfrac{1}{3}\left(K_{l,l+1} + K_{l+1,l+2}\right)\right)\left(1 - P_{l,l+1}\right) , \\
\mathcal{P}_l^{(\mathbf{4})} &= \mathcal{P}_l^{(\bar{\mathbf{4}})} = \tfrac{1}{10}\left(K_{l,l+1} + K_{l+1,l+2}\right)\left(1 + P_{l,l+1}\right) , \\
\mathcal{P}_l^{(\mathbf{4}')} &= \mathcal{P}_l^{(\bar{\mathbf{4}}')} = \tfrac{1}{6}\left(K_{l,l+1} + K_{l+1,l+2}\right)\left(1 - P_{l,l+1}\right) .
\end{aligned} \tag{3.44}$$

To determine the coefficients c_i, we will use the Bethe equations (3.41) to calculate the energy of a few simple states of length $L = 2$. The length two states fall in the irreducible representations[5]

$$\mathbf{4} \otimes \bar{\mathbf{4}} \otimes \mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{1}^2 \oplus \mathbf{15}^4 \oplus \mathbf{20}' \oplus \mathbf{45} \oplus \bar{\mathbf{45}} \oplus \mathbf{84} . \tag{3.45}$$

Not all of these representations correspond to gauge invariant operators, since they do not satisfy the zero-momentum condition (3.39). In [135], the spectrum of dimension two operators was analysed from the Bethe ansatz. The result is summarized in table 3.1

To analyse the spectrum of the Hamiltonian (3.43), we will start by considering the four operators

$$\mathcal{O}_{\mathbf{84}} = \mathrm{tr}\left(Y^1 Y_4^\dagger Y^1 Y_4^\dagger\right) , \tag{3.46}$$

$$\mathcal{O}_{\mathbf{20}'} = \mathrm{tr}\left(Y^1 Y_4^\dagger Y^2 Y_3^\dagger\right) - \mathrm{tr}\left(Y^1 Y_3^\dagger Y^2 Y_4^\dagger\right) , \tag{3.47}$$

$$\mathcal{O}_{\mathbf{1}^+} = \mathrm{tr}\left(Y^I Y_I^\dagger Y^J Y_J^\dagger\right) + \mathrm{tr}\left(Y^I Y_J^\dagger Y^J Y_I^\dagger\right) , \tag{3.48}$$

$$\mathcal{O}_{\mathbf{1}^-} = \mathrm{tr}\left(Y^I Y_I^\dagger Y^J Y_J^\dagger\right) - \mathrm{tr}\left(Y^I Y_J^\dagger Y^J Y_I^\dagger\right) . \tag{3.49}$$

This choice is very convenient, since each of these operators is annihilated by all but one of the projectors in the Hamiltonian:

$$\begin{aligned}
\mathcal{H}\,\mathcal{O}_{\mathbf{84}} &= 4\bar{\lambda}^2 c_0\, \mathcal{O}_{\mathbf{84}} , \quad \mathcal{H}\,\mathcal{O}_{\mathbf{20}'} = 4\bar{\lambda}^2 c_1\, \mathcal{O}_{\mathbf{20}'} , \\
\mathcal{H}\,\mathcal{O}_{\mathbf{1}^+} &= 4\bar{\lambda}^2 c_2\, \mathcal{O}_{\mathbf{1}^+} , \quad \mathcal{H}\,\mathcal{O}_{\mathbf{1}^-} = 4\bar{\lambda}^2 c_3\, \mathcal{O}_{\mathbf{1}^-} .
\end{aligned} \tag{3.50}$$

[5] The representation **20′** that appears in this tensor product is *not* the same as the **20** in (3.42). For instance, the former is real while the latter is complex.

To reproduce the spectrum in table 3.1, we need to have $c_0 = 0$ and $c_1 = 2$. Since there are two singlets with different energies, there are two possibilities for the coefficients c_2 and c_3:

$$c_2 = \frac{1}{2}\,,\; c_3 = \frac{5}{2}\,, \qquad \text{or} \qquad c_2 = \frac{5}{2}\,,\; c_3 = \frac{1}{2}\,. \tag{3.51}$$

Let us define the corresponding two Hamiltonians

$$\begin{aligned} \mathcal{H}_1 =& \bar{\lambda}^2 \sum_{l \text{ even}} \left(2\mathcal{P}_l^{(\mathbf{20})} + \tfrac{1}{2}\mathcal{P}_l^{(\mathbf{4})} + \tfrac{5}{2}\mathcal{P}_l^{(\mathbf{4'})}\right) \\ &+ \bar{\lambda}^2 \sum_{l \text{ odd}} \left(2\mathcal{P}_l^{(\bar{\mathbf{20}})} + \tfrac{1}{2}\mathcal{P}_l^{(\bar{\mathbf{4}})} + \tfrac{5}{2}\mathcal{P}_l^{(\bar{\mathbf{4}}')}\right)\,, \end{aligned} \tag{3.52}$$

$$\begin{aligned} \mathcal{H}_2 =& \bar{\lambda}^2 \sum_{l \text{ even}} \left(2\mathcal{P}_l^{(\mathbf{20})} + \tfrac{5}{2}\mathcal{P}_l^{(\mathbf{4})} + \tfrac{1}{2}\mathcal{P}_l^{(\mathbf{4'})}\right) \\ &+ \bar{\lambda}^2 \sum_{l \text{ odd}} \left(2\mathcal{P}_l^{(\bar{\mathbf{20}})} + \tfrac{5}{2}\mathcal{P}_l^{(\bar{\mathbf{4}})} + \tfrac{1}{2}\mathcal{P}_l^{(\bar{\mathbf{4}}')}\right)\,. \end{aligned} \tag{3.53}$$

Both $\mathcal{H}_1$ and $\mathcal{H}_2$ give the correct spectrum for the above four operators. Hence, we need an additional operator to distinguish them. A good choice is the operator

$$\mathcal{O}_{\mathbf{15}} = \operatorname{tr}\left(Y^1 Y_4^\dagger Y^I Y_I^\dagger\right) - \operatorname{tr}\left(Y^1 Y_I^\dagger Y^I Y_4^\dagger\right)\,, \tag{3.54}$$

in the adjoint representation **15**. Acting with the two Hamiltonians, we get

$$\mathcal{H}_1\, \mathcal{O}_{\mathbf{15}} = \frac{26\bar{\lambda}^2}{3}\, \mathcal{O}_{\mathbf{15}}\,, \qquad \mathcal{H}_2\, \mathcal{O}_{\mathbf{15}} = 6\bar{\lambda}^2\, \mathcal{O}_{\mathbf{15}}\,. \tag{3.55}$$

Hence the spectrum from the Bethe equations is correctly reproduced only by $\mathcal{H}_2$. To compare this Hamiltonian with the perturbative result of last section, we write out the projectors in terms of the operators P and K using (3.44). The result,

$$\mathcal{H}_2 = \bar{\lambda}^2 \sum_l \left(1 - P_{l,l+2} + \tfrac{1}{2} K_{l,l+1} P_{l,l+2} + \tfrac{1}{2} K_{l+1,l+2} P_{l,l+2}\right)\,, \tag{3.56}$$

exactly agrees with the two-loop dilatation operator (3.32). This shows that the two-loop spin-chain Hamiltonian in the SU(4) sector is integrable.

3.4 More fields

In the last two sections we concentrated on the scalar SU(4) sector of ABJ(M). It is straightforward to conjecture a set of two-loop Bethe equations for the full theory, by applying the construction of Ogievetsky and Wiegmann [144] to the symmetry algebra $\mathfrak{osp}(6|4)$.[6] The resulting equations were given in [135].

In [137] the two-loop Hamiltonian of a spin-chain containing one fermionic excitation, in addition to scalar excitations, was computed. The spectrum of this Hamiltonian exactly agrees with the prediction from the $\mathfrak{osp}(6|4)$ Bethe equations. An extension of the perturbative calculation to the full theory would require a lot of work. Fortunately, we do not need to obtain the full Hamiltonian from perturbation theory to check the validity of the Bethe equations. In [175], the Hamiltonian of the $\mathfrak{osp}(4|2)$ subsector of the theory was shown to possess an extended Yangian symmetry. This guarantees that the Hamiltonian of the sector is integrable.[7] Moreover, the $\mathfrak{osp}(4|2)$ Hamiltonian can be constructed using the R-matrix formalism, and uniquely lifts to the full theory. A similar construction of the $\mathfrak{osp}(4|2)$ dilatation operator from the R-matrix was also performed in [137]. The obtained Hamiltonian agrees with the known perturbative results.

3.5 More loops and the power of $\mathfrak{su}(2|2)$

In [92], a conjectured all-loop extension of the Bethe equations discussed above was put forward. The structure of these equations in many ways mimics the all-loop Bethe equations of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality [36]. As discussed in section 2.8, the spin-chain ground state breaks the $\mathfrak{osp}(6|4)$ symmetry to $\mathfrak{su}(2|2)$. This symmetry is an essential ingredient in the extension of the integrable two-loop Hamiltonian to higher loops.

Two of the most important components in understanding the spectrum of an integrable quantum model are

1. the S-matrix describing the scattering of two excitations; and
2. the dispersion relation relating the momentum of a single excitation to its energy.

The two-particle S-matrix enters the right-hand-side of the Bethe equations, which in turn determines the quantization of the momenta of the excitations. The dispersion relation give us the energy carried by the excitations. As we will now see, both the S-matrix and the dispersion relation for the ABJ(M) spin-chain are highly constrained by the $\mathfrak{su}(2|2)$ symmetry.

THE S-MATRIX. Since the excitations of the ABJ(M) spin-chain can be divided into those sitting on odd (A-particles) and even (B-particles) sites, it is

[6] Only Lie-algebras are treated in detail in [144], but the construction has been shown to work also for many superalgebras, see, *e.g.*, [134, 135]

[7] A similar argument showing the integrability of the one-loop spin-chain Hamiltonian of $\mathcal{N}=4$ super Yang-Mills was given in [64].

natural to split the two-particle S-matrix into three parts: scattering of two A-particles, of two B-particles, and of one particle of each kind. This structure was analyzed in [8]. The $\mathfrak{su}(2|2)$ symmetry forces each of these parts to be proportional to the S-matrix of AdS_5/CFT_4 [40, 21]. The phases of the different terms are constrained by unitarity and crossing symmetry [108]. The authors of [8] found a solution based on the BES dressing phase [38]. The resulting two-particle S-matrix reproduces the all-loop Bethe equations described above [8]. Hence the S-matrix of ABJ(M) is very closely related to that of $\mathcal{N}=4$ super Yang-Mills. The differences between the two theories is in the coupling constant dependence. The S-matrix in [40, 21] depends on the 't Hooft coupling $\lambda_{YM}=g_{YM}^2 N$. In ABJ(M) this coupling is not directly replaced by the $\bar{\lambda}$, but by a function of the couplings $\bar{\lambda}$ and σ

$$\lambda_{YM} \to 4\pi^2 h^2(\bar{\lambda},\sigma)\,. \tag{3.57}$$

An interesting property of the proposed S-matrix of ABJ(M) is that the mixed scattering between A- and B-particles is reflectionless [9, 12]. This has been checked perturbatively at both weak [10] and strong [173] coupling. The S-matrix of ABJ(M) is discussed in more detail in Paper IV and in section 6.5.3.

THE DISPERSION RELATION. As a consequence of $\mathfrak{su}(2|2)$ symmetry, the dispersion relation of a magnon with R-charge Q is given by [40]

$$\epsilon(p)=\sqrt{Q^2+4h^2(\bar{\lambda},\sigma)\sin^2\frac{p}{2}}\,. \tag{3.58}$$

Note again the appearance of the function $h^2(\bar{\lambda},\sigma)$, replacing the simple dependence on λ_{YM} in $\mathcal{N}=4$ SYM. For the fundamental magnon in ABJM $Q=1/2$, while the magnon of $\mathcal{N}=4$ SYM has $Q=1$. The magnons of the two theories transform in the same representation of $SU(2|2)$. The different values of Q for the dispersion relations can be traced back to a difference in the way the central charge $\Delta-J$ of $\mathfrak{su}(2|2)$ is embedded in $\mathfrak{osp}(6|4)$ and $\mathfrak{psu}(2,2|4)$.

Both the S-matrix and magnon dispersion relation of ABJ(M) depend on the coupling constants of the theory through the function $h^2(\bar{\lambda},\sigma)$. Also the Bethe equations depend on the couplings via this function. Hence, an anomalous dimension calculated from these equations is given as a function of h, and not of λ and $\hat{\lambda}$. To express such results in terms of the parameters of the action, we need to determine the form of $h^2(\bar{\lambda},\sigma)$.

By comparing the general form (3.58) with the weak coupling result (3.36) we see that the leading expansion of $h^2(\bar{\lambda},\sigma)$ is given by

$$h^2(\bar{\lambda},\sigma) = \bar{\lambda}^2 + \mathcal{O}(\bar{\lambda}^4)\,. \tag{3.59}$$

In chapter 6 we will see that the strong coupling behavior is different,

$$h^2(\bar{\lambda},\sigma) = \frac{\bar{\lambda}}{2} + \mathcal{O}(1)\,. \tag{3.60}$$

Hence, there is a non-trivial change in the coupling dependence between weak and strong coupling. The next chapter, as well as Paper I, Paper III and Paper II, is devoted to the perturbative calculation of the next term in the above expansion.

4. The four-loop dispersion relation of ABJ(M)

In the last chapter we saw that both the magnon dispersion relation and the two-particle S-matrix have a non-trivial dependence on the coupling constants $\bar{\lambda}$ and σ, parametrized by the function $h^2(\bar{\lambda},\sigma)$. Since only Feynman diagrams with an even number of loops are divergent in three dimensions, the next-to-leading correction to the dilatation operator will appear at four loops. Provided the ABJ(M) spin-chain is integrable, these corrections should be captured by the next term in the weak coupling expansion of the function $h^2(\bar{\lambda},\sigma)$.

In Paper I and Paper II this four-loop correction to $h^2(\bar{\lambda},\sigma)$ was calculated using perturbation theory. The perturbative calculations were performed both in a component formalism (Paper I) and using $\mathcal{N}=2$ superfields (Paper II) and result in the expansion

$$h^2(\bar{\lambda},\sigma) = \bar{\lambda}^2 - \bar{\lambda}^4(4+\sigma^2)\zeta_2 + \mathcal{O}(\bar{\lambda}^6)\,. \tag{4.1}$$

The four-loop coefficient of this expansion has a few interesting properties:

- Both terms in (4.1) are proportional to ζ_2. In general we would expect a four-loop result to also contain a rational part. Such terms appear in individual diagrams but cancel, leaving a final result of "maximal transcendentality". In the component calculation this cancelation seems almost miraculous, but in the superspace calculation there is a correlation between the rational coefficient of single and double poles of each diagram. Renormalization requires the double poles to cancel, which also removes the rational part of the single poles that contribute to the final result.
- Both coefficients are integer valued. Individual diagrams contain terms where ζ_2 is multiplied by various rational numbers, but in the final result these add up to integers.
- The result (4.1) is even in σ and hence preserves parity. From both the gauge theory side and string theory side of the AdS_4/CFT_3 duality we expect parity to be broken in the ABJ model where $\sigma \neq 0$. Moreover, in [55] it was shown that the non-planar corrections to the two-loop dilatation operator of ABJ does not preserve parity. It remains to see if parity is an accidental symmetry of the planar four-loop operator or remains preserved at higher loops.

- The result for $h^2(\bar{\lambda},\sigma)$ up to four loops can be written in a particularly simple form if we rewrite it in terms of the couplings λ and $\hat{\lambda}$,

$$h^2(\bar{\lambda},\sigma) = \lambda\hat{\lambda}(1-(\lambda+\hat{\lambda})^2\zeta_2) + \mathcal{O}(\bar{\lambda}^6)\,. \tag{4.2}$$

  At higher loops each term appearing in the expansion of $h^2(\bar{\lambda},\sigma)$ should contain a factor $\lambda\hat{\lambda}$ times some combination of the two couplings. The reason for this structure is that in the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ sector all diagrams with non-trivial flavor structure involves at least three neighboring fields. Hence the color loops in the diagram will always give a factor NM times some combination of N and M. In terms of the couplings this translates to an overall factor of $\lambda\hat{\lambda}$. However, there does not seem to be a reason to expect the rest of the coupling dependence to combine into powers of $\lambda+\hat{\lambda}$.

The rest of this chapter contains some comments on various aspects of the perturbative calculations presented in Paper I and Paper II.

