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Approximating the Binomial Distribution by the Normal Distribution – Error and Accuracy

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Abstract

Different rules of thumb are used when approximating the binomial distribution by the normal distribution. In this paper an examination is made regarding the size of the approximations errors. The exact probabilities of the binomial distribution is derived and then compared to the approximated value of the normal distribution. In addition a regression model is done. The result is that the different rules indeed gives rise to errors of different sizes. Furthermore, the regression model can be used in order to get guidance of the maximum size of the error.

Acknowledgenment

Thank you Professor Sven Erick Alm!

Contents

1 Introduction

Neither is any extensive examination found, regarding the rules of thumb used when approximating the binomial distribution by the normal distribution, nor of the accuracy and the error which they result in. The scope of this paper is the most common approximation of a Binomial distributed random variable by the normal distribution. We let $X \sim \text{Bin}(n, p)$, with expectation $E(X) = np$ and variance $V(X) = np(1 - p)$, be approximated by Y, where $Y \sim \text{N}(np, np(1 - p))$. We denote, $X \approx Y$.

The rules of thumb, is a set of different guidelines, minimum values or limits, here denoted L for $np(1 - p)$, in order to get a good approximation, that is, $np(1-p) \geq L$. There are various kinds of rules found in the literature and any extensive examination of the error and accuracy has not been found. Reasonable approaches when comparing the errors are, the maximum error and the relative error, which both are investigated.

The main focus lies on two related topics. First, there is a shorter section, where the origin of the rules, where they come from and who is the originator, is discussed. Next comes an empirical part, where the error affected by the different rules of thumb is studied. The result is both plotted and tabled. An analysis of regression is also made, which might be useful as a guideline when estimating the error in situations not covered here. In addition to the main topics, a section dealing with the preliminaries, notation and definitions of probability theory and mathematical statistics is found. Each of the sections will be more explanatory themselves regarding their topics. I presume the reader to be familiar with some basic concepts of mathematical statistics and probability theory, otherwise the theoretical part would range way to far. Therefor, also proofs and theorems are just referred to. Finally there is a summarizing section, where the results of the empirical part are discussed.

2 Theory and methodology

First of all, the reader is assumed to be familiar with basic concepts in mathematical statistics and probability theory. Furthermore there are, as stated above, some theory that instead of being explicitly explained, only is referred to. Regarding the former, I suggest the reader to view for instance [1] or [4] and concerning the latter the reader may want to read [7].

2.1 Characteristics of the distributions

As the approximation of a binomial distributed random variable by a normal distributed random variable is the main subject, a brief theoretical introduction about them is made. We start with a *binomial distributed random*

variable, X and denote,

$$X \sim \text{Bin}(n,p), \text{ where } n \in \mathbb{N} \text{ and } p \in [0,1].$$

The parameters p and n are the probability of an outcome and the number of trials. The expected value and variance of X are,

$$E(X) = np \text{ and } V(X) = np(1-p),$$

respectively. In addition, X has got the probability function

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ where } 0 \leq k \leq n,$$

and the cumulative probability function, or distribution function,

$$F_X(k) = P(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{k-i}. \tag{1}$$

The variable X is approximated by a normal distributed random variable, call it Y, we write,

$$Y \sim \text{N}(\mu, \sigma^2), \text{ where } \mu \in \mathbb{R} \text{ and } \sigma^2 < \infty.$$

The parameters μ and σ^2 are the mean value and variance, $E(Y)$ and $V(Y)$, respectively. The density function of Y is

$$f_Y(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

and the distribution function is defined by,

$$F_Y(k) = P(Y \leq x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2} dt. \tag{2}$$

2.2 Approximation

Thanks to De Moivre, among others, we know by the central limit theorem that a sum of random variables converges to the normal distribution. A binomial distributed random variable X may be considered as a sum of Bernoulli distributed random variables. That is, let Z be a Bernoulli distributed random variable,

$$Z \sim \text{Be}(p) \text{ where } p \in [0,1],$$

with probability distribution,

$$p_Z = P(Z = k) = \begin{cases} p & \text{for } k = 1 \\ 1 - p & \text{for } k = 0 \end{cases}.$$

Consider the sum of n independent identically distributed Z_i's, i.e.