4.1 Extraction of $h^2(\bar{\lambda},\sigma)$

To find the magnon dispersion relation it is enough to focus on the closed $\mathrm{SU}(2)\times\mathrm{SU}(2)$ sector of ABJ(M). In this sector there are two different types of magnons – corresponding to the two factors of $\mathrm{SU}(2)$ – which sit at even and odd sites of the spin-chain, respectively. From the $\mathrm{SU}(2|2)$ symmetry it follows that the dispersion relation of a magnon sitting at an odd site should have the form

$$\epsilon_{\mathrm{odd}}(p) = \sqrt{\frac{1}{4} + 4h^2(\bar{\lambda},\sigma)\sin^2\frac{p}{2}}\,. \tag{4.3}$$

The dispersion relation of the even magnon has the same form, but with the parameter σ replaced by $-\sigma$:

$$\epsilon_{\mathrm{even}}(p) = \epsilon_{\mathrm{odd}}(p)|_{\sigma\to-\sigma}\,, \tag{4.4}$$

This difference is easy to understand in a perturbative calculation, where we get the diagrams contributing to ϵ_{even} by taking the diagrams for ϵ_{odd} and shifting them one spin-chain site. This shift corresponds to an exchange of the couplings λ and $\hat{\lambda}$, and hence to a change in the sign of σ. This relation is discussed in [138].

The expansion of $h^2(\bar{\lambda},\sigma)$ can be written as

$$h^2(\bar{\lambda},\sigma) = \bar{\lambda}^2 + \bar{\lambda}^4 h_4(\sigma) + \mathcal{O}(\bar{\lambda}^6)\,. \tag{4.5}$$

Inserting this into (4.3) gives

$$\epsilon_{\mathrm{odd}}(p) = \frac{1}{2} + 4\bar{\lambda}^2\sin^2\frac{p}{2} - \bar{\lambda}^4\left(16\sin^4\frac{p}{2} - 4h_4(\sigma)\sin^2\frac{p}{2}\right) + \mathcal{O}(\bar{\lambda}^6)\,. \tag{4.6}$$

We also expand the dilatation operator

$$\mathfrak{D} = \mathfrak{D}^{(0)} + \bar{\lambda}^2 \mathfrak{D}^{(2)} + \bar{\lambda}^4 \mathfrak{D}^{(4)} + \mathcal{O}(\bar{\lambda}^6) . \tag{4.7}$$

Each term in this expansion can further be split into parts acting on a number of sites with the first site being odd or even

$$\mathfrak{D}^{(l)} = \mathfrak{D}^{(l)}_{\text{odd}} + \mathfrak{D}^{(l)}_{\text{even}} . \tag{4.8}$$

The only flavor structures that can appear in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector dilatation operator are permutations. In Paper I and Paper II we use the notation

$$\{a_1, a_2, \ldots, a_m\} = \sum_{l=1}^{L} P_{2l+a_1, 2l+a_1+2} P_{2l+a_2, 2l+a_2+2} \cdots P_{2l+a_m, 2l+a_m+2} , \tag{4.9}$$

where the operator $P_{a,b}$ is the two-site permutation operator used in the last chapter. Note that if all the coefficients a_i are odd (even) the above structure only contains permutations acting on odd (even) sites. At four loops the dilatation operator acts on at most five consecutive sites. However, due to supersymmetry we know that the anomalous dimension of any chiral primary operator should vanish. Hence we can restrict ourselves to combinations of permutations that annihilate such operators. Such structures, or *chiral functions*, were used in the four-loop calculation of wrapping interactions in $\mathcal{N}=4$ SYM [70, 71], and in the superspace calculation in Paper II. Here we will need the structures

$$\chi(a,b) = \{a,b\} - \{a\} - \{b\} + \{\} , \qquad \chi(a) = \{a\} - \{\} , \tag{4.10}$$

in terms of which the two-loop dilatation operator can be written as

$$\mathfrak{D}^{(2)} = -\bar{\lambda}^2 (\chi(1) + \chi(2)) . \tag{4.11}$$

This has the same form as the Hamiltonian of the Heisenberg spin-chain

The dispersion relation (4.6) can be reproduced by acting with the dilatation operator (4.7) on the single magnon state[1]

$$\mathcal{O}_p = \sum_{l=0}^{L} e^{ipl} (Y^1 Y_4^\dagger)^l (Y^2 Y_4^\dagger)(Y^1 Y_4^\dagger)^{L-l-1} . \tag{4.12}$$

provided

$$\begin{aligned} \mathfrak{D}^{(2)}_{\text{odd}} &= -\chi(1) , \\ \mathfrak{D}^{(2)}_{\text{even}} &= -\chi(2) , \\ \mathfrak{D}^{(4)}_{\text{odd}} &= -\chi(1,3) - \chi(3,1) + (2 - h_4(\sigma))\chi(1) , \\ \mathfrak{D}^{(4)}_{\text{even}} &= -\chi(2,4) - \chi(4,2) + (2 - h_4(-\sigma))\chi(2) . \end{aligned} \tag{4.13}$$

[1] The singel magnon operator for a spin-chain of length L, $\mathcal{O}_p$, is not gauge invariant unless $p = 0$. Still we can consider it as a state of the spin-chain.

We can now extract the function $h_4(\sigma)$ by comparing the results of the perturbative calculations with (4.13).

Note that the above dilatation operator does not contain any flavor structures involving both odd and even sites. Hence the two types of magnons are decoupled at this order of perturbation theory. This feature is explicitly verified in Paper I and Paper II, and was also previously noted in [32]. From the Bethe equation we expect the magnons to interact through the dressing phase which appears at eight loops [92].

4.2 Comments on the perturbative calculations

The perturbative calculations of $h_4(\sigma)$ using component and superspace Feynman diagrams are presented in Paper I and Paper II, respectively.[2] Here I will not go into the full details of the calculations, but only discuss a few central points.

COMPONENTS VERSUS SUPERSPACE. A big advantage of performing the calculation of renormalization factors in superspace is that the Feynman diagrams behave better in the ultraviolet. This greatly reduces the number of diagrams that we need to consider. For example, the component calculation in Paper I involves on the order of 100 diagrams, while 15 diagrams are needed in the superspace calculation in Paper II.

However, the better ultraviolet behavior of the superspace diagrams comes at the expense of worse behavior in the infrared. In the component calculation all infrared divergences can be avoided by an insertion of non-zero external momentum in two vertices. In the superspace calculation there are diagrams that contain both ultraviolet and infrared divergences. To ensure that the final result is free of infrared divergences we need to include extra diagrams that are finite in the ultraviolet region but divergent in the infrared, thus canceling the overall infrared divergence. Another possibility for taking care of the infrared divergences would be to use a non-standard gauge fixing procedure. For the ABJ(M) case this was analysed in [122], based on a four-dimensional procedure introduced in [2]. While this would get rid of the infrared divergences, it would complicate the Feynman diagrams involving gauge fields.

RENORMALIZATION SCHEME. In both Paper I and Paper II local operators are renormalized using minimal subtraction [164]. To perform the regularization

[2] Reference [138] gives the full details of the component calculation, including the calculation of the involved scalar integrals. Paper I contains the results for the relevant Feynman diagrams, as well as a discussion of the final results.

we use the BPH procedure, in which we take care of the subdivergences of each diagram in order to extract only the overall divergence of the diagram [49, 101, 174].[3] This means that we never need to introduce explicit counter terms in the action. In particular we do not distinguish between subdivergences appearing at a vertex and at a local operator. The ABJ(M) model is conformal and therefore has zero beta-function. Hence all subdivergences related to a particular vertex should cancel in the final result. As a check of our calculation we show in Paper I that the two-loop beta-function of the six-vertex indeed vanishes.

FLAVOR STRUCTURES. In order to reduce to number of diagrams we can make use of supersymmetry. The dimension of a chiral primary does not receive any quantum corrections. Hence the four-loop corrections to the dilatation operator should annihilate such states. In the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector the only flavor structures that appear in the dilatation operator are permutations. In particular there is a term proportional to the trivial permutation, *i.e.*, the identity operator. In the component formalism there are many Feynman diagrams that contribute to this term. However, we do not need to calculate these diagrams explicitly. Instead we calculate the part of the dilatation operator that involves non-trivial permutations, and act with it on a chiral primary. The condition that the total four-loop result vanishes for such operators allows us to determine the coefficient of the identity term.

Due to the structure of the superpotential, chiral primary operators are annihilated by each individual diagram in the superspace calculation. Hence we automatically get the identity part when we calculate the non-trivial permutations in this formalism.

COUPLING DEPENDENCE. The ABJ model contains two ’t Hooft coupling constants, $\lambda = N/k$ and $\hat{\lambda} = M/k$. In a diagrammatic calculation the factors of N and M that contribute to these couplings come from color traces in the diagrams. In a planar theory these factors can be read off directly from each diagram. Each face of the diagram gives a factor N or M. Any two loops that share a matter propagator will give rise to a factor NM, since the matter fields of ABJ transform in the bifundamental representation of the gauge group and the color traces in the two loops are taken over the fundamental representation of $\mathrm{U}(N)$ and $\mathrm{U}(M)$, respectively. On the other hand, if two loops share a gauge propagator they will give a factor of either N^2 or M^2, depending on which gauge boson appears in the loops.

[3] See [61] for a nice introduction to the BPH procedure.

Cancelation of double poles. In dimensional reduction we perform all integrals in $D = 3 - 2\epsilon$ dimensions and extract the dilatation operator from the renormalization of the local operators using the prescription

$$\delta\mathfrak{D} = \lim_{\epsilon \to 0} 2\epsilon \frac{d \log \mathcal{Z}(\bar{\lambda}, \epsilon)}{d \log \bar{\lambda}} . \tag{4.14}$$

For this limit to be convergent, $\log \mathcal{Z}$ cannot contain higher poles in $1/\epsilon$. Up to four loops $\log \mathcal{Z}$ has the expansion

$$\log \mathcal{Z} = \bar{\lambda}^2 \mathcal{Z}_2 + \bar{\lambda}^4 \left(\mathcal{Z}_4 - \frac{1}{2} \mathcal{Z}_2^2 \right) + \mathcal{O}(\bar{\lambda}^6) . \tag{4.15}$$

The two-loop result $\mathcal{Z}_2$ contains only single poles while $\mathcal{Z}_4$ contains both single and double poles. From the above expression we see that the double poles are not arbitrary, but should be proportional to the square of the two-loop result. We can use this requirement as a consistency check on the four-loop calculation. For the superspace calculation this is presented in Paper II. The check for the component calculation is not included in Paper I, but in the extended paper [138].

Note that $\mathcal{Z}_2$ and $\mathcal{Z}_4$ are operators acting on the spin-chain. Hence the above equation should be read as an operator equation and the term quadratic in $\mathcal{Z}_2$ should be interpreted as the result of acting twice with $\mathcal{Z}_2$ on a spin-chain state.

4.3 Wrapping interactions

The range of the spin-chain Hamiltonian grows with each order in perturbation theory. The two-loop Hamiltonian presented in the last chapter acts on three consecutive spin-chain sites, and the four-loop corrections discussed in this chapter involve up to five neighboring sites. This means that we need to be careful if we want to calculate the anomalous dimension of short operators. Consider for example the operator

$$\mathcal{O}_{\mathbf{20}} = \operatorname{tr} Y^{[1} Y^{\dagger}_{[4} Y^{2]} Y^{\dagger}_{3]} , \tag{4.16}$$

of length four and in the **20** representation of $\mathrm{SU}(4)$. This operator is in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector, but since it is made of only four fields we need to be careful when applying the four-loop dilatation operator $\mathfrak{D}^{(4)}$ to it. Some of the Feynman diagrams contributing to $\mathfrak{D}^{(4)}$ involve five neighboring fields, but when we consider $\mathcal{O}_{\mathbf{20}}$ these diagrams should not be included. On the other hand we need to consider additional diagrams called the *wrapping diagrams* [37, 157]. These are diagrams that would be non-planar if the operator had length larger than four, but become planar for short operators.

The calculations in Paper I and Paper II show that the anomalous dimension of $\mathcal{O}_{\mathbf{20}}$ is[4]

$$\gamma_{\mathbf{20}} = 8\bar{\lambda}^2 - (48\zeta_2 + 8\zeta_2\sigma^2)\bar{\lambda}^4 + \mathcal{O}(\bar{\lambda}^4)\,. \tag{4.17}$$

This result agrees with a prediction by Gromov, Kazakov, and Vieira [94], provided we plug in the expansion of $h^2(\bar{\lambda},\sigma)$ from (4.1) in their result. These authors find their prediction from the Thermodynamic Bethe ansatz (TBA), which is based on the asymptotic Bethe equations [92]. The TBA was originally introduced in the AdS$_5$/CFT$_4$ case in [15, 16]. Using the string hypothesis for the mirror theory [19], the TBA can be formulated as a set of difference equations, known as the Y-system [95, 51, 18, 52, 89], which can efficiently be solved order by order.

4.4 An interesting limit.

The appearance of two coupling constants in the ABJ limit enables us to explore various limits of the theory. In particular in Paper I we discuss the case

$$\hat{\lambda} \ll \lambda \ll 1\,. \tag{4.18}$$

In this limit we can keep only the leading order terms in $\hat{\lambda}$ and perform a perturbative expansion in λ. We can then consider the rescaled dilatation operator

$$\mathcal{D} = \frac{1}{\hat{\lambda}}\delta\mathfrak{D} = -\lambda(1-\zeta_2\lambda^2)(\chi(1)+\chi(2)) + \mathcal{O}(\lambda^5)\,. \tag{4.19}$$

Note that $\mathcal{D}$ has the form of the Heisenberg $XXX_{1/2}$ spin-chain Hamiltonian and that the effect of the four-loop result is captured by the overall normalization of $\mathcal{D}$ without any new interactions appearing. In particular it only acts on three sites in the spin-chain at a time. From the structure of the Feynman diagrams we expect this structure to continue to higher loops. For a diagram to be proportional to $\hat{\lambda}$, as opposed to higher powers of $\hat{\lambda}$, it can involve at most three neighboring fields. The only flavor structures that can appear is then the identity operator and next-to-neighbor permutations. As discussed above, supersymmetry fixes the coefficient of the identity term to be minus that of the permutation. Hence we end up with a result of the form

$$\mathcal{D} = -\lambda f(\lambda)(\chi(1)+\chi(2))\,. \tag{4.20}$$

Since this Hamiltonian has range three there are no wrapping interactions for states with four or more sites even at higher orders in λ. Moreover, since

[4] Note that the result of the wrapping diagrams by themselves differ in the two calculation, while the results for the physical dimension $\gamma_{\mathbf{20}}$ agree. The reason is that the contribution to the asymptotic dilatation operator from diagrams of range five is different in the two formalisms.

$\mathcal{D}$ is proportional to the $XXX_{1/2}$ Hamiltonian the system is integrable to any order of perturbation theory.

At first sight the absence of wrapping interactions seems to be a very strong result, since the asymptotic Bethe equations now give the full answer. However, the couplings λ and $\hat{\lambda}$ can not take arbitrary independent values. As noted in chapter 2 consistency of the ABJ model requires $|\lambda - \hat{\lambda}| \leq 1$. At weak coupling this still allows us to take the limit (4.18), but at strong coupling λ and $\hat{\lambda}$ are always required to be of the same order. Hence it is not consistent to take the limit $\hat{\lambda} \ll \lambda$ when λ and $\hat{\lambda}$ are large and the absence of wrapping corrections is a purely weak coupling result.

5. String theory on $AdS_4 \times \mathbb{CP}^3$

Planar ABJM theory is dual to free type IIA string theory in a background of $AdS_4 \times \mathbb{CP}^3$. A nice way to understand this background is from M-theory, where it arises in the low energy limit of the world-sheet theory on a stack of N M2-branes at a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold point. This M-theory description clarifies the origin of the parameters N and k and explains some of the numerical factors that appear in the string theory formulation. However, since the focus of this thesis is on the integrable aspects of the AdS_4/CFT_3 duality, which on the gravity side only appear in the string theory limit, I will first give a brief introduction to type IIA string theory in $AdS_4 \times \mathbb{CP}^3$, and come back to the M-theory picture in section 5.5.

5.1 The background

Four-dimensional anti-de Sitter space is a solution to the equations of motion of Einstein gravity in vacuum with a negative cosmological constant. The space has a constant negative curvature and can be embedded as $AdS_4 \subset \mathbb{R}^{2,3}$ by imposing the constraint

$$\vec{z}^2 = z_\mu z^\mu = -(z^{-1})^2 - (z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2 = -1\,, \tag{5.1}$$

on the embedding coordinates $\vec{z}$. We can solve this constraint by

$$\begin{gathered} z^{-1} = \cosh\rho\,\sin t\,, \quad z^0 = \cosh\rho\,\cos t\,, \\ z^1 = \sinh\rho\,\sin\phi\,\sin\theta\,, \quad z^2 = \sinh\rho\,\cos\phi\,\sin\theta\,, \quad z^3 = \sinh\rho\,\cos\theta\,, \end{gathered} \tag{5.2}$$

which gives the metric

$$ds^2_{AdS_4} = -\cosh^2\rho\,dt^2 + d\rho^2 + \sinh^2\rho(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{5.3}$$

The coordinates take values[1] $-\infty < t < \infty$, $0 \le \rho < \infty$, $0 \le \theta < \pi$ and $0 \le \phi < 2\pi$.

[1] Note that the time direction t appears to be compact in these coordinates. We will always consider AdS_4 to be the *universal cover* of this space, where t can take on any real value.

To obtain the complex projective space $\mathbb{CP}^n$ we consider $\mathbb{C}^{n+1}\setminus\{0\}$, which we write in terms of the coordinates

$$\vec{y}=(y^1,y^2,\ldots,y^{n+1})\in\mathbb{C}^{n+1}\setminus\{0\}\,, \tag{5.4}$$

and identify points that are related by multiplication by an overall constant.

$$(y^1,y^2,\ldots,y^{n+1})\cong(\kappa y^1,\kappa y^2,\ldots,\kappa y^{n+1})\,,\quad \kappa\in\mathbb{C}\setminus\{0\} \tag{5.5}$$

By a suitable choice of κ we can rescale the coordinates $\vec{y}$ so that

$$|\vec{y}|^2\equiv y^1\bar{y}_1+y^2\bar{y}_2+\cdots+y^{n+1}\bar{y}_{n+1}=1\,. \tag{5.6}$$

This defines an S^{2n+1} submanifold of $\mathbb{C}^{n+1}$. To get $\mathbb{CP}^n$ we still need to identify points related by an overall phase. Hence we can write $\mathbb{CP}^n$ as the quotient $\mathrm{S}^{2n+1}/\mathrm{S}^1$, where S^1 indicates simultaneous rotation of all coordinates y^I by the same phase. $\mathbb{CP}^n$ is a *Kähler manifold* [141], with Kähler metric

$$ds^2_{\mathbb{CP}^n}=\frac{dy^I d\bar{y}_I}{|\vec{y}|^2}-\frac{|y^I d\bar{y}_I|^2}{|\vec{y}|^4}\,. \tag{5.7}$$

This is called the *Fubini-Study metric* of $\mathbb{CP}^n$. The corresponding Kähler form is given by

$$d\omega=i\,d\left(\frac{y^I}{|y|}\right)\wedge d\left(\frac{\bar{y}_I}{|y|}\right) \tag{5.8}$$

In the case of $\mathbb{CP}^3$ we can parametrize y^I in terms of the six angles $0\le\xi<\frac{\pi}{2}$, $0\le\eta<2\pi$, $0\le\vartheta_i<\pi$ and $0\le\varphi_i<2\pi$, with $i=1,\,2$, as

$$\begin{aligned}
y^1&=\sin\xi\,\cos\tfrac{\vartheta_2}{2}\,e^{+i\varphi_2/2}\,e^{-i\eta/2}\,,\\
y^2&=\cos\xi\,\cos\tfrac{\vartheta_1}{2}\,e^{+i\varphi_1/2}\,e^{+i\eta/2}\,,\\
y^3&=\cos\xi\,\sin\tfrac{\vartheta_1}{2}\,e^{-i\varphi_1/2}\,e^{+i\eta/2}\,,\\
y^4&=\sin\xi\,\sin\tfrac{\vartheta_2}{2}\,e^{-i\varphi_2/2}\,e^{-i\eta/2}\,.
\end{aligned} \tag{5.9}$$

The metric in these coordinates is given by

$$\begin{aligned}
ds^2_{\mathbb{CP}^3}=d\xi^2&+\frac{1}{4}\sin^2 2\xi\left(d\eta+\frac{1}{2}\cos\vartheta_1\,d\varphi_1-\frac{1}{2}\cos\vartheta_2\,d\varphi_2\right)^2\\
&+\frac{1}{4}\cos^2\xi\left(d\vartheta_1^2+\sin^2\vartheta_1\,d\varphi_1^2\right)+\frac{1}{4}\sin^2\xi\left(d\vartheta_2^2+\sin^2\vartheta_2\,d\varphi_2^2\right)\,,
\end{aligned} \tag{5.10}$$

and the Kähler form by

$$d\omega=\frac{1}{2}d\left(\cos 2\xi\,d\eta+\cos^2\xi\,\cos\vartheta_1\,d\varphi_1+\sin^2\xi\,\cos\vartheta_2 d\varphi_2\right)\,. \tag{5.11}$$

The angles ϑ_i and φ_i parametrize two two-spheres inside $\mathbb{CP}^3$. We will discuss various other submanifolds in chapter 6.

The full metric of the background is now

$$ds^2 = \frac{R^2}{4} ds^2_{\mathrm{AdS}_4} + R^2 ds^2_{\mathbb{CP}^3}\,, \tag{5.12}$$

where R is the $\mathbb{CP}^3$ radius. Note that the AdS radius is half of that of $\mathbb{CP}^3$. The radius is related to the integer parameters N and k as

$$R^2 = \sqrt{\frac{32\pi^2 N}{k}} = 4\pi\sqrt{2\lambda}\,, \tag{5.13}$$

where we again have introduced the 't Hooft coupling $\lambda = N/k$.

Isometries. From the form of the metric on AdS_4 and $\mathbb{CP}^3$ in (5.3) and (5.10) it is clear that this background admits five Killing vectors

$$\Delta = -i\partial_t\,, \quad S = -i\partial_\phi\,, \quad Q = -i\partial_{\varphi_1}\,, \quad J = -i\partial_{\varphi_2}\,, \quad J_3 = -i\partial_\eta\,. \tag{5.14}$$

These generators make up the Cartan subalgebra of the isometries of the background. The notation for the charges has been chosen to match the notation for the $\mathrm{OSp}(6|4)$ charges in chapter 2. We can read off the full isometry from the embedding $\mathrm{AdS}_4 \times \mathbb{CP}^3 \subset \mathbb{R}^{2,3} \times \mathbb{C}^4$. The conditions in (5.1) and (5.6) are preserved by $\mathrm{SO}(3,2)$ and $\mathrm{SU}(4)$ rotations of the coordinates, respectively.

From (5.14) we see that the charge Δ gives the *space-time energy* of a string state. In the AdS/CFT correspondence this is dual to the dimension of the corresponding operator on the gauge theory side. In chapter 2 we also introduced the spin-chain energy $\Delta - J$. In the string theory this charge plays the role of the *world-sheet Hamiltonian*, provided we use light-cone gauge.

In addition to the isometries the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background is preserved by 24 supersymmetries [142]. Together these charges generate the supergroup $\mathrm{OSp}(6|4)$.

The dilaton and string coupling. The background contains a constant dilaton, which can be expressed in terms of the parameters of the theory as

$$e^\phi = \frac{R}{k} = \left(\frac{32\pi^2 N}{k^5}\right)^{1/4} = \frac{\sqrt{\pi}}{N}(2\lambda)^{5/4}\,. \tag{5.15}$$

Since the string coupling is given by the dilaton, $g_s = e^\phi$, we note that the planar limit $N, k \to \infty$ with λ fixed (see also (2.47)), is a weak coupling limit on the string theory side. To leading order in $1/N$, we only need to consider *freely* propagating strings. Note that this still is a highly non-trivial problem, especially when the coupling λ is small and hence the curvature of the background large.

If we do not consider the planar limit, but let k be a small number, the theory becomes strongly coupled for any value of N. But strongly coupled IIA string theory has a more natural description in terms of M-theory. Hence, the above expression for the dilaton gives us a hint about the M-theory origin of the theory. This will be explored in more detail in section 5.5.

FIELD STRENGTH FLUXES. In addition to the metric and the dilaton, the background contains non-trivial Ramond–Ramond field strength fluxes. There are N units of four-form flux $F^{(4)}$ through AdS_4, with $F^{(4)}$ proportional to the volume form,

$$F^{(4)} = \frac{3kR^2}{8}\epsilon_{\mathrm{AdS}_4} = \frac{3\pi N}{\sqrt{2\lambda}}\epsilon_{\mathrm{AdS}_4}\,. \tag{5.16}$$

The two-form $F^{(2)}$ wraps a $\mathbb{CP}^1$ inside $\mathbb{CP}^3$ and is proportional to the Kähler form

$$F^{(2)} = k\,d\omega\,. \tag{5.17}$$

There is also a non-vanishing Neveu-Schwarz–Neveu-Schwarz B-field, which will be discussed in section 5.4.

5.2 The Green-Schwarz action

So far we have only discussed the bosonic sectors of the string theory. The $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background contains non-trivial Ramond–Ramond fields, and hence it is useful to describe the full string theory by a Green-Schwarz action [83]. The Green-Schwarz action for IIA superstrings in an arbitrary background was written down in [87], but the general expression is very complicated to work with, since it contains terms up to 32$^{\mathrm{nd}}$ order in the fermions. Still, some progress has been made in the formulation of the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ case [81, 82], see also [160].

The Green-Schwarz action contains two Majorana-Weyl fermions, with a total of 32 degrees of freedom, but half of these are unphysical, and can be gauged away by gauge fixing the κ-symmetry. Hence we end up with 16 physical fermionic degrees of freedom.

5.3 The coset sigma-model

The manifold AdS_4 is equivalent to the coset $\mathrm{SO}(3,2)/\mathrm{SO}(3,1)$. Similarly, $\mathbb{CP}^3$ is the coset $\mathrm{SO}(6)/\mathrm{U}(3)$. Since $\mathrm{SO}(3,2)\times\mathrm{SO}(6)$ is the bosonic subgroup of $\mathrm{OSp}(6|4)$ it is natural to consider the super-coset [17, 161]

$$\frac{\mathrm{G}}{\mathrm{H}_0}=\frac{\mathrm{OSp}(6|4)}{\mathrm{SO}(3,1)\times\mathrm{U}(3)}\,, \tag{5.18}$$

in close analogy to the description of superstrings in $\mathrm{AdS}_5\times\mathrm{S}^5$ by the coset $\mathrm{PSU}(2,2|4)/\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ [132].

The supergroup $\mathrm{OSp}(6|4)$ is *semi-symmetric*, *i.e.*, it is invariant under a $\mathbb{Z}_4$ automorphism. This invariance allows us to introduce a grading of the generators of the algebra

$$\mathfrak{osp}(6|4)=\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\mathfrak{h}_2\oplus\mathfrak{h}_3\,. \tag{5.19}$$

This grading is consistent with the commutation relations

$$[\mathfrak{h}_m,\mathfrak{h}_n\}\subset\mathfrak{h}_{(m+n)\ \mathrm{mod}\ 4}\,. \tag{5.20}$$

The elements of $\mathfrak{h}_l$ have eigenvalue i^l under the automorphism. The bosonic subalgebra lives in $\mathfrak{h}_0\oplus\mathfrak{h}_2$, while the generators in $\mathfrak{h}_1\oplus\mathfrak{h}_3$ are fermionic. The $\mathbb{Z}_4$-invariant subgroup H_0 coincides with the denominator of the coset (5.18).

The main object in the coset sigma-model are the Lie-algebra valued currents

$$j=g^{-1}dg=j^{(0)}+j^{(1)}+j^{(2)}+j^{(3)}\,. \tag{5.21}$$

Here g is a group element of $\mathrm{OSp}(6|4)$. On the right hand side in the above equation j has been decomposed into components of definite $\mathbb{Z}_4$ charge, *i.e.*, $j^{(n)}\in\mathfrak{h}_n$. By construction j is a flat current

$$dj+j\wedge j=0\,. \tag{5.22}$$

The coset sigma-model action can now be written as [17, 161]

$$\mathcal{S}=-\sqrt{2\lambda}\int d\sigma\,d\tau\,\mathrm{STr}\left[\sqrt{-h}h^{\alpha\beta}j^{(2)}_\alpha j^{(2)}_\beta+\epsilon^{\alpha\beta}j^{(1)}_\alpha j^{(3)}_\beta\right]\,. \tag{5.23}$$

This action possesses a gauge symmetry

$$g(\sigma,\tau)\to g(\sigma,\tau)h(\sigma,\tau)\,, \tag{5.24}$$

where $h(\sigma,\tau)$ takes values in H_0. It is also invariant under global multiplication from the left with any group element of $\mathrm{OSp}(6|4)$.

The group $\mathrm{OSp}(6|4)$ contains 24 odd generators, corresponding in the coset sigma-model to 24 fermionic degrees of freedom. This means that it can not be equivalent to the Green-Schwarz action, which contains 32 fermions. However, like the Green-Schwarz action, the coset sigma-model is invariant under a fermionic κ-symmetry. By gauge fixing this symmetry, we get rid of eight of the fermions. Hence, the coset sigma-model also contains 16 physical degrees of freedom [17, 161].

For generic string configurations, the Green-Schwarz action and the coset sigma-model are equivalent. However, for certain singular backgrounds, such as a string moving only in AdS_4, the rank of the κ-gauge symmetry is enhanced from eight to twelve [17]. For such classical solutions, the coset sigma-model only contains twelve physical fermions, and hence can not be used in a semi-classical quantization of the string.[2]

5.3.1 The algebraic curve

We will now concentrate on the bosonic part of the coset sigma-model, so that j takes values in $\mathfrak{h}_2$. The equations of motion following from (5.23) are

$$d * j = 0\,. \tag{5.25}$$

The flatness condition (5.22) for j together with (5.25) implies that the Lax connection

$$L(x) = \frac{1}{1-x^2}\, j + \frac{x}{1-x^2} * j\,, \quad x \in \mathbb{C}\,, \tag{5.26}$$

is flat for any value of the parameter x, known as the *spectral parameter*.

Using $L(x)$ we define the *monodromy matrix*

$$\Omega(x) = P\exp \int d\sigma\, L_\sigma(x)\,, \tag{5.27}$$

where we integrate over a closed loop at constant τ. Since $L(x)$ is flat, the eigenvalues of $\Omega(x)$ are independent of τ for any x. Hence the eigenvalues of $\Omega(x)$ generate an infinite set of conserved charges, showing the classical integrability of the model [17, 91, 161].[3]

Let us denote the eigenvalues for the $\mathbb{CP}^3$ part of $\Omega(x)$ by $e^{i\tilde{p}_i}$, $i = 1,\ldots,4$ and those for the AdS_4 part as $e^{i\hat{p}_i}$, $i = 1,\ldots,4$.[4] The functions $\tilde{p}_i$ and $\hat{p}_i$

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2 Note that the folded string spinning in AdS_4 provides an example of such a singular string background. In [13] it was shown that the correct one-loop result can be obtained from the coset sigma-model by considering a more general solution with a non-zero angular momentum J on $\mathbb{CP}^3$ and taking the smooth $J \to 0$ limit. See also [130, 120, 90, 131].

3 The construction of an infinite tower of conserved charges can be straightforwardly extended to the full $\mathrm{OSp}(6|4)$ sigma-model [17, 91, 161]. In [159], a Lax connection for the AdS_4 sector of the Green-Schwarz action, which is not contained in the coset, was constructed, indicating that the classical integrability extends to the full string theory.

4 In addition to the eigenvalues $e^{i\hat{p}_i}$ the AdS_4 part of $\Omega(x)$ always has a fifth eigenvalue of value one.

are referred to as *quasimomenta*. The quasimomenta are not independent but satisfies [91]

$$\tilde{p}_1+\tilde{p}_2+\tilde{p}_3+\tilde{p}_4=0\,,\quad \hat{p}_1+\hat{p}_4=0\,,\quad \hat{p}_2+\hat{p}_3=0\,. \tag{5.28}$$

It is convenient to combine $\tilde{p}_i$ and $\hat{p}_i$ into the ten functions q_i defined by

$$\begin{gathered} q_1=\tfrac{1}{2}(\hat{p}_1+\hat{p}_2)\,,\quad q_2=\tfrac{1}{2}(\hat{p}_1-\hat{p}_2)\,,\\ q_3=\tilde{p}_1+\tilde{p}_2\,,\quad q_4=\tilde{p}_1+\tilde{p}_3\,,\quad q_5=\tilde{p}_1+\tilde{p}_4\,,\\ q_{10-(i-1)}=-q_i\,,\quad i=1,\dots,5\,. \end{gathered} \tag{5.29}$$

These ten function define a ten-sheeted Riemann surface, or *algebraic curve*. In [91], a number of properties of these functions were derived:

1. Along a square root branch cut C_{ij} connecting the sheets i and j, continuity of the eigenvalues of $\Omega(x)$ requires $q_i^+(x)-q_j^-(x)\in 2\pi\mathbb{Z}$, where the superscript $\pm$ indicate that the functions are to be evaluated just above/below the cut.
2. The Noether charges corresponding to a solution can be read off from them large x behavior of the quasimomenta:

$$\begin{pmatrix} q_1\\ q_2\\ q_3\\ q_4\\ q_5 \end{pmatrix}=\frac{1}{x}\sqrt{\frac{2}{\lambda}}\begin{pmatrix} \Delta+S\\ \Delta-S\\ J+Q\\ J-Q\\ J_3 \end{pmatrix}+\mathcal{O}(1/x^2)\,. \tag{5.30}$$

3. The functions q_1, q_2, q_3 and q_4 have poles at $x=\pm1$, while q_5 does not. The Virasoro constraint requires all these poles to have the same residue $\alpha_\pm/2$.
4. Different sheets of the Riemann surface are related by inversion: $q_1(1/x)=-q_2(x)$, $q_3(1/x)=2\pi m-q_4(x)$ ($m\in\mathbb{Z}$), and $q_5(1/x)=q_5(x)$.