$$X = \sum_{i=0}^{n} Z_i$$

and note that $X \sim \text{Bin}(n, p)$. For instance one can realize that the probability of the sum being equal to k, $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$. Hence, we know that when $n \to \infty$, the distribution of X will be normal and for large n approximately normal. How large n should be in order to get a good approximation also depends, to some extent, on p. Because of this, it seems reasonable to define the following approximations. Again, let $X \sim \text{Bin}(n, p)$ and $Y \sim \text{N}(\mu, \sigma^2)$. The most common approximation, $X \approx Y$, is the one where $\mu = np$ and $\sigma^2 = np(1 - p)$, this is also the one used here. Regarding the distribution function we get

$$F_X(k) \approx \Phi \left(\frac{k - np}{\sqrt{np(1 - p)}} \right), \tag{3}$$

where $F_X(k)$ is defined in (1) and Φ is the standard normal distribution function. We extend the expression above and get that,

$$F_X(b) - F_X(a) = P(a < X \leq b) \approx \Phi \left(\frac{b - np}{\sqrt{np(1 - p)}} \right) - \Phi \left(\frac{a - np}{\sqrt{np(1 - p)}} \right). \tag{4}$$

2.3 Continuity correction

We proceed with the use of continuity correction, which is recommended by [1], suggested by [4] and advised by [9], in order to decrease the error, the approximation (3) will then be replaced by

$$F_X(k) \approx \Phi \left(\frac{k + 0.5 - np}{\sqrt{np(1 - p)}} \right) \tag{5}$$

and hence (4) is written as

$$F_X(b) - F_X(a) = P(a < X \leq b) \approx \Phi \left(\frac{b + 0.5 - np}{\sqrt{np(1 - p)}} \right) - \Phi \left(\frac{a + 0.5 - np}{\sqrt{np(1 - p)}} \right). \tag{6}$$

This gives, for a single probability, with the use of continuity correction, the approximation,

$$p_X(k) = F_X(k) - F_X(k-1) \approx \Phi\left(\frac{k + 0.5 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{(k-1) + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

(7)

and further we note that it can be written

$$F_X(k) - F_X(k-1) \approx \int_{k-0.5}^{k+0.5} f_Y(t)dt.$$

(8)

2.4 Error

There are two common ways of measuring an error, the absolute error and the relative error. In addition another usual measure of how close, so to speak, two distributions are to each other, is the *supremum norm*

$$\sup_A |P(X \in A) - P(Y \in A)|.$$

However, from a practical point of view, we will study the absolute error and relative error of the distribution function. Let a denote the exact value and $\bar{a}$ the approximated value. The *absolute error* is the difference between them, the real value and the one approximated. The following notation is used,

$$\varepsilon_{abs} = |a - \bar{a}|.$$

Therefor, the *absolute error of the distribution function*, denoted $\varepsilon_{F_{abs}}(k)$, for any fixed p and n, where $k \in \mathbb{N} : 0 \leq k \leq n$, without use of continuity correction, is

$$\varepsilon_{F_{abs}}(k) = \left|F_X(k) - \Phi\left(\frac{k - np}{\sqrt{np(1-p)}}\right)\right|.$$

(9)

Regarding the relative error, in the same way as before, let a be the exact value and $\bar{a}$ the approximated value. Then the *relative error* is defined as

$$\varepsilon_{rel} = \left|\frac{a - \bar{a}}{a}\right|.$$

This gives the *relative error of the distribution function*, denoted $\varepsilon_{F_{rel}}(k)$, for any fixed p and n, where $k \in \mathbb{N} : 0 \leq k \leq n$, without use of continuity correction, is

$$\varepsilon_{F_{rel}}(k) = \frac{\varepsilon_{F_{abs}}(k)}{F_X(k)},$$

or equivalently, inserting $\varepsilon_{F_{abs}}(k)$ from (9),

$$\varepsilon_{F_{rel}}(k) = \frac{\left| F_X(k) - \Phi\left(\frac{k - np}{\sqrt{np(1-p)}}\right) \right|}{F_X(k)}.$$

2.5 Method

The examination is done in the statistical software R. The software provides predefined functions for deriving the distribution function and probability function of the normal and binomial distributions. The examination is split into two parts, where the first part deals with the absolute error of the approximation of the distribution function and the second part concerns the relative error. The conditions under which the calculations are made, are those found as guidelines in [4]. The calculations will be made with the help of a two-step algorithm. At the end of each section a linear model is fitted to the error. Finally, an overview, where a table and a plot of how the value of npq, where $q = 1 - p$, affects the maximum approximation error for different probabilities are presented.