In chapter 6 we will use the algebraic curve to find giant magnon solutions in $\mathbb{CP}^3$ and calculate the leading correction to their classical energy at finite angular momentum.

5.4 Discrete torsion and the ABJ model

The string theory background described above is parametrized by the two integers N and k, and is the gravity dual of the ABJM model. As noted by Aharony, Bergman, and Jafferis [4], we can construct a more general background where the Neveu-Schwarz–Neveu-Schwarz two-form $B^{(2)}$ acquire a non-trivial holonomy on a $\mathbb{CP}^1$ inside $\mathbb{CP}^3$ [6],

$$b^{(2)} = \frac{1}{2\pi} \int_{\mathbb{CP}^1 \subset \mathbb{CP}^3} B^{(2)} = \frac{M-N}{k} + \frac{1}{2}. \tag{5.31}$$

The additional parameter M is integer valued and $b^{(2)}$ takes values in $\mathbb{Z}_k$. Hence configurations that differ in an integer shift of $b^{(2)}$ should be considered equivalent. Note that the B-field is non-vanishing even for $N = M$, *i.e.*, in the gravity dual of ABJM. As will be discussed in the next section, in M-theory the B-field originates from fractional M2-branes which gives rise to a discrete torsion.

Under world-sheet parity $b^{(2)}$ changes sign. Hence a non-vanishing B-field generally breaks the world-sheet parity symmetry of the theory. In the ABJM case of $b^{(2)} = 1/2$, parity takes $b^{(2)} \to -b^{(2)} = -1/2$. But we can aways shift $b^{(2)}$ by one, so the result of the parity transformation is equivalent to $b^{(2)} = 1/2$ and for this value the theory is parity invariant.

In the Green-Schwarz action this B-field enters as a θ-angle, which only affects the D-brane spectrum and the world-sheet instantons. These are suppressed by $\exp(-\sqrt{\lambda})$, and hence the perturbative analysis of the string theory is unaffected by the inclusion of the B-field. However, in the full non-perturbative theory it becomes important. In particular the question of whether the integrability of the model holds at quantum level could depend on the value of the θ-angle. This happens in the $\mathrm{O}(N)$ sigma-model, which is known to be integrable for the two parity preserving cases of $\theta = 0$ and $\theta = \pi$. The physics at low energy is very different for these two values of θ and there is no known model that interpolates between these two points. Hence the question of the integrability of the $\mathrm{O}(N)$ model for generic θ is unresolved.

5.5 From M-theory to IIA strings

As mentioned in the beginning of this chapter, the gravity dual of the ABJM theory at small k has a natural description in terms of M-theory. Let us consider a stack of $\tilde{N}$ M2-branes in flat space. The near horizon geometry of these branes is $\mathrm{AdS}_4 \times \mathrm{S}^7$ [125]. This background is maximally supersymmetric, preserving 32 supersymmetries. The seven-sphere can be written as a Hopf fibration with an S^1 fiber over $\mathbb{CP}^3$. The metric on S^7 then takes the form

$$ds^2_{\mathrm{S}^7} = ds^2_{\mathbb{CP}^3} + (d\tilde{\psi} + \omega)^2, \tag{5.32}$$

where $0 \leq \tilde{\psi} < 2\pi$. In terms of the coordinates y^I the last term is

$$d\tilde{\psi} + \omega = \frac{i}{2} \frac{y^I \, d\bar{y}_I - \bar{y}_I \, dy^I}{|y|^2}. \tag{5.33}$$

Note that

$$d\left(d\tilde{\psi}+\omega\right)=d\omega=i\,d\left(\frac{dy^I}{|y|}\right)\wedge d\left(\frac{d\bar{y}_I}{|y|}\right), \tag{5.34}$$

so ω is the one-form potential for the Kähler form (5.8).

In the beginning of this chapter we also introduced a set of angular coordinates on $\mathbb{CP}^3$. We can extend these coordinate to the seven-sphere by simply introducing an overall phase $\tilde{\psi}$ to all four complex coordinates y^I.

The full eleven-dimensional metric is

$$ds_{11}^2=\frac{L^2}{4}ds^2_{\mathrm{AdS}_4}+L^2 ds^2_{\mathrm{S}^7}\,. \tag{5.35}$$

In units of the eleven-dimensional Planck length, the radius of the sphere is related to the number of M2-branes $\tilde{N}$ as [125]

$$L=(32\pi^2\tilde{N})^{1/6}\,. \tag{5.36}$$

The M-theory background also contains $\tilde{N}$ units of four-form field strength

$$F^{(4)}=\frac{3L^3}{8}\epsilon_{\mathrm{AdS}_4}\,, \tag{5.37}$$

where $\epsilon_{\mathrm{AdS}_4}$ is the unit radius volume form on AdS_4.

Orbifolding. We now want to deform the above setup by introducing an $\mathbb{Z}_k$ orbifold identification in the directions transverse to the M2-branes. In the coordinates y^I the orbifold action is

$$y^I\to e^{\frac{2\pi i}{k}}y^I\,,\quad k\in\mathbb{Z}\,, \tag{5.38}$$

i.e., we perform an simultaneous rotation in the four orthogonal planes spanned by y^I. Since the total phase of $\vec{y}$ is $\tilde{\psi}$ we introduce a rescaled coordinate $\psi=\tilde{\psi}/k$, which has period 2π. The resulting geometry is $\mathrm{AdS}_4\times\mathrm{S}^7/\mathbb{Z}_k$ with the metric

$$ds^2_{\mathrm{AdS}_4\times\mathrm{S}^7/\mathbb{Z}_k}=\frac{L^2}{4}ds^2_{\mathrm{AdS}_4}+L^2ds^2_{\mathbb{CP}^3}+\frac{L^2}{k^2}(d\psi+k\omega)^2\,. \tag{5.39}$$

Since the S^1 fiber is shortened by a factor $1/k$ the total volume is decreased the same factor. To ensure that the four-form flux is properly quantized, we need the integer $\tilde{N}$ to be an integer multiple of k: $\tilde{N}=Nk$ for some integer N.

The background metric now contains two scales – the $\mathbb{CP}^3$ radius $L\propto(Nk)^{1/6}$, and the radius of the S^1, $L/k\propto(N/k^5)^{1/6}$. For $k^5\ll N$, the circle is large and M-theory provides a good description. However, when $k^5\gg N$, the geometry becomes essentially ten-dimensional, and hence the theory reduces to weakly coupled IIA string theory.

In general, the metric of eleven-dimensional supergravity compactified on a circle is written in terms of ten-dimensional quantities as [150]

$$ds_{11}^2 = e^{-\frac{2\phi}{3}} ds_{IIA}^2 + e^{\frac{4\phi}{3}} (d\psi + A^{(1)})^2 . \tag{5.40}$$

Comparing (5.39) and (5.40) we identify the string frame metric, dilaton and two-form field strength as

$$ds_{IIA}^2 = \frac{L^3}{k} \left(\frac{1}{4} ds_{\mathrm{AdS}_4}^2 + ds_{\mathbb{CP}^3}^2 \right) , \quad e^{2\phi} = \frac{L^3}{k^3} , \quad F^{(2)} = dA^{(1)} = k\, d\omega . \tag{5.41}$$

We also note that the radius of $\mathbb{CP}^3$ can be expressed as

$$R^2 = \frac{L^3}{k} = \frac{\sqrt{32\pi^2 \tilde{N}}}{k} = \sqrt{\frac{32\pi^2 N}{k}} = 4\pi\sqrt{2\lambda} . \tag{5.42}$$

The four-form flux remains the same as in eleven dimensions

$$F^{(4)} = \frac{3L^3}{8} \epsilon_{\mathrm{AdS}_4} = \frac{3kR^2}{8} \epsilon_{\mathrm{AdS}_4} . \tag{5.43}$$

These results exactly match the expressions given in section 5.1.

The above derivation of the IIA background form M-theory explains the origin of the parameters N and k, as well as some of the numerical coefficients appearing in the first part of this chapter.

Supersymmetry. As noted above the $\mathrm{AdS}_4 \times \mathrm{S}^7$ solution in eleven-dimensional supergravity is maximally supersymmetric, preserving 32 supersymmetries [125]. The isometries of AdS_4 and S^7 are generated by $\mathfrak{so}(8)$ and $\mathfrak{sp}(4)$, respectively. Including also the supersymmetries, the full symmetry of the background is the supergroup $\mathrm{OSp}(8|4)$.

The orbifold action (5.38) breaks $\mathrm{SO}(8)$ to $\mathrm{SU}(4) \times \mathrm{U}(1)$. An $\mathrm{SO}(8)$ spinor transform under $\mathbb{Z}_k$ as

$$\Psi \to e^{2\pi i (s_1 + s_2 + s_3 + s_4)/k} \Psi , \tag{5.44}$$

where $s_i = \pm 1/2$ are the spinor weights. Due to chirality, the sum of s_i has to be even, leaving the $\mathbf{8}_c$ representation of $\mathrm{SO}(8)$. For a spinor to survive the orbifold projection it must satisfy

$$\sum_{i=1}^{4} s_i = 0 \pmod{k} . \tag{5.45}$$

For $k > 2$, six of the eight spinors are left invariant under the orbifold action, and hence the $\mathcal{N}=8$ supersymmetry is broken to $\mathcal{N}=6$.[5] For $k=1$ and $k=2$ all eight supersymmetries remain unbroken [5].

FRACTIONAL M2-BRANES. As already mentioned, a more general M-theory background was introduced in [4]. In addition to the N M2-branes, the authors considered $M-N$ *fractional* M2-branes at the orbifold point.[6] This results in a discrete torsion for the three-form gauge field $C^{(3)}$

$$\frac{1}{2\pi}\int_{S^3/\mathbb{Z}_k \subset S^7/\mathbb{Z}_k} C^{(3)} = \frac{M-N}{k} + \frac{1}{2}. \tag{5.46}$$

In local coordinates $C^{(3)}$ can be expressed as

$$C^{(3)} \propto \frac{M-N}{k} d\omega \wedge d\psi. \tag{5.47}$$

Since one of the legs of $C^{(3)}$ points along the M-theory circle it reduces to the NS–NS two-form B in the string theory limit, explaining the origin of the non-trivial B-field in the ABJ theory.

An interesting consequence of the brane construction of the background is the equivalence between the Chern-Simons theories with gauge groups $\mathrm{U}(N)_k \times \mathrm{U}(M)_{-k}$ and $\mathrm{U}(M)_k \times \mathrm{U}(2M-N+k)_{-k}$, see [4]. In terms of the 't Hooft couplings λ and $\hat{\lambda}$ this corresponds to the equivalence between the theories with couplings $(\lambda, \hat{\lambda})$ and $(\hat{\lambda}, 2\hat{\lambda}-\lambda+1)$.

As mentioned in chapter 2 unitarity on the gauge theory side requires that the integers M and N satisfy the inequality $|M-N| \leq k$. As a simple consistency check let us consider the case $k=1$. The gauge theory constraint means that M and N can differ by at most one. But according to the above identification the cases $M = N \pm 1$ and $M = N$ are equivalent for $k=1$, and there is only a single configuration. In the brane picture $k=1$ corresponds to M2-branes in flat $\mathbb{C}^4$. In this case there are no orbifold point and hence there can be no fractional branes, so also from this point of view we only expect a single configuration for a given N.

GAUGE THEORY, STRING THEORY AND M-THEORY. M-theory in the orbifold background $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_7$ is parametrized by the integers N and k. In different regions of this parameter space different description of this theory become natural. Table 5.1 summarizes the condition these parameters have to satisfy for the respective description to be weakly coupled.

[5] If we use the other spinor representation of SO(8) all the supersymmetries are broken [142].

[6] The fractional M2-branes are obtained by wrapping M5-branes on a vanishing three-cycle at the orbifold point.

Table 5.1: An overview of the different regions of the parameters N and k where the AdS_4/CFT_3 correspondence is described by weakly coupled field theory, string theory or M-theory.

Region	Weakly coupled theory
$1 \ll N \ll k$	Planar gauge theory
$1 \ll N^{1/5} \ll k \ll N$	IIA string theory
$1 \ll k \ll N^{1/5}$	11D SUGRA

6. Magnons at strong coupling

In this chapter I discuss the world-sheet excitation in $\mathrm{AdS}_4 \times \mathbb{CP}^3$, focusing on the giant magnon regime first studied in $\mathrm{AdS}_5 \times \mathrm{S}^5$ by Hofman and Maldacena [102]. These string states can be seen as gravity duals of gauge theory states represented by a spin-chain with a single magnon excitation.

The basic giant magnon is a classical string living in $\mathbb{R} \times \mathrm{S}^2$ and carries a large angular momentum J. In space-time the string is rigidly moving with the center of mass of the string traveling around the equator, and on the world-sheet the solution is solitonic. It can be embedded in $\mathrm{AdS}_5 \times \mathrm{S}^5$ in a unique way up to global rotations of the five-sphere. In $\mathrm{AdS}_4 \times \mathbb{CP}^3$, on the other hand, there are several inequivalent giant magnon solutions. The different possibilities are discussed below, together with various *dyonic* generalizations of these solutions by including an additional non-zero angular momentum, as well as the finite-size corrections for finite, but large, angular momentum J. The results in this chapter are based on Paper IV and Paper V.

6.1 The string theory ground state and the Penrose limit

The bosonic part of the string Lagrangian reads[1]

$$\mathcal{L} = \frac{1}{4\pi}\sqrt{-h}h^{\alpha\beta}G_{MN}\partial_\alpha X^M \partial_\beta X^N \tag{6.1}$$

$$= \frac{\sqrt{8\lambda}}{2}\sqrt{-h}h^{\alpha\beta}\left(\frac{1}{4}\hat{G}_{\mu\nu}\partial_\alpha z^\mu \partial_\beta z^\nu + \tilde{G}_{IJ}\partial_\alpha y^I \partial_\beta y^J\right), \tag{6.2}$$

where the metric has been split into the AdS_4 part ($\hat{G}$) and the $\mathbb{CP}^3$ part ($\tilde{G}$), and a factor of the $\mathbb{CP}^3$ radius R^2 has been pulled out, here expressed in terms of the 't Hooft parameter λ.

Let us start by considering a classical string solution consisting of a point-like string moving along a null-geodesic in $\mathrm{AdS}_4 \times \mathbb{CP}^3$. The geodesic sits in the center of AdS_4 ($\rho = 0$) and I use a timelike conformal gauge with

[1] We use units where $\alpha' = 1$.

$t = 2\tau$.[2] The $\mathbb{CP}^3$ part of the solution is given in terms of the embedding coordinates (5.6) as

$$\vec{y} = \tfrac{1}{\sqrt{2}}(e^{i\tau}, 0, 0, e^{-i\tau}) , \tag{6.3}$$

or in terms of the angles (5.9)

$$\xi = \vartheta_1 = \vartheta_2 = \frac{\pi}{2}, \quad \eta = \varphi_1 = 0 , \quad \varphi_2 = 2\tau . \tag{6.4}$$

It is easy to check that this is a solution to the equations of motion and the Virasoro constraints obtained from (6.1).

The energy Δ and angular momentum J of this solution is given by

$$\Delta = \int d\sigma \frac{\partial \mathcal{L}}{\partial \dot{t}} = \sqrt{2\lambda} \int d\sigma , \qquad J = \int d\sigma \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = \sqrt{2\lambda} \int d\sigma , \tag{6.5}$$

while the charges S, Q and J_3 all vanish. Hence this string solution satisfies the BPS condition

$$\Delta - J = 0 . \tag{6.6}$$

The charges of the point-like string are right for it to be the string theory dual of the ground state operator (2.53) in the gauge theory. Hence we can think of this solution as a ground state on the string side. Note that the energy, as well as the angular momentum, of this string is proportional to the length of the world-sheet.