2.5.1 Algorithm

The two step algorithm below is used. The values of npq mentioned in the literature are, in all cases said to be equal or larger than some limit, here denoted L. The worst case scenario, as to speak, is the case where they are equal, that is, $npq = L$. Therefor equalities are chosen as limits. We know that $n \in \mathbb{N}$, which means that p must be semi-fixed if the equality should hold, this means that the values of p are adjusted, but still remain close to the ones initially chosen. The way of doing this is a two-step algorithm. First a reasonable set of different initial probabilities, $\tilde{p}_i$'s are chosen, whereafter the corresponding $\tilde{n}_i$ values, which in turn will be rounded to n_i, are derived. These are used to adjust $\tilde{p}_i$ to p_i so that the equality will hold.

1. (a) Chose a set $\tilde{\mathbf{P}}$ of different initial probabilities, $\tilde{p}_i \in [0, 0.5]$, where $i \in \mathbb{N} : 0 < i < \left| \tilde{\mathbf{P}} \right|$.

   (b) Derive the corresponding $\tilde{n}_i \in \mathbb{R}^+$ so that $\tilde{n}_i \tilde{p}_i (1 - \tilde{p}_i) = L$,

   (c) continue by deriving $n_i \in \mathbb{N}$, in order to get a integer,

   $$n_i(p_i) := \min\{n \in \mathbb{N} : n\tilde{p}_i(1 - \tilde{p}_i) \geq L\}. \tag{10}$$

   Now we got a set of $n_i \in \mathbb{N}$, denote it $\mathbf{N}$.

2. Chose a set $\mathbf{P}$ so that for every $p_i \in \mathbf{P}$,

$$n_i p_i (1 - p_i) = L.$$

The result is that we always keep the limit L fixed. Let us take a look at an example. Let $L = 10$, use continuity correction and the initial $\tilde{\mathbf{P}} = 0.1(0.1)0.5$,

Exemplifying table of algorithm values					
i	1	2	3	4	5
$\tilde{p}_i$	0.1	0.2	0.3	0.4	0.5
$\tilde{n}_i$	111.11	62.50	47.62	41.67	40.00
n_i	112	63	48	42	40
p_i	0.099	0.198	0.296	0.391	0.500

Different rules of thumb are suggested by [4]. Using approximation (3) the authors say that $np(1 - p) \geq 10$ gives reasonable approximations and in addition, using (5), it may even be sufficient using $np(1 - p) \geq 3$. The investigation takes place under three different conditions,

- $np(1 - p) = 10$ without continuity correction, suggested in [4],

- $np(1 - p) = 10$ with continuity correction, suggested in [2],

- $np(1 - p) = 3$ with continuity correction, suggested in [4].

The investigation of the rules is made only for $p_i \in [0, 0.5]$ due to symmetry. As we see, $np(1 - p)$ simply gets the same values for $p \in [0, 0.5]$ as for $p \in [0.5, 1]$. So, for every p, $n_i(p_i)$ is derived, this in turn, means that we get $n_i(p_i) + 1$ approximations. For every $n_i(p_i)$, and of course p_i as well, we define *the maximum absolute error of the approximation of the distribution function*,

$$M_{F_{abs}} = \max\{\varepsilon_{F_{abs}}(k) : 0 \leq k \leq n_i(p_i)\}, \tag{11}$$

and in addition *the maximum relative error*

$$M_{F_{rel}} = \max\{\varepsilon_{F_{rel}}(k) : 0 \leq k \leq n_i(p_i)\}. \tag{12}$$

The results are both tabled and plotted.

2.5.2 Regression

Beforehand, some plots where made which indicated that the maximum absolute error could be a linear function of p. Regarding the relative maximum error, a quadratic or cubic function of p seemed plausible. Because of that, a regression is made. The model assumed to explain the absolute error is

$$M_\varepsilon = \alpha + \beta p + \epsilon_l, \tag{13}$$

where M_ε is the maximum error, α is the intercept, β the slope and ϵ_l the error of the linear model. For the relative error, the two additional regression models are,

$$M_\varepsilon = \alpha + \beta p + \gamma p^2 + \epsilon_l \tag{14}$$

and

$$M_\varepsilon = \alpha + \beta p + \gamma p^2 + \delta p^3 + \epsilon_l. \tag{15}$$

3 Background

In the first basic courses in mathematical statistics, the approximations (3) and (5) are taught. Students have learned some kind of rules of thumb they should use when applying the approximations, myself included, for example the rules suggested by Blom [4],

$$np(1-p) \geq 10,$$
$$np(1-p) \geq 3 \text{ with continuity correction.}$$

Any motivation why the limit L is set to be $L = 10$ and $L = 3$ respectively is not found in the book. On the other hand, in 1989 Blom claims that the approximation "gives decent accuracy if npq is approximately larger than 10" with continuity correction [2]. Further, it is interesting, that Blom changes the suggestion between the first edition of [3] from 1970, where it says, similarly as above, that it "gives decent accuracy if $np(1-p)$ is approximately larger than 10" *with* continuity correction, and in the second edition from 1984 the same should yield, but now instead *without* use of continuity correction, the conclusion is that there has been some fuzziness regarding the rules. Neither have I, nor my advisor Sven Erick Alm, found any examination of the accuracy of these rules anywhere else. With Blom [4] as starting-point, I begun backtracking, hoping that I could find the source of the rules of thumb. It is worth mentioning that among authors, slightly different rules have been used. For instance Alm himself and Britton, present a schema with rules for approximating distributions, in which $np(1-p) > 5$ with continuity correction is suggested [1]. Even between countries, or from an international point of view, so to speak, differences are found. Schader and Schmid [10] says that "by far the most popular are"

$$np(1-p) > 9$$

and

$$np > 5 \text{ for } 0 < p \le 0.5,$$
$$n(1-p) > 5 \text{ for } 0.5 < p < 1,$$

which I am not familiar with and I have not found in any Swedish literature. In the mid-twentieth century, more precise 1952, Hald [9] wrote,

> An exhaustive examination on the accuracy of the approximation formulas has not yet been made, and we can therefore only give rough rules for the applicability of the formulas.

With these words in mind, the conclusion is that there probably does not exist any earlier work made about the accuracy of the approximation. However, Hald himself made an examination in the same work for $npq > 9$. Further he also points out that in cases where the binomial distribution is very skew, $p < \frac{1}{n+1}$ and $p > \frac{n}{n+1}$, the approximation cannot be applied. Some articles have been found that briefly discuss the accuracy and error of the distributions. Mainly, the focus of the articles lies on some more advanced method of approximating than (3) or (5). An update of [2] has been made by Enger, Englund, Grandell and Holst in 2005, [4]. The writers have been contacted and Enger was said to be the one that assigned the rules. Hearing this made me believe that the source could be found. However, Enger could not recall from where he had got it [6]. That is how far I could get. Nevertheless, the examination remains as interesting as beforehand.

Discussing rules for approximating, one can not avoid at least mentioning the Berry-Esseen theorem. The theorem gives a conservative estimate, in the sense that it gives the largest possible size of the error. It is based upon the rate of convergence of the approximation to the normal distribution. The Berry-Esseen theorem will not be further examined here, but there are several interesting articles due to that the theorem is improved every now and then, most recently in May 2010 [11].

4 The approximation error of the distribution function

The errors of the approximations, $M_{F_{abs}}$ and $M_{F_{rel}}$, defined in (11), and (12) respectively, are plotted and tabled. The cases that are examined are those mentioned earlier, suggested by [4].

4.1 Absolute error

We examine the absolute maximum errors of the approximation of the distribution function, $M_{F_{abs}}$ defined in (11), here in the first part. In addition to

that a regression is made, defined in (13), to see if we might find any linear
trend.

Case 1: $npq = 10$, without continuity correction

First, the case where $L = 10 = npq$, without continuity correction. $\tilde{\mathbf{P}}$, the
set of different initial probabilities is chosen to be $\tilde{p}_i = 0.01(0.01)0.50$. This
means that we use 50 equidistant $\tilde{p}_i$. The smallest probability is $p_1 = 0.0100$
and it has the largest error $M_{F_{abs}} = 0.0831$. $M_{F_{abs}}$ decreases the closer to
0.5 we get, which is natural since the binomial distribution tends to be skew.
The points make a pattern which is a bit curvy, but still the points are close
to the straight line in Figure 1. Another remark made, is that the distance
between the probabilities decreases the closer to 0.5 we get. The fact that
there are several $\tilde{n}_i$ rounded to the same value of n_i, which in turn gives
equal values on p_i, makes several $M_{F_{abs}}$ the same, and plotted in the same
spot. So they are all there, but not visible due to that reason. Next we try
to fit a linear model for $M_{F_{abs}}$. The result is