The string state above is very similar to the point-like string in $AdS_5 \times S^5$ studied by Berenstein, Maldacena, and Nastase [44]. In that paper, the authors studied fluctuations around the point-like string in the limit J, $\lambda \to \infty$ keeping J^2/λ fixed, known today as the BMN limit. The corresponding limit of the $AdS_4 \times \mathbb{CP}^3$ solution studied here was considered in [143, 76, 85] (see also [1]). The resulting spectrum consists of four light bosons on $\mathbb{CP}^3$ with dispersion relation

$$\epsilon(p) = \sqrt{\tfrac{1}{4} + \tfrac{\lambda}{2} p^2} , \tag{6.7}$$

as well as four heavy bosons – one on $\mathbb{CP}^3$ and three on AdS_4 – with

$$\epsilon(p) = \sqrt{1 + \tfrac{\lambda}{2} p^2} . \tag{6.8}$$

The fermionic spectrum matches the bosonic one, with four light fermions with mass $1/2$, and four heavy fermions with mass 1. The world-sheet momentum of the BMN excitations takes the form $p = 2\pi n/J$, for an integer n. Since the angular momentum J is very large the momentum is small: $p \ll 1$.

[2] As we will see, the length of the world-sheet is not fixed to 2π in this gauge, but proportional to the energy Δ.

Hence the above dispersion relations (6.7) and (6.8) are compatible with the small momentum limit of the general $\mathrm{SU}(2|2)$ dispersion relation

$$\epsilon(p) = \sqrt{Q^2 + 4h^2(\lambda)\sin^2\frac{p}{2}}\,, \tag{6.9}$$

for $Q = 1/2$ and $Q = 1$, respectively, provided the leading order expansion of $h(\lambda)$ at strong coupling is given by

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} + \mathcal{O}(1)\,. \tag{6.10}$$

Note that the leading term in this strong coupling expansion differs from the weak coupling result found in chapter 3, indicating that the function $h(\lambda)$ is far more non-trivial in ABJ(M) than in $\mathcal{N} = 4$ super Yang-Mills.

6.2 The giant magnon

In the previous section we considered the BMN limit, where J and λ are taken to be large with J^2/λ fixed and the momentum p of excitations scaling as $1/J$. Hofman and Maldacena [102] considered another limit, where the angular momentum J is again taken to be very large, but with the 't Hooft coupling λ, as well as the world-sheet momentum p, kept fixed.

In the timelike conformal gauge, the length of the world-sheet is proportional to the energy of the string state. In the above limit, the energy, and hence the world-sheet, is infinitely large. It then makes sense to consider an open string state. To get a physical, closed, string we can glue several such open strings together at the end-points. Since these are infinitely far apart on the world-sheet, there will be no interactions between the various components. For finite J, on the other hand, such interactions are important.

To find the *giant magnon* solution, we consider a sigma-model on $\mathbb{R}\times \mathrm{S}^2$ in the Hofman-Maldacena limit, and look for solutions with $\Delta - J$ finite. For our purpose, it is convenient to write the solution by embedding $\mathrm{S}^2 \subset \mathbb{C}^2$. It then reads

$$\vec{w} = \begin{pmatrix} e^{i\phi_{\mathrm{mag}}(\sigma,\tau)}\sin\theta_{\mathrm{mag}}(\sigma,\tau) \\ \cos\theta_{\mathrm{mag}}(\sigma,\tau) \end{pmatrix} = \begin{pmatrix} e^{i\tau}\left(\cos\frac{p}{2} + i\sin\frac{p}{2}\tanh u\right) \\ \sin\frac{p}{2}\,\mathrm{sech}\,u \end{pmatrix}\,, \tag{6.11}$$

with

$$\cos\theta_{\mathrm{mag}}(\sigma,\tau) = \sin\frac{p}{2}\,\mathrm{sech}\,u\,, \qquad \tan(\phi_{\mathrm{mag}}(\sigma,\tau) - \tau) = \tan\frac{p}{2}\tanh u\,. \tag{6.12}$$

Here the coordinate u is the spatial coordinate boosted with velocity $v = \cos\frac{p}{2}$, $u = \gamma(\sigma - v\tau)$. Hence the solution contains as a single parameter the momentum p, which now has a geometrical interpretation as the opening angle between the two end-points of the string along the equator of the sphere:

$$\Delta\phi_{\mathrm{mag}} = \phi_{\mathrm{mag}}(\sigma \to +\infty, \tau) - \phi_{\mathrm{mag}}(\sigma \to -\infty, \tau) = p\,. \tag{6.13}$$

Calculating the charges of this string we get

$$\Delta - J = \frac{R^2}{\pi}\sin\frac{p}{2}\,, \tag{6.14}$$

where R is the radius of the two-sphere. We note that the same periodicity of the momentum dependence that we found for the spin-chain in chapter 3.

6.2.1 The dyonic giant magnon

The giant magnon can be extended to a solution on $\mathbb{R}\times \mathrm{S}^3$, by introducing an additional angular momentum Q that is large but finite [59, 66]. In terms of embedding coordinates $\mathrm{S}^3 \subset \mathbb{C}^2$, this *dyonic* giant magnon can be written as [1]

$$\vec{w} = \begin{pmatrix} e^{i\tau}\left(\cos\frac{p}{2} + i\sin\frac{p}{2}\tanh U\right) \\ e^{iV}\sin\frac{p}{2}\,\mathrm{sech}\,U \end{pmatrix}, \tag{6.15}$$

where

$$\begin{aligned} U &= (\sigma\cosh\beta - \tau\sinh\beta)\cos\alpha\,, \quad &\cot\alpha &= \frac{2r}{1-r^2}\sin\frac{p}{2}\,, \\ V &= (\tau\cosh\beta - \sigma\sinh\beta)\sin\alpha\,, \quad &\tanh\beta &= \frac{2r}{1+r^2}\cos\frac{p}{2}\,. \end{aligned} \tag{6.16}$$

The momentum p still has the interpretation of the opening angle between the end-points along the equator in the w_1 plane, though the world-sheet velocity is not given by $\cos p/2$. The extra parameter r parametrizes the additional angular momentum. When $r \to 1$, V vanishes and U becomes identical with the u in (6.12). The dyonic solution then reduces to the non-dyonic giant magnon.

As with the ordinary giant magnon, the energy Δ and the angular momentum J diverge, but their difference, as well as Q, remain finite

$$\Delta - J = \frac{R^2}{\pi}\frac{1+r^2}{2r}\sin\frac{p}{2}\,, \qquad Q = \frac{R^2}{\pi}\frac{1-r^2}{2r}\sin\frac{p}{2}\,. \tag{6.17}$$

These two relations can be combined to give

$$\Delta - J = \sqrt{Q^2 + \frac{R^4}{\pi^2}\sin^2\frac{p}{2}}\,. \tag{6.18}$$

We recognize the form of this relation from the general SU(2|2) dispersion relation discussed in chapter 3.

For the dyonic giant magnon to make sense as a classical string solution, Q can not be arbitrary, but needs to be much large than one. For Q of order one, the additional angular momentum is not visible for the classical string, and the dispersion relation reduces to that for the giant magnon.

6.3 Giant magnons in $\mathbb{CP}^3$

To construct a giant magnon solution in $\mathbb{R}\times\mathbb{CP}^3$, we need to find a suitable two-dimensional subspace of $\mathbb{CP}^3$ for it to live in. There are two inequivalent possibilities, $\mathbb{CP}^1$ and $\mathbb{RP}^2$. For a nice discussion of these solutions see [1].

- *The* $\mathbb{CP}^1$ *magnon.* By setting $y^2 = y^3 = 0$, or $\xi = \pi/2$, we get a $\mathbb{CP}^1 \cong \mathrm{S}^2$ submanifold inside $\mathbb{CP}^3$. The resulting metric reads

$$ds^2_{\mathbb{CP}^1} = \frac{1}{4}\left(d\vartheta_2^2 + \sin^2\vartheta_2\, d\varphi_2^2\right) . \tag{6.19}$$

  Since this sphere has radius $1/2$, conformal gauge requires us to set

$$\varphi_2 = \phi_{\mathrm{mag}}(2\sigma, 2\tau) , \qquad \vartheta_2 = \theta_{\mathrm{mag}}(2\sigma, 2\tau) . \tag{6.20}$$

  This solution was first found in [76]. It has the dispersion relation

$$\Delta - J = \sqrt{2\lambda}\sin\frac{p}{2} . \tag{6.21}$$

- *The* $\mathbb{RP}^2$ *magnon.* We get another interesting submanifold by setting $\bar{y}^4 = y^1$ and $\bar{y}^3 = y^2$, or $\eta = 0$, $\vartheta_1 = \vartheta_2 = \pi/2$. This gives an $\mathbb{RP}^3 \subset \mathbb{CP}^3$. We can further reduce this space by requiring y^2 to be real, *i.e.*, $\varphi_1 = 0$, to obtain $\mathbb{RP}^2 \subset \mathbb{CP}^3$. The giant magnon solution (6.11) can then be embedded as

$$\vec{y} = (w_1, w_2, \bar{w}_2, \bar{w}_1) , \quad \text{or} \quad \xi = \theta_{\mathrm{mag}}(\sigma,\tau) , \quad \varphi_2 = \phi_{\mathrm{mag}}(\sigma,\tau) . \tag{6.22}$$

  The $\mathbb{RP}^2$ giant magnon was studied in [76, 85], and has the dispersion relation

$$\Delta - J = 2\sqrt{2\lambda}\sin\frac{p}{2} . \tag{6.23}$$

  An interesting feature of this solution is that since it is embedded in $\mathbb{RP}^2$ and not S^2 a magnon with momentum $p = \pi$ describes a *closed* string.

By setting the momentum of the two magnons to zero, we see that both solutions reduce to the point-like string (6.3). Hence it is natural to consider them both to be excitations above the same ground state.

6.3.1 Dyonic giant magnons in $\mathbb{CP}^3$

There are several extensions of the giant magnon solutions in $\mathbb{CP}^3$. For both the $\mathbb{RP}^2$ and $\mathbb{CP}^1$ magnon an additional angular momentum can be introduced producing dyonic magnons similar to the $\mathbb{R} \times S^3$ solution by Dorey *et al.* [59]. In addition, there is an extension of the $\mathbb{RP}^2$ magnon which introduces an extra parameter in the solutions, but still only has a single non-vanishing angular momentum. This solution has no correspondence in $AdS_5 \times S^5$.

- *The* $\mathbb{RP}^3$ *magnon.* As we saw in the last section, there is an $\mathbb{RP}^3$ submanifold of $\mathbb{CP}^3$. The dyonic giant magnon solution of Dorey *et al.* [59] can straightforwardly be embedded in this space by setting

$$y^1 = \bar{y}^4 = w_1 , \qquad y^2 = \bar{y}^3 = w_2 , \tag{6.24}$$

  where w_i are the $\mathbb{C}^2$ embedding coordinates in (6.15). Its dispersion relation reads

$$\Delta - J = \sqrt{Q^2 + 8\lambda \sin^2 \frac{p}{2}} . \tag{6.25}$$

  This solution was studied in the $\mathbb{CP}^3$ case in [11, 154]. Like the $\mathbb{RP}^2$ magnon, its dyonic sibling corresponds to a closed string when $p = \pi$.

- *The* $\mathbb{CP}^2$ *magnon.* There is no way to extend the $\mathbb{CP}^1$ submanifold to a three-dimensional space in which the ordinary dyonic magnon can be embedded. Still there are many reasons to expect a solution with an additional angular momentum to exist. In Paper V we found a special case of such a magnon for the particular momentum configuration $p = \pi$. This solution lives in the $\mathbb{CP}^2$ obtained by setting $y^3 = 0$, and is charged under both Q and J_3, with the two charges related by $J_3 = 2Q$. We conjectured the dispersion relation of the general $\mathbb{CP}^2$ magnon to be

$$\Delta - J = \sqrt{Q^2 + 2\lambda \sin^2 \frac{p}{2}} . \tag{6.26}$$

  A second solution is obtained if we instead set $y^2 = 0$. This solution has $J_3 = -2Q$, but is otherwise identical to the first.

  The full $\mathbb{CP}^2$ magnon was later constructed by Hollowood and Miramontes [103], using the dressing method [170, 171], a method for constructing multi-soliton solutions in integrable field theories, confirming the above charges. The explicit form of the solution is not very illuminating, so I refer to [103] for the full details.

- *The dressed magnon.* There is another two-parameter solution living in $\mathbb{CP}^2 \subset \mathbb{CP}^3$. This solution was obtained more or less simultaneously by three different groups [104, 111, 162] using the dressing method. In contrast to the $\mathbb{CP}^2$ magnon above, this solution is not charged under Q nor under J_3. The solution is written down in Paper V. Parametrizing the solution by the parameter[3] Q_f introduced in that paper, the dispersion relation reads

$$\Delta - J = \sqrt{Q_f^2 + 8\lambda \sin^2 \frac{p}{2}}\,. \tag{6.27}$$

Hence the dispersion relation of the dressed magnon has the same form as that for the $\mathbb{RP}^3$ magnon. However, the other $\mathbb{CP}^3$ charges differ between the two solutions. Since this solution only has a single non-vanishing angular momentum, it is not really correct to refer to it as "dyonic".

The properties of the various giant magnon solutions presented here (except for the full $\mathbb{CP}^2$ dyonic magnon), are summarized in Table I of Paper V.[4]

6.4 Giant magnons in the algebraic curve

Giant magnons in S^3 were first studied in the algebraic curve formalism by Minahan, Tirziu, and Tseytlin [136]. These authors showed that the magnon corresponds to a *logarithmic* branch cut connecting two sheets. Later an alternative description was found, in which the magnon is obtained starting with a two square root branch cuts configuration and taking the singular limit where the endpoints of the cuts coincide Vicedo [167]. As we will see later, a combination of the two pictures is very useful for finite-J giant magnons.

6.4.1 Embedding giant magnons in the algebraic curve

In Paper IV an ansatz for an algebraic curve for strings moving non-trivially on $\mathbb{CP}^3$ was introduced

$$\begin{aligned}
q_1(x) &= \frac{\alpha x}{x^2-1}\,, \\
q_2(x) &= \frac{\alpha x}{x^2-1}\,, \\
q_3(x) &= \frac{\alpha x}{x^2-1} + G_r(x) + G_r(1/x) - G_v(1/x) - G_u(1/x) \\
&\qquad - G_r(0) + G_v(0) + G_u(0)\,, \\
q_4(x) &= \frac{\alpha x}{x^2-1} + G_v(x) + G_u(x) - G_r(x) - G_r(1/x) + G_r(0)\,, \\
q_5(x) &= -G_v(x) + G_u(x) - G_v(1/x) + G_u(1/x) + G_v(0) - G_u(0).
\end{aligned} \tag{6.28}$$

[3] The subscript f stands for "false", indicating that Q_f is *not* an angular momentum, but a parameter of the solution.

[4] Note that the charges J and Q used in Paper V are twice the charges used in this summary.

The ansatz contains three functions $G_v(x)$, $G_u(x)$ and $G_r(x)$, called resolvents, which describe the different ways we can introduce cuts that connect the different sheets of the Riemann surface. The various inversion symmetries of the curve are automatically satisfied provided $G_u(0)+G_v(0)=2\pi m$ for some integer m. This can be interpreted as a level matching condition. In the giant magnon limit we will however relax this condition, and introduce the magnon momentum $p=2\pi m$.

THE GROUND STATE. The simplest possibility is to set

$$G_u(x)=G_v(x)=G_r(x)=0\,. \tag{6.29}$$

We then obtain the quasimomenta

$$q_1(x)=q_2(x)=q_3(x)=q_4(x)=\frac{\alpha x}{x^2-1}\,, \qquad q_5(x)=0\,. \tag{6.30}$$

Expanding this at large x, and comparing with the expected asymptotics from section 5.3.1 we get

$$\Delta=J\,, \qquad S=Q=J_3=0\,, \tag{6.31}$$

corresponding to the point-like string considered in section 6.1. Hence the above solution can be thought of as a ground state of the algebraic curve.

THE SMALL MAGNON. We can now add excitations to the above ground state by turning on one or more of the resolvents. To get a giant magnon we use the resolvent

$$G_{\text{mag}}(x)=-i\log\frac{x-X^+}{x-X^-}\,. \tag{6.32}$$

This function has a logarithmic cut with endpoints $X^\pm$, where X^- is the complex conjugate of X^+.. Let us choose

$$G_v(x)=G_{\text{mag}}(x)\,, \qquad G_u(x)=G_r(x)=0\,, \tag{6.33}$$

corresponding to a cut connecting the sheets q_4 and q_5. This solution was first studied by Shenderovich [156], who named it the *small* giant magnon. It carries the charges

$$\begin{aligned} \Delta-J&=\frac{i}{2}\sqrt{\frac{\lambda}{2}}\left(X^+-\frac{1}{X^+}-X^-+\frac{1}{X^-}\right), \qquad p=-i\log\frac{X^+}{X^-}\,,\\ Q&=-\frac{i}{2}\sqrt{\frac{\lambda}{2}}\left(X^++\frac{1}{X^+}-X^--\frac{1}{X^-}\right), \quad J_3=2Q\,. \end{aligned} \tag{6.34}$$

Solving this for $X^\pm$ we get

$$X^\pm = \frac{Q+\sqrt{Q^2+2\lambda\sin^2\frac{p}{2}}}{\sqrt{2\lambda}} e^{\pm\frac{ip}{2}} \csc\frac{p}{2}\,, \tag{6.35}$$

which leads to the dispersion relation

$$\Delta - J = \sqrt{Q^2+2\lambda\sin^2\frac{p}{2}}\,, \tag{6.36}$$

The charges of this solution perfectly matches those of the dyonic $\mathbb{CP}^2$ dyonic magnon. Note that in the $Q\to 0$ limit the two branch points $X^\pm$ approach $e^{\pm\frac{ip}{2}}$ and hence the unit circle. In this limit we get a solution with the same charges as the $\mathbb{CP}^1$ magnon.

Another option is to set

$$G_u(x) = G_{\mathrm{mag}}(x)\,, \qquad G_v(x) = G_r(x) = 0\,. \tag{6.37}$$

Now the cut connects the sheets q_6 and q_4. The only difference in the charges is that J_3 changes sign. Hence this solution corresponds to the second orientation of the $\mathbb{CP}^2$ magnon.

Two more magnons can be constructed by also turning on the $G_r(x)$ resolvent. The resulting solutions are closely related to the small magnons above, but the charge Q has the opposite sign. Such solutions, or “anti-magnons” were also found in [103] using the dressing method. These excitations correspond a different choice $\mathrm{SU}(2)\times\mathrm{SU}(2)$ subgroup of the $\mathrm{SU}(4)$ isometry group of $\mathbb{CP}^3$ [65].

A PAIR OF SMALL MAGNONS. We can also turn on both kinds of small magnons at the same time, by setting

$$G_u(x) = G_v(x) = G_{\mathrm{mag}}(x)\,, \qquad G_r(x) = 0\,. \tag{6.38}$$

All the charges Δ, J, Q, J_3 and p will now be a sum of the charges from the two small magnons that make up the solution. Hence the total charges are

$$\Delta - J = \sqrt{Q^2+8\lambda\sin^2\frac{p}{4}}\,, \qquad J_3 = 0\,. \tag{6.39}$$

This is almost the same as for the $\mathbb{RP}^3$ magnon, except for a factor of two in the momentum dependence. However, since the above solution is constructed as a sum of two small magnons with equal momentum, it is natural to instead write the charges in terms of the momentum $\tilde{p}$ of one of them, so that the dispersion relation is

$$\Delta - J = \sqrt{Q^2 + 8\lambda \sin^2 \frac{\tilde{p}}{2}}\,, \tag{6.40}$$

matching the $\mathbb{RP}^3$ magnon.

THE BIG MAGNON. Reference [156] contains yet another solution, which is obtained by setting

$$G_u(x) = G_r(x) = G_v(x) = G_{\mathrm{mag}}(x)\,. \tag{6.41}$$

For this solution both the charges Q and J_3 vanish. However, we can still introduce a parameter Q_u that is related to the branch points $X^{\pm}$ in the same way as in (6.34). The dispersion relation can then be written as

$$\Delta - J = \sqrt{Q_u^2 + 8\lambda \sin^2 \frac{p}{4}}\,. \tag{6.42}$$

As with the solution in the previous paragraph, the big magnon is constructed as the sum of two small magnons, but in this case both Q and J_3 of the two magnons have different signs. We can again introduce the momentum of a single magnon, $\tilde{p} = p/2$, in terms of which the dispersion relation reads

$$\Delta - J = \sqrt{Q_u^2 + 8\lambda \sin^2 \frac{\tilde{p}}{2}}\,, \tag{6.43}$$

matching the "dressed" magnon in $\mathbb{CP}^2$, provided we identify the parameters Q_u and Q_f.

6.5 Finite-size corrections

In the giant magnon limit, we send both the energy Δ and the angular momentum J to infinity, keeping their difference finite. For finite, but large, Δ and J we expect $\Delta - J$ to receive corrections. Since the size of the world-sheet in the gauge we use is proportional to the energy, we refer to these corrections as finite-size corrections. In [15] it was argued that these corrections arise due to virtual particles circling the world-sheet, and that their contribution to the energy is exponentially suppressed. The finite-size giant magnon was first analyzed in detail in $\mathrm{AdS}_5 \times \mathrm{S}^5$ by Arutyunov, Frolov, and Zamaklar [20], who found exponentially suppressed corrections to the energy. They also found these corrections to be gauge dependent. This should, however, not come as a surprise. When we consider a single giant magnon with an arbitrary momentum, we describe an open string which in the infinite-J limit we can consider as a building block of closed strings. This cutting of the world-sheet works since the size of world-sheet is infinite in the giant magnon limit. For a finite-J magnon there is no way to cut the world-sheet without breaking reparametrization invariance, and hence gauge invariance. A gauge invariant configuration can be obtained by considering magnons on a $\mathbb{Z}_n$ orbifold of the sphere [23, 151], or by considering strings consisting of several magnons with the total momentum vanishing [133, 116, 145].

6.5.1 String theory solutions

The analysis in [20] was based on an explicita solution to the equations of motion for the string in $AdS_5 \times S^5$ describing a giant magnon with finite J. In the same spirit, the finite-size version of the $\mathbb{RP}^2$ magnon was derived in [84]. The resulting leading exponential correction to the dispersion relation reads[5]

$$\delta(\Delta - J) = -4(\Delta - J)\sin^2\frac{p}{2}e^{-\frac{2\Delta}{\Delta - J}}\,. \tag{6.44}$$

This expression has exactly the same form as in the S^5 case in [20]. Also the finite version of the $\mathbb{CP}^1$ magnon has been found [121]. In terms of Δ and J, the correction again takes the same form as the expression above.

The dyonic $\mathbb{RP}^3$ magnon is very similar to the corresponding S^3 solution, and this similarity remains for finite J. The energy correction in the S^3 case was first computed in [98], though the result is implicitly given already in [20]. The $\mathbb{RP}^3$ case was studied in [11, 7], and the results in the two backgrounds were found to agree perfectly.

At the moment, finite-size solutions of the $\mathbb{CP}^2$ and dressed giant magnons are not known. However, as we will see below, we can calculate the energy corrections for these magnons using the algebraic curve.

6.5.2 Algebraic curve

To describe a giant magnon with finite angular momentum J using the algebraic curve we need to generalize the resolvent (6.32). To see how this can be done, we need to better understand the appearance of the logarithmic cut. The full algebraic curve for the giant magnon was analysed in detail by Vicedo [167]. Let us consider an algebraic curve that has two square root branch cuts connecting the sheets q_4 and q_5, with branch points $X^\pm$ and $Y^\pm$, respectively.[6]

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[5] The correction given here is obtained in conformal gauge and coincide with the physical result in [23].

[6] Due to the symmetries of the curve discussed in section 5.3.1, there will also be two cuts between the sheets q_3 and q_6 with branch points inside the unit circle.

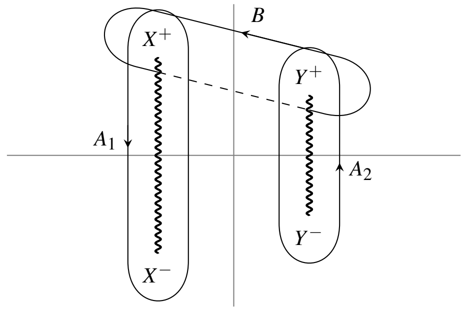


Figure 6.1: The periods A_1, A_2 and B. A_1 and A_2 lie in the sheet q_5 while B goes through the two square root branch cuts, crossing from q_5 to q_4 and back to q_5.

For the function q_5 to be single valued, the integral of[7] dq around any single cut needs to vanish. Introducing the periods A_1 and A_2 circling the two cuts in q_5 (see figure 6.1) we find

$$\oint_{A_i} dq = 0\,. \tag{6.45}$$

In section 5.3.1 we saw that the value of the quasimomenta can jump with an integer multiple of 2π when we cross a cut from one sheet to another. Hence the integral of dq around the period B circling two branch points X^+ and Y^+ should give

$$\oint_{B} dq = 2\pi n\,, \quad n \in \mathbb{Z}. \tag{6.46}$$

As argued in [167], the giant magnon is obtained from the general two cut solution above in the singular limit where $Y^\pm \to X^\pm$. To satisfy (6.46) in this limit, dq has to have a simple pole with residue $-in$ at X^+. The charges of the giant magnon are straightforwardly reproduced from this solution [167, 147]. Thus a single giant magnon can be seen as a singular limit of an elliptic string state.

I have now presented two configurations corresponding to a giant magnon – a single logarithmic cut or two square root cuts in a singular limit. To see the relation between these descriptions we note that the choice of which two branch points are associated to a particular branch cut is arbitrary. In particular we could have chosen to let one of the square root cuts above connect X^+ and Y^+, with the other sitting between X^- and Y^-. But this change does not preserve the integrals of dq around the periods A_1, A_2 and B. To compensate for this difference we need to add a logarithmic cut to q_5. In the limit $X^+ \to Y^+$ we then reproduce the resolvent (6.32).

From the above construction of the giant magnon in the algebraic curve there is a natural generalization for finite values of J: we need to consider a

[7] In this section dq indicates the differential on the Riemann surface spanned by the sheets q_i.

solution with two square root branch cuts whose branch points are very close to each other but do not coincide. By the same argument as above we can rewrite such a solution as a function with a single logarithmic cut which has a square root branch cut attached at each end. A solution with the correct analytical properties was constructed in [133] (see also [93]). It can be written in terms of the resolvent [145]

$$G_{\text{finite}}(x) = -2i\log\frac{\sqrt{x-X^+}+\sqrt{x-Y^+}}{\sqrt{x-X^-}+\sqrt{x-Y^-}}\,. \tag{6.47}$$

As a simple check of this solution we note that by setting $Y^\pm = X^\pm$ we get back the original giant magnon solution (6.32).

It is now straightforward to construct finite-size magnons in $\mathbb{CP}^3$ – we just need to go back to the different cases we considered in the last section and replace the infinite-J resolvent with (6.47). Since we are interested in the leading correction for large but finite J we can do the calculation perturbatively by expanding in the size of the square root branch cuts, which we consider to be small. The details of the calculation are given in Paper IV and Paper V. In summary the calculation consists of the following steps:

1. The branch points $Y^\pm$ are given by[8]

$$Y^\pm = X^\pm(1 \pm i\epsilon e^{\pm i\phi})\,, \tag{6.48}$$

   with $\epsilon \ll 1$. The phase ϕ is discussed below.
2. We want to consider a magnon with given R-charge Q and momentum p. From the expression of these charges in terms of the quasimomenta we determine the location of $X^\pm$.
3. The charge $\Delta - J$ is obtained from the asymptotic behavior of the quasimomenta as an expansion in ϵ.
4. To fix the parameter ϵ in terms of the charges we require that the cut between $X^\pm$ and $Y^\pm$ really is a square root branch cut connecting the sheets q_i and q_j. As mentioned in section 5.3.1, for x on such a cut the quasimomenta should satisfy

$$q_i(x^+) - q_j(x^-) = 2\pi n\,, \tag{6.49}$$

   where the subscript $\pm$ indicates evaluation just above or below the cut. Plugging the obtained expression for ϵ back into $\Delta - J$ we can read off the leading correction to the dispersion relation.

I will now briefly discussed the finite-size corrections to the different magnons discussed in section 6.5.2.

[8] The small parameter ϵ is called δ in Paper IV and Paper V.

THE SMALL MAGNON. Using the above procedure for the small dyonic magnon we get

$$\delta(\Delta - J) = -8\lambda \frac{Q}{\mathcal{E}(p)\mathcal{S}(p)} \sin^3 \frac{p}{2} \cos 2\phi \, e^{-\frac{\Delta}{\mathcal{S}(p)} \frac{\mathcal{E}(p)}{\mathcal{S}(p)}}, \tag{6.50}$$

where $\mathcal{E}(p)$ is the dispersion relation at infinite J,

$$\mathcal{E}(p) = \sqrt{Q^2 + 2\lambda \sin^2 \frac{p}{2}}, \tag{6.51}$$

and $\mathcal{S}(p)$ is given by

$$\mathcal{S}(p) = \sqrt{\frac{Q^2}{\sin^2 \frac{p}{2}} + 2\lambda \sin^2 \frac{p}{2}}. \tag{6.52}$$

In the non-dyonic limit $Q \to 0$ this correction vanishes. This result was given in Paper IV. But the non-dyonic small magnon is supposed to describe the Hofman-Maldacena magnon embedded in $\mathbb{CP}^1$. Since $\mathbb{CP}^1$ is a two-sphere, we would expect the finite-size corrections to be the same as in $\mathrm{AdS}_5 \times \mathrm{S}^5$, as first derived by Arutyunov, Frolov, and Zamaklar [20]. In Paper V we show that the above result is valid only for Q on the order of $\sqrt{\lambda}$. The reason is that when the algebraic curve is expanded in ϵ there is an implicit assumption that $\epsilon \ll Q/\sqrt{\lambda}$. Hence we cannot expect to get the correct result by just setting $Q = 0$ in the above expression. Instead we need to do a more careful expansion assuming that Q is of the same size as ϵ. This leads to the expected AFZ form

$$\delta(\Delta - J) = -4(\Delta - J) \sin^2 \frac{p}{2} \cos 2\phi \, e^{-\frac{2\Delta}{\Delta - J}}, \tag{6.53}$$

matching the result of [121]. The difference in the order of the limits $Q \to 0$ and $\epsilon \to 0$ explains the discrepancy between the results of the algebraic curve calculations for the non-dyonic small magnon in Paper IV and Paper V.

This problem does not appear in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ analysis in [133]. The difference between the two cases is that in the algebraic curve for the small magnon the quasimomentum q_5 contains extra cuts inside the unit circle which appear due to the inversion symmetry of the curve. When $Q \to 0$ all cuts approach the unit circle. The approximation used above assumes that all we can isolate the cut between X^+ and Y^+ since all other cuts are far away in the complex plane. This is not necessarily true in the $Q \to 0$ limit and hence we need to do a more careful analysis. The algebraic curve for $\mathrm{AdS}_5 \times \mathrm{S}^5$ also contains an inversion symmetry that leads to extra cuts inside the unit circle, but these cuts always appear on a different sheet than where we put the magnon. Hence the limit $Q \to 0$ in that case is well behaved.

To understand origin of the phase ϕ appearing in the above expressions we can consider a string state consisting of several magnons with the same charge Q and momentum p. For M magnons the total momentum vanishes provided the momentum of each magnon is of the form $p = 2\pi m/M$ for some integer m. In Paper IV we show that we then can choose the integer n in (6.49) in such a way that the phase ϕ vanishes. For multi-magnon states this phase can be interpreted as the relative angle between neighboring magnons, with $\cos 2\phi = \pm 1$ corresponding to the two types of helical strings considered in [146, 98].

A PAIR OF SMALL MAGNONS. The finite-size algebraic curve for a pair of small magnons with opposite charges J_3 is simpler than that for a single small magnon. The reason is that the additional cuts that appear inside the unit circle due to the inversion symmetry of the curve cancel on the sheet q_5. This leads to a solution that is essentially identical to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case. The resulting correction to the energy is given by (see Paper IV and Paper V)

$$\delta(\Delta - J) = -32\lambda \frac{1}{\mathcal{E}(p)} \sin^4 \frac{p}{4} \cos 2\phi \, e^{-\frac{\Delta}{\mathcal{S}(p)} \frac{\mathcal{E}(p)}{\mathcal{S}(p)}}, \tag{6.54}$$

where p is the momentum of one of the small magnons and

$$\mathcal{E}(p) = \sqrt{Q^2 + 8\lambda \sin^2 \frac{p}{2}}, \qquad \mathcal{S}(p) = \sqrt{\frac{Q^2}{\sin^2 \frac{p}{2}} + 8\lambda \sin^2 \frac{p}{2}}. \tag{6.55}$$

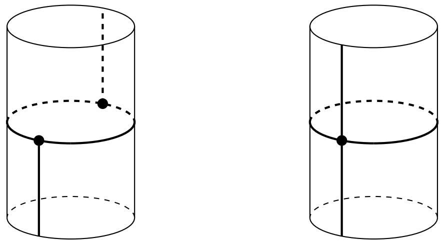
This matches the result for the $\mathbb{RP}^3$ magnon in [11, 7]. In this case there is no problem in taking the $Q \to 0$ limit, which again reproduces the AFZ result and matches the results for the $\mathbb{RP}^2$ magnon [84].

THE BIG MAGNON. The big magnon is only considered in Paper V. The solution is very similar to a single small magnon. In particular there are cuts within the unit circle and we need to be careful when taking the $Q \to 0$ limit.[9] The resulting correction takes the form

$$\delta(\Delta - J) = -128\lambda \frac{S(p)^2}{Q^2 \mathcal{E}(p)} \sin^2 \frac{p}{2} e^{-\frac{\Delta}{\mathcal{S}(p)} \frac{\mathcal{E}(p)}{\mathcal{S}(p)}}, \tag{6.56}$$

where the functions $\mathcal{E}$ and $\mathcal{S}$ take the same form as for the pair of small magnons. The above result is again expressed in terms of the momentum p of one of the small magnons that make up the full solution.