$$M_{F_{abs}} = 0.0836 - 0.0417p + \epsilon_l.$$

The regression line is the straight line in Figure 1. The slope of the line
shows that the size of $M_{F_{abs}}$ changes moderately. Note that the sum of the
errors of the regression line, $\sum |\epsilon_l|$, is relatively small, the result should be
somewhat precise estimates of $M_{F_{abs}}$ for probabilities which are not taken in
consideration here.
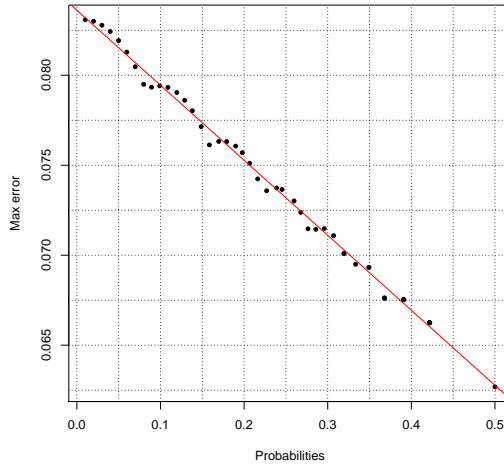
Figure 1: Maximum absolute error for $npq = 10$ *without* continuity correc-
tion. The straight line is the regression line, $M_{F_{abs}} = 0.0836 - 0.0417p$.

Case 2: $npq = 10$, with continuity correction

Under these circumstances $M_{F_{abs}}$ decreases and is about four times smaller than without continuity correction. The regression line,

$$M_{F_{abs}} = 0.0209 - 0.0416p + \epsilon_l, \tag{16}$$

also has got a four times smaller intercept than in the first case. What is interesting is that, the slope is approximately the same in both cases, this in turn, means that for every $\tilde{p}_i = 0.01(0.01)0.50$, it holds that $M_{F_{abs}}$ also is four times smaller. This can be seen in Figure 2.
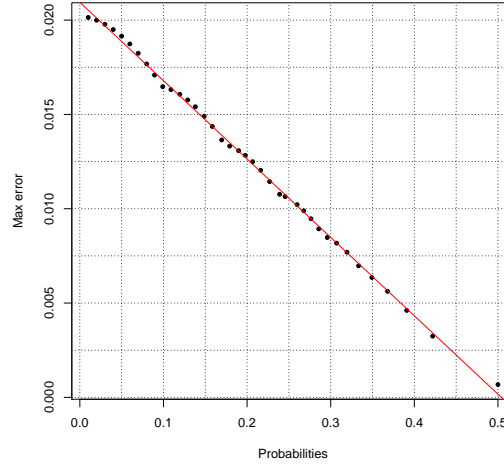
Figure 2: Maximum absolute error for $npq = 10$ *with* continuity correction. The straight line is the regression line, $M_{F_{abs}} = 0.0209 - 0.0416p + \epsilon_l$.

Case 3: $npq = 3$, with continuity correction

Finally we take a look at the last case, regarding the absolute error, where $L = 3 = npq$ and continuity correction is used. The plot is seen in Figure 3. $\tilde{\mathbf{P}}$ is the same as above. In this case the regression line is

$$M_{F_{abs}} = 0.0373 - 0.0720p + \epsilon_l.$$

The largest error, $M_{F_{abs}} = 0.0355$ appears at $p_1 = 0.0100$ and is about twice the size compared to the largest $M_{F_{abs}}$ for $L = 10$. The slope of the line is more aggressive here, which in turn results in errors, one order of magnitude less than in the Case 1 for probabilities close to 0.5. Also here the sum of

discrepancy from the regression line is relatively small which should result in fairly good estimations of $M_{F_{abs}}$.
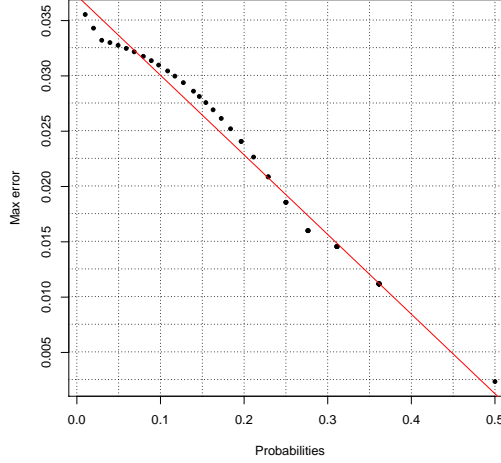

Figure 3: Maximum absolute error for $npq = 3$, *with* continuity correction. The straight line is the regression line, $M_{F_{abs}} = 0.0373 - 0.0720p$.