[9] Remember that for the big magnon Q is not an angular momentum but an additional parameter of the solution. If we consider the big magnon as consisting of two small magnons with opposite sign of the charge Q, the parameter can be interpreted as the angular momentum of one of the small magnons.



(a) μ-term *(b)* F-term

Figure 6.2: The Lüscher formulae arise due to virtual particles wrapping the world-sheet. In the μ-term the incoming particle splits into two on-shell particles that travel around the world-sheet and then recombine. The F-term is given by diagrams in which a single virtual particle travel around the circumference of the cylinder.

A careful study of the $Q \to 0$ limit shows that the $\mathbb{RP}^2$ result from the pair of small magnons is again reproduced. This is expected since in this limit there is no way of distinguishing the different charges Q carried by the two small magnons that make up these two solutions.

6.5.3 The Lüscher μ-term

For finite values of the energy Δ the world-sheet of the giant magnon has a finite length. Hence the world-sheet no longer has the topology of a plane, but rather that of a cylinder. A general approach for studying the leading order effects of putting a two-dimensional quantum field theory on a finite world-sheet is by using the so called Lüscher formulae. Physically these corrections appear due to processes where virtual particles travel around the circumference of the cylinder. There are two different kinds of processes that can occur. Figure 6.2 (a) depicts a particle splitting into two *on-shell* particle, which travel around the world-sheet before they recombine. The resulting correction is known as the μ-term. In figure 6.2 (b) the physical particle interacts with a single virtual particle that travels around the cylinder, leading to the F-term correction. Since the μ-term arises from an on-shell process it leads to a correction to the classical energy of the incoming state. The F-term involves an off-shell virtual particle and hence gives a quantum correction to the energy.[10] Since I only discuss finite-size effects for classical giant magnons in this thesis, I will only consider the μ-term.

The original derivation by Lüscher [123] is valid for a particle with a relativistic dispersion relation and relates the corrections of the energy to the

[10] The F-term is suppressed in $1/\sqrt{\lambda}$ compared to the μ-term. Still the F-term can be more important since the exponential suppression of the two terms differ [93].

two-particle S-matrix [115]. The generalization to a non-relativistic dispersion relation was first discussed by Ambjørn, Janik, and Kristjansen [15] and later analyzed in detail by Janik and Łukowski [107], who calculated the leading correction to the classical finite-size giant magnon in $AdS_5 \times S^5$ using the μ-term. The original Lüscher formulae give the shift of the mass for a single physical particle. Here we will also need a more general μ-term formula for multi-particle states, which was independently derived in [99] and in [28].

THE S-MATRIX. Before we can use the μ-term to calculate corrections to the energies of giant magnons in $\mathbb{CP}^3$ we need some facts about the S-matrix of ABJ(M). As explained in chapter 3 there are two kinds of excitations that enter the S-matrix. I will refer to them as A- and B-particles. For each kind of particle there are two bosonic and two fermionic excitations transforming in the fundamental representation of $SU(2|2)$. The S-matrix can be divided into four parts depending on the type of particles involved – the matrices S^{AA} and S^{BB} describing scattering of particles of the same type, and the matrices S^{AB} and S^{BA} describing scattering of particles of different types. The matrix structure of these matrices is the same in all four cases [8] and is fully determined by the $SU(2|2)$ symmetry [40]. Hence they only differ in the overall scalar factor. We can write these S-matrices as

$$S^{AA}(p_1,p_2) = S^{BB}(p_1,p_2) = S_0(p_1,p_2)\hat{S}(p_1,p_2)\,, \tag{6.57}$$

$$S^{AB}(p_1,p_2) = S^{BA}(p_1,p_2) = \tilde{S}_0(p_1,p_2)\hat{S}(p_1,p_2)\,. \tag{6.58}$$

Here $\hat{S}$ is the common $SU(2|2)$ matrix part and the two scalars S_0 and $\tilde{S}_0$ are given by[11]

$$S_0(p_1,p_2) = \frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\sigma(p_1,p_2)\,, \quad \tilde{S}_0(p_1,p_2) = \frac{x_1^- - x_2^+}{x_1^+ - x_1^-}\sigma(p_1,p_2)\,, \tag{6.59}$$

where $\sigma(p_1,p_2)$ is the BES dressing phase [38]. The S-matrix of ABJ(M) is closely related to that of $\mathcal{N}=4$ SYM. In the latter case the excitations form a single multiplet of $SU(2|2)\otimes SU(2|2)$ and the S-matrix consists of two copies of the $SU(2|2)$ matrix $\hat{S}(p_1,p_2)$ with the scalar factor equaling $S_0(p_1,p_2)\tilde{S}_0(p_1,p_2)$.

[11] The complex parameters $x_i^\pm$ are related to the momentum p_i by

$$x_i^+ + \frac{1}{x_i^+} - x_i^- - \frac{1}{x_i^-} = \frac{\sqrt{2}i}{\sqrt{\lambda}}\,, \qquad \frac{x_i^+}{x_i^-} = e^{ip_i}\,.$$

THE $\mathbb{RP}^2$ MAGNON. As we have seen above, we can think of the $\mathbb{RP}^2$ magnon as being composed of two fundamental magnons, with one from each SU(2) sector. Hence we need to use the two-excitation μ-term of Hatsuda and Suzuki [98]. I will not write down the full expression here, but refer to Paper IV. The μ-term contains a sum over all intermediate states and depends on the residue of the S-matrix for scattering between the incoming particle and the intermediate particle at any poles that correspond to physical bound states. Since the $\mathbb{RP}^2$ magnon is constructed from one A-particle and one B-particle there will be two copies of the fundamental S-matrix discussed above. Depending on the type of the intermediate particle, one of the S-matrix factors will be of type S^{AA} or S^{AB} while the other will be either S^{AB} or S^{BB}. In both cases we need to consider the poles of a factor of the form

$$S^{AA}(p,q)S^{BA}(p,q) = S^{AB}(p,q)S^{BB}(p,q) = S_0(p,q)\tilde{S}_0(p,q)\hat{S}(p,q)\hat{S}(p,q)\,.$$

This factor has precisely the same form as a single copy of the S-matrix of $\mathcal{N}=4$ SYM. Hence we can directly import the results from the $\mathrm{AdS}_5 \times \mathrm{S}^5$ calculation in [107]. The result is given in Paper IV, and independently in [50], and agrees with corresponding results in the previous sections of this thesis.

Also for the dyonic $\mathbb{RP}^3$ magnon the μ-term takes the same form as in the corresponding calculation in $\mathcal{N}=4$ SYM, as shown in [7].

THE $\mathbb{CP}^1$ MAGNON. In Paper IV we also considered the $\mathbb{CP}^1$ magnon. Since there is now just a single fundamental magnon we can use the one-particle μ-term of Janik and Łukowski [107]. If the magnon is a particle of type A the only S-matrix that enters the calculation is S^{AA}. The result of the calculation in Paper IV (see also [50]) is

$$\delta(\Delta - J) = 2i\alpha \sin\frac{p}{2} e^{-\frac{\Delta}{\Delta - J}}\,, \tag{6.60}$$

where the factor α formula depends on whether we use the *string frame* ($\alpha = 1$) or *spin-chain frame* ($\alpha = e^{-ip/2}$) of the S-matrix [21].[12] Note that this correction is not real valued. The same thing was observed in the calculation of the correction to the dyonic magnon in $\mathrm{AdS}_5 \times \mathrm{S}^5$ from the μ-term in [98]. In that paper it was argued that the physical correction is obtained as the real part of the μ-term. Even with that modification, the above result does not match the results of the algebraic curve and sigma-model, for any of the values of α. It does however match the result of the algebraic curve calculation for the small magnon in Paper IV, which is obtained by setting $Q = 1/2$ in (6.50).

[12] The two cases differ in the phase of various S-matrix elements. In the first case the S-matrix satisfies the ordinary Yang-Baxter equation while the second case satisfies a twisted Yang-Baxter equation [21].

As we noted above and in Paper V, this result is incorrect since there is an order-of-limits problem in the expansion of the curve.

At the moment this discrepancy between the μ-term and the algebraic curve and sigma model is unresolved. I will end this section with a few remarks on this result.

- Comparing the result (6.60) with the expected form (6.44) there are two main differences: Firstly, the prefactor of the former correction is of lower order in $1/\sqrt{\lambda}$. Hence the corrections in (6.60) has the form of a quantum correction rather than a classical correction. Secondly the two expressions differ by a factor of two in the exponential, with (6.44) being more strongly suppressed than (6.60).
- In the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case the result for the finite-size giant magnon obtained from the μ-term agrees with the AFZ result [21] only if the string frame of the S-matrix is used [107]. This corresponds to $\alpha = 1$, and makes the real part of (6.60) vanish.

One possible way of explaining the discrepancy is to trust the string frame result that the correction calculated in Paper IV and [50] vanishes. Then the leading correction should come from another pole in the S-matrix that was not considered in these papers. Such a pole would give a stronger exponential suppression, in accordance with (6.44).

6.6 Identification of magnons

In this chapter I have listed a number of different giant magnon solutions in $\mathbb{CP}^3$ that have been obtained both from the sigma-model and from the algebraic curve. By comparing the charges carried by the different solutions it is straightforward to match the magnons in the various subspaces of $\mathbb{CP}^3$ to the corresponding algebraic curve solution. The result is collected in table 6.1.

We can also compare the charges of the giant magnons to the charges of the spin-chain excitations considered in chapter 2. As noted in the beginning of this chapter, the point-like string considered here is dual to the spin-chain ground state $\mathcal{O}^L_{gs} = \mathrm{tr}(Y^1 Y_4^\dagger)^L$. To excite the spin-chain we replace a pair of fields $Y^1 Y_4^\dagger$ by some other combination of one bifundamental and one anti-bifundamental field. The charges of these excitations can be read off from table 2.2 of chapter 2. We can then make the identifications given in table 6.2. Note in particular that the small giant magnon corresponds to fundamental excitations of the spin-chain while the big magnon and a pair of small giant magnons are interpreted as multi-excitations on the gauge theory side. The parameter Q_f of the big magnon counts the number of scalars Y^2 (or Y^3) in the spin-chain excitation. The total charge Q vanishes since the state contains an additional Q_f scalars $Y_2^\dagger$ ($Y_3^\dagger$).

Table 6.1: Various giant magnon solutions in $\mathbb{CP}^3$. This table shows the different magnons in the algebraic curve and the corresponding subspace of $\mathbb{CP}^3$ that the magnon lives in. For the big magnon and the pair of magnons, which can be considered to be two-magnon states, the dispersion relation is expressed in terms of the momentum of one small magnons.

Curve solution	Subspace	$\Delta - J$	Q	J_3
Small magnon	$\mathbb{CP}^1$	$\sqrt{2\lambda}\sin\frac{p}{2}$	0	0
	$\mathbb{CP}^2$	$\sqrt{Q^2+2\lambda\sin^2\frac{p}{2}}$	Q	$2Q$
	$\mathbb{CP}^2$	$\sqrt{Q^2+2\lambda\sin^2\frac{p}{2}}$	Q	$-2Q$
Big magnon	$\mathbb{RP}^2$	$\sqrt{8\lambda}\sin\frac{p}{2}$	0	0
	$\mathbb{CP}^2$	$\sqrt{Q_f^2+8\lambda\sin^2\frac{p}{2}}$	0	0
Pair of small magnons	$\mathbb{RP}^2$	$\sqrt{8\lambda}\sin\frac{p}{2}$	0	0
	$\mathbb{RP}^3$	$\sqrt{Q^2+8\lambda\sin^2\frac{p}{2}}$	Q	0

Table 6.2: Giant magnons and spin-chain excitations. From the charges of each giant magnon solution in $\mathbb{CP}^3$ we can identify the corresponding spin-chain excitation. The $\mathbb{CP}^1$ giant magnon corresponds to a single excitation while all other giant magnons correspond to multi-excitations of the spin-chain.

Curve	Magnon	Spin-chain excitation
Small magnon	$\mathbb{CP}^1$ / $\mathbb{CP}^2$	$Y^2Y_4^\dagger$ or $Y^1Y_3^\dagger$
Pair of small magnons	$\mathbb{RP}^2$ / $\mathbb{RP}^3$	$Y^2Y_3^\dagger$
Big magnon	Dressed $\mathbb{CP}^2$	$Y^2Y_2^\dagger$ or $Y^3Y_3^\dagger$

7. Epilogue

The focus of this thesis has been the planar spectrum of the ABJ(M) theory at weak and strong coupling, and in particular the dimensions of local gauge invariant operators and the energies of their dual string states. At both sides of the AdS_4/CFT_3 duality the determination of the spectrum is greatly simplified by the appearance of integrable structures, which allow us to compute the dimensions of long operators consisting of many fields and the energies of strings with large angular momenta.

The fundamental excitations of the integrable model of AdS_4/CFT_3 are the magnons. In chapter 3 and 4 I discussed the dynamics of weakly coupled magnons in the spin-chain picture of the local operators. When we go to the strong coupling side the spin-chain magnons evolve to world-sheet solitons known as giant magnons. The giant magnons appearing in $\mathbb{CP}^3$ were discussed in chapter 6.

7.1 The function $h^2(\bar{\lambda},\sigma)$

The most important ingredients in understanding the spectrum of an integrable model are the two-particle S-matrix and the magnon dispersion relation. In ABJ(M) both these objects are highly constrained by the $SU(2|2)$ symmetry of the system. In particular the dispersion relation takes the form

$$\Delta - J = \sqrt{\frac{1}{4} + 4h^2(\bar{\lambda},\sigma)\sin^2\frac{p}{2}}\,. \tag{7.1}$$

The only quantity that is not fixed by symmetry is the function $h^2(\bar{\lambda},\sigma)$ that encodes the dependence on the coupling constants. The same function also appears in the ABJ(M) S-matrix.

The weak coupling expansion $h^2(\bar{\lambda},\sigma)$ starts with

$$h^2(\bar{\lambda},\sigma) = \bar{\lambda}^2 - \bar{\lambda}^4(4+\sigma^2)\zeta_2 + \mathcal{O}(\bar{\lambda}^6)\,, \tag{7.2}$$

and at strong coupling, it takes the form

$$h(\bar{\lambda},\sigma) = \sqrt{\frac{\bar{\lambda}}{2}} + c + \mathcal{O}\left(\frac{1}{\sqrt{\bar{\lambda}}}\right). \tag{7.3}$$

Hence the function $h^2(\bar{\lambda},\sigma)$ interpolates non-trivially between the weak and strong coupling limits. In particular, since the weak coupling expansion is even in $\bar{\lambda}$ there should be a square root branch cut along the negative real axis.

The constant term c in (7.3) corresponds to a one-loop correction in the string theory. Calculations of such corrections to spinning folded strings in AdS_4 have been performed both from the sigma-model [130, 13, 120, 131] and from the algebraic curve [90]. However, the resulting coefficient c differ in the two approaches, taking a value of $c = -\frac{\log 2}{2\pi}$ or $c = 0$, respectively. The origin of this discrepancy is a difference in regularization prescriptions in the sum over mode numbers [90, 33, 3]. In the sigma-model analyses all modes are treated equally, while the algebraic curve calculation distinguishes between light and heavy excitations, with the motivation that the heavy modes do not correspond to fundamental excitations but are composite states [173]. This leads to different cutoffs for the light and heavy modes. The same two prescriptions have also been used to study one-loop corrections to giant magnons and result in the same values for c as above [156, 3]. The calculation of the finite size corrections to the pp-wave Hamiltonian in [24] seems to favor the $c = 0$ result. There are two possibile resolutions of the discrepancy in the value of c:

1. One of the two prescriptions used in the literature could be unphysical. For example it might break some of the symmetries of the model. The other prescription would then be physically favored and would give a unique value for the one-loop coefficient c.
2. The alternative is that both methods of regularizing are valid and that the function $h^2(\bar{\lambda},\sigma)$ is regularization dependent. This might seem like a surprising conclusion, since the magnon dispersion relation is closely related to the dimensions of gauge invariant operators and string energies, which both are physical quantities. However, the form of the perturbative expansion is not unique but depends on how the coupling constants are defined. One way of avoiding this ambiguity is to define the couplings in terms of a particular physical quantity, such as the cusp anomalous dimension [131].

The magnon dispersion relation in $\mathcal{N} = 4$ super Yang-Mills is also constrained by the SU(2|2) symmetry of the spin-chain ground state, which forces it to take the form

$$\sqrt{Q^2 + 4h^2_{\mathrm{SYM}}(\lambda)\sin^2\frac{p}{2}}\,. \tag{7.4}$$

However, in this case the coupling dependence is much simpler and all indications point to the function $h^2_{\mathrm{SYM}}(\lambda)$ taking the form $h^2_{\mathrm{SYM}}(\lambda) = \lambda/4\pi^2$. This has been confirmed in perturbation theory to three-loops at weak coupling [158] and to two-loops at strong coupling [153, 80]. Moreover, arguments have been given for why the all-loop function should have this form from both the string theory [43] and the gauge theory [155].

Another difference between ABJ(M) and $\mathcal{N}=4$ SYM is that the radius of $\mathbb{CP}^3$ receives quantum corrections [45], while the S^5 radius is protected by supersymmetry [132]. These corrections are relevant for two-loop calculations at strong coupling.

7.1.1 Possible scenarios for an all-loop function

In Paper II and Paper III we discuss two possible all-loop expressions for the function $h^2(\bar{\lambda},\sigma)$ for two different regions of the parameter space of the couplings.

For the ABJM case of $\sigma=0$ it is convenient to define $t \equiv 2\pi i\lambda$, which naturally appear also in the expressions for supersymmetric Wilson loops in ABJ(M) [113, 127, 68], and to introduce the rescaled function $g(\lambda) = (2\pi)^2 h^2(\lambda)$. The proposed function is then given by

$$g(t) = -(1-t)\log(1-t) - (1+t)\log(1+t)\,. \tag{7.5}$$

The weak coupling expansion of this function matches the perturbative two- and four-loop results and the leading strong coupling behavior on the string theory side, while the one-loop correction vanishes. Hence this function corresponds to the $c=0$ result in (7.3) observed in [90] and [24].

There does not seem to be a apparent way of generalizing this function to the case $\lambda \neq \hat{\lambda}$. However the extra parameter gives us more information on the structure of the function. In particular it should be symmetric under [4]

$$\lambda \to \hat{\lambda}\,, \qquad \hat{\lambda} \to 2\hat{\lambda} - \lambda + 1\,. \tag{7.6}$$

Moreover the ABJ model is only physically consistent for $|\lambda - \hat{\lambda}| \leq 1$.

In Paper III we consider the limit $\hat{\lambda} \ll \lambda$. As we have seen in chapter 4 the dilatation operator simplifies considerably in this limit, taking the form

$$\delta\mathfrak{D} \approx \lambda\hat{\lambda} f(\lambda)(\chi(1)+\chi(2))\,, \tag{7.7}$$

where the function $f(\lambda)$ now encodes the remaining coupling dependence. Up to four-loops we have

$$f(\lambda) \approx 1 - \zeta_2\lambda^2 = 1 - \frac{\pi^2}{6}\lambda^2\,, \tag{7.8}$$

which suggest an all-loop function

$$f(\lambda) = \frac{1}{\pi}\sin(\pi\lambda)\,. \tag{7.9}$$

This function has several of the features we expect from the full answer. Firstly $f(\lambda)$ becomes negative for $\lambda > 1$. But then the anomalous dimensions also becomes negative. In particular some short operators such as $\mathrm{tr}\,Y^I Y_I^\dagger$, with classical dimension one will get a dimension slightly less then one. This however breaks the unitary bound for superconformal theories in three dimensions [140], and hence the theory becomes unphysical. In the limit under consideration this perfectly agrees with the constraint $|\lambda - \hat{\lambda}| \approx \lambda \leq 1$.

The expansion of $h^2(\bar{\lambda}, \sigma)$ up to four loops is parity invariant. Let us assume that this will hold also at higher orders. Then the full function should be symmetric under exchange of λ and $\hat{\lambda}$. Combining this with the symmetry (7.6) we get the transformation

$$\lambda \to 2\hat{\lambda} - \lambda + 1\,, \qquad \hat{\lambda} \to \hat{\lambda}\,. \tag{7.10}$$

For $\hat{\lambda} \ll \lambda$ this reduces to $\lambda \to 1 - \lambda$, which is consistent with the function $f(\lambda)$ in (7.9).

7.2 Wrapping interactions

If our final goal was to determine the spectrum of operator consisting of infinitely many fields or of strings for which the world-sheet was the complex plane, integrability would give us the full answer and the only remaining task would be to determine the function $h^2(\bar{\lambda}, \sigma)$. However, physical operators consists of a finite number of fields and string theory in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is a theory of closed strings. To obtain the full answer we therefore need to add finite-size corrections to the result we get from the asymptotic Bethe equations.

In this thesis I have discussed the explicit calculation of such corrections in two different regimes. At weak coupling they take the form of wrapping Feynman diagrams, which are planar only for short operators. In Paper I and Paper II we calculate the contribution from these diagrams at four loops. At strong coupling there are corrections due to intermediate states wrapping the world-sheet cylinder. Their leading, exponentially suppressed, contribution to the classical energy of giant magnons in $\mathbb{CP}^3$ was computed in Paper IV and Paper V.

The methods for obtaining finite-size corrections that I have discussed in this thesis are useful at the leading few orders in perturbation theory at weak or strong coupling. To get beyond this we need some other approach. Recently there has been a lot of progress in understanding the full planar spectrum of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ using the Thermodynamic Bethe Ansatz (TBA) [172], see for example the recent reviews by Bajnok [29] and by Gromov and Kazakov [88], and references therein. Also in the case of ABJ(M) some progress towards a formulation of the TBA have been made. In [94] a set of Y-system equations were conjectured and using these equations the leading wrapping corrections to a scalar operator of length four in the **20** representation of $\mathrm{SU}(4)_R$ was calculated. Furthermore, the TBA equations for both the ground-state and excited states have been formulated in [52] and in [89].

7.3 Future work

Writing a thesis is an exercise in looking back. Still I want to end this summary with a few remarks on possible extensions of the results described here.

The four-loop calculation discussed in chapter 4 only involved the closed $\mathrm{SU}(2)\times\mathrm{SU}(2)$ sector. The resulting dilatation operator is trivially integrable, in the sense that any result we could have obtained would have been compatible with integrability [34]. To actually get a check of integrability of ABJ(M) at four loops we would need to consider a larger sector. Such a calculation will in general require the evaluation of a very large number of additional Feynman diagrams. A more tractable problem is obtained in the limit $\hat{\lambda} \ll \lambda$, where we only need to consider diagrams contributing to the $\hat{\lambda}\lambda^3$ term of the dilatation operator. The result would give us a check on the integrability of the ABJ model, under the assumption that ABJM is integrable.

In section 6.5.3 we saw that the leading finite-size correction to the energy of the giant magnon in $\mathbb{CP}^1$ obtained from the Lüscher formula does not have the expected form. The difference between the algebraic curve results in Paper IV and Paper V arises due to an order-of-limits problem. It would be interesting to see if there is a similar problem in the μ-term calculation.

No direct tests of quantum integrability at the strong coupling side of the duality have been performed. One possibility would be to check that the worldsheet three-particle S-matrix factorizes. For the full sigma-model such a calculation would be very involved, but it should be doable in a simplifying limit such as the near-flat-space limit [124]. In the case of $\mathrm{AdS}_5\times\mathrm{S}^5$ we performed such a check in [78]. It would also be interesting to calculate the corrections to the dispersion relation in this limit. For related calculations in $\mathrm{AdS}_5/\mathrm{CFT}_4$ see [117, 118].

Further investigations of the one-loop corrections at strong coupling are needed in order to understand the origin of the regularization scheme dependence discussed above.

Finally the function $h^2(\bar{\lambda},\sigma)$ calls for further investigation. A six-loop calculation is a daunting task, but may be feasible using the superspace techniques from Paper II, in particular in the extreme ABJ limit discussed above. However, it is not clear that another coefficient in the weak coupling expansion would help us find the full function. It would be interesting to examine in more detail the constraints put on the function $h^2(\bar{\lambda},\sigma)$ from the equivalence of the ABJ model at various values of the parameters M, N and k discussed in this section. Perhaps these symmetries together with the information on the asymptotic behavior will allow us to obtain an exact result.

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Summary in Swedish

Fundamentala excitationer i ABJM-modellen

En central fråga i modern teoretisk fysik är hur kvantmekanik och gravitation kan förenas i en teori. En sådan beskrivning behövs om vi vill förstå oss på hur vårt universum såg ut när det var mycket ungt, eller hur ett svart hål ser ut på nära håll.

Kvantmekaniken ger en beskrivning av fysik på mycket korta avstånd. I synnerhet är partikelfysikens standardmodell en kvantfältteori som beskriver hur punktformiga elementarpariklar, till exempel elektroner och kvarkar, växelverkar med varandra genom tre av de fundamentala krafterna: de starka och svaga kärnkrafterna och den elektromagnetiska kraften. Växelverkan förmedlas genom utbyte av budbärarpartiklar – gluoner, W- och Z-bosoner, och fotoner.

Den fjärde fundamentala kraften, gravitationen, ger upphov till attraktion mellan massiva kroppar. Denna kraft är mycket svagare än de övriga tre naturkrafterna, men är viktig på stora avstånd. De två kärnkrafterna har en mycket kort räckvidd. Både gravitationen och elektromagnetismen når däremot långt. Elektriska laddningar kan vara både positiva och negativa. En stor kropp, som en planet eller en katt, består av en stor mängd partiklar vars totala elektriska laddning är mycket nära noll. Den elektromagnetiska attraktionen eller repellationen mellan katten och planeten är därför försumbar. En massa är däremot alltid positiv och gravitationen är alltid attraktiv och drar därför katten och planeten mot varandra. Gravitationen beskrivs av Einsteins allmänna relativitetsteori.

Den idag dominerande teorin för kvantgravitation är *strängteorin*. De fundamentala objekten är här inte punktformiga partiklar utan vibrerande strängar. Beroende på hur de vibrerar kan de tolkas som olika partiklar. Bland dessa återfinner vi alla partiklar från standardmodellen, men också gravitonen – budbärarpartikeln för gravitation.

En annan viktig olöst fråga är hur vi effektivt kan beskriva kvantfältteori vid stark koppling, det vill säga när partiklarna växelverkar starkt med varandra. Protonerna och neutronerna i en atomkärna är uppbyggda av kvarkar och hålls samman av den starka kärnkraften, vilken beskrivs av kvantkromodynamiken (QCD), en kvantfältteori som är en del av den ovan nämnda standardmodellen.

Med hjälp av QCD kan vi förstå resultaten från experiment där två protoner krockar med varandra med mycket hög hastighet, som vid den nyss startade acceleratorn LHC vid Cern i Schweiz. En annan fråga som QCD borde kunna ge oss svaret på är hur kärnpartiklar som protoner och neutroner får sin massa. Här stöter vi dock på problem. Anledningen till detta är att de gluoner som överför växelverkan mellan kvarkarna växelverkar mycket starkt med varandra inuti kärnpartiklarna. Detta medför att de gängse metoderna för att utföra beräkningar i en kvantfältteori, såsom störningsteori, bryter ihop. En full förståelse av fysiken inuti en atomkärna kräver därför helt nya beräkningstekniker.

Dessa två frågor – beskrivningen av kvantgravitation och av starkt kopplad kvantfältteori – möts i AdS/CFT-dualiteten. Två duala fysikaliska teorier tycks på ytan beskriva två helt olika system men ger i själva verket två olika beskrivningar av ett och samma system. De två teorier som relateras till varandra i AdS/CFT-dualiteten är strängteori i ett visst krökt rum och kvantfältteori med så kallad konform symmetri.

Förkortningen "AdS" står för anti-de Sitter-rum, en lösning till Einsteins rörelseekvationer som har formen av en hyperboloid. På den ena sidan av AdS/CFT-dualiteten har vi supersymmetriska strängar som rör sig i ett tiodimensionellt rum som delvis utgörs av just ett anti-de Sitter-rum. Ett sådant rum är oändligt stort men har ändå en rand. På denna rand lever den duala fältteorin. En konform fältteori (CFT) ser likadan ut oavsett från vilken skala vi betraktar den. En sådan symmetri begränsar starkt hur teorin kan bete sig. De centrala objekten är inte längre partiklar utan lokala operatorer som fås som en produkt av de olika fälten i teorin. En viktig egenskap hos dessa operatorer är deras dimension, som säger hur de beter sig när vi gör en omskalning av teorin. När vi tar med kvanteffekter kan en operators dimension få korrektioner – en anomal dimension. Lokala operatorer är enligt AdS/CFT duala till strängtillstånd i AdS-rummet och operatorernas dimensioner motsvarar energin hos dessa tillstånd. Det första steget för att i detalj förstå teorierna på båda sidorna av AdS/CFT-dualiteten är att beräkna spektrumet av operatorer och deras dimensioner samt energin hos de motsvarande strängarna.

Den mest studerade konkretiseringen av AdS/CFT-dualiteten brukar kallas $\mathrm{AdS}_5/\mathrm{CFT}_4$. Siffran fyra anger att fältteorin är fyrdimensionell. Närmare bestämt har vi att göra med maximalt supersymmetrisk Yang-Mills-teori med gaugegrupp $\mathrm{SU}(N)$. Den duala strängteorin är av typ IIB och lever i ett rum bestående av femdimensionellt anti-de Sitter-rum, AdS_5, samt en femdimensionell sfär, S^5.

Ett annat exempel på AdS/CFT föreslogs sommaren 2008 av Aharony, Bergman, Jafferis och Maldacena: $\mathrm{AdS}_4/\mathrm{CFT}_3$. Denna dualitet är i fokus i denna avhandling. Fältteorin, som kallas för ABJM-modellen efter de ursprungliga författarna, är nu en tredimensionell supersymmetrisk Chern-Simons-teori med gaugegrupp $\mathrm{U}(N) \times \mathrm{U}(N)$. Denna lever på randen av ett rum vars geometri är $\mathrm{AdS}_4 \times \mathbb{CP}^3$ och som beskrivs av supersymmetrisk strängteori av typ IIA. AdS_4 är igen ett anti-de Sitter-rum, men av en dimension lägre än ovan och $\mathbb{CP}^3$ är en sexdimensionell generalisering av en sfär.

En viktig egenskap hos AdS/CFT-dualiteten är att den relaterar en starkt kopplad och en svagt kopplad teori till varandra. Detta kan användas som ett kraftfullt verktyg för att utföra beräkningar. Genom att använda störningsteori i strängteori kan vi lära oss något om starkt kopplad kvantfältteori, och en perturbativ analys av fältteorin oss information om kvantgravitation i ett starkt krökt rum. Denna egenskap gör dock duliteten svår att bevisa – det finns ingen regim där vi kan hantera både strängteorin och fältteorin fullt ut.

Trots detta har det under de senaste åren gjorts stora framsteg när det gäller att förstå både $\mathrm{AdS}_5/\mathrm{CFT}_4$ och $\mathrm{AdS}_4/\mathrm{CFT}_3$. Anledningen till att detta har varit möjligt är att teorierna på båda sidorna av dessa dualiteterna i den så kallade planära gränsen är *integrerbara*. I en integrerbar teori finns det, utöver rörelsemängd och energi, en oändlig mängd extra bevarade storheter. Med dessa till hjälp kan vi i princip lösa modellen *exakt*. De strukturer som leder till integrerbarhet i $\mathrm{AdS}_5/\mathrm{CFT}_4$ och $\mathrm{AdS}_4/\mathrm{CFT}_3$ liknar varandra i mycket, men det senare fallet är mer komplicerat och inte lika väl förstått.

Min avhandling består av fem av artiklar där vi undersöker dessa strukturer både i ABJM-modellen och i den duala strängteorin på $\mathrm{AdS}_4 \times \mathbb{CP}^3$. I synnerhet studerar vi de fundamentala excitationerna i de integrerbara modellerna, så kallade *magnoner*. Dessa är mer komplicerade i $\mathrm{AdS}_4/\mathrm{CFT}_3$ än i $\mathrm{AdS}_5/\mathrm{CFT}_4$, dels för att det finns olika typer av magnoner med olika massor, och dels för att magnonernas dispersionsrelation, det vill säga relationen mellan rörelsemängd och energi, får kvantmekaniska korrektioner.

I artikel I–III presenteras två perturbativa beräkningar av dessa korrektioner till dispersionsrelationen hos magnonerna i ABJM-modellen. Beräkningarna är gjorda till fjärde ordningen i störningsteori. Jag finner att våra resultat är en viktig utgångspunkt för att förstå hur operatorerna i teorin förändras när vi går från svag till stark koppling.

Artikel IV och V handlar om magnoner på strängteorisidan av dualiteten, så kallade “giant magnons”. I $\mathrm{AdS}_4/\mathrm{CFT}_3$ finns det flera olika typer av sådana tillstånd. Genom att beräkna deras energier med flera olika metoder klassificerar vi dessa lösningar och identifierar vilken excitation på fältteorisidan av dualiteten de motsvarar. Denna identifikation är central för att förstå relationen mellan strängteori i $\mathrm{AdS}_4 \times \mathbb{CP}^3$ och ABJM-modellen.

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