4.2 Relative Error

Here, the maximum relative error of the approximation of the distribution function, $M_{F_{rel}}$, defined in (12) is examined. The regression models (14) and (15) are both tested.

Case 1: $npq = 10$, without continuity correction

In the first case we perform the calculations under, $L = 10 = npq$ without continuity correction. The result is shown in Figure 4. As we see $M_{F_{rel}}$ increases very rapidly. The smallest value of $M_{F_{rel}}$, 16.97317 is at p_1. The largest 138.61756 at p_{50}. As we see in Table 4, it is $k = 0$ that gives the largest error. For other values of k the error is much smaller. Furthermore we note that $M_{F_{rel}}$ is very large. If we look at a specific example where $p = 0.2269$, which means that $n = 57$, then $X \sim \text{Bin}(57, 0.2269)$. Let X be approximated, according to (3), by $Y \sim \text{N}(12.933, 3.162078)$. We get that $P(X \leq 1) = 7.55 \cdot 10^{-6}$ and $P(Y \leq 1) = 8.04 \cdot 10^{-5}$. Under these circumstances we get,

$$M_{F_{rel}} = \frac{|P(X \leq 1) - P(Y \leq 1)|}{P(X \leq 1)} = 9.64.$$

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The result is shown in Table 4. So the relative error is, as we also can see, large, for small k and small probabilities. The regression curves, defined in (14) and (15) are,

$$M_{F_{rel}} = 14.66 + 69.86p + 416.14p^2 + \epsilon_l$$

and

$$M_{F_{rel}} = 21.53 - 92.26p + 1246.60p^2 - 1136.07p^3 + \epsilon_l$$

respectively. We note that there are not any larger differences in accuracy depending on the choice of model. Naturally, the discrepancy of the second model is lower.
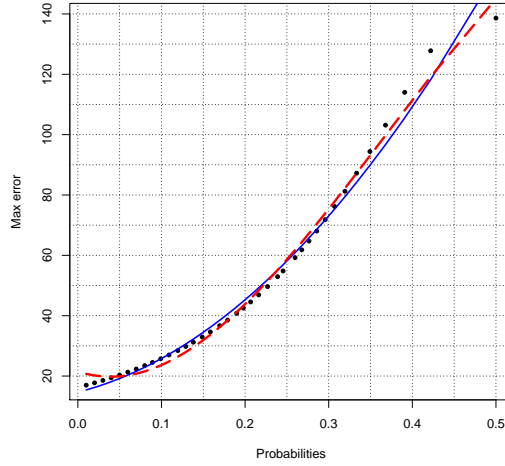
Figure 4: Maximum relative error for $npq = 10$ *without* continuity correction. The solid line is the regression curve, $M_{F_{rel}} = 14.66 + 69.86p + 416.14p^2$ and the dashed line, $M_{F_{rel}} = 21.53 - 92.26p + 1246.60p^2 - 1136.07p^3$.

Case 2: $npq = 10$, with continuity correction

We continue by looking at the same case as above, but here continuity correction is used. This gives somewhat remarkable results, $M_{F_{rel}}$ is actually about two times larger than without continuity correction. Let us study the same numeric example as above, except that we use continuity correction. We got $p = 0.2269$ which again means that $n = 57$, then $X \sim \text{Bin}(57, 0.2269)$. We let X be approximated, according to (5), by $Y \sim \text{N}(12.933, 3.162078)$. It results in, $P(X \leq 1) = 7.55 \cdot 10^{-6}$ and $P(Y \leq 1 + 0.5) = 0.000150$. Under

these circumstances we get,

$$M_{F_{rel}} = \frac{|P(X \leq 1) - P(Y \leq 1 + 0.5)|}{P(X \leq 1)} = 18.84,$$

which fits the values in Table 5. $M_{F_{abs}}$ gets dramatically worse when we use continuity correction than without. Hence, also $M_{F_{rel}}$ becomes worse. In Figure 5 one can judge that the results gets worse as we get closer to probabilities near 0.5. The regression curves, defined in (14) and (15) are,

$$M_{F_{rel}} = 34.9 - 69.8p + 1597.1p^2 + \epsilon_l$$

and

$$M_{F_{rel}} = 37.4 - 127.3p + 1891.8p^2 - 403.2p^3 + \epsilon_l,$$

respectively. Looking at Figure 5, we see that the difference between the two models is insignificant.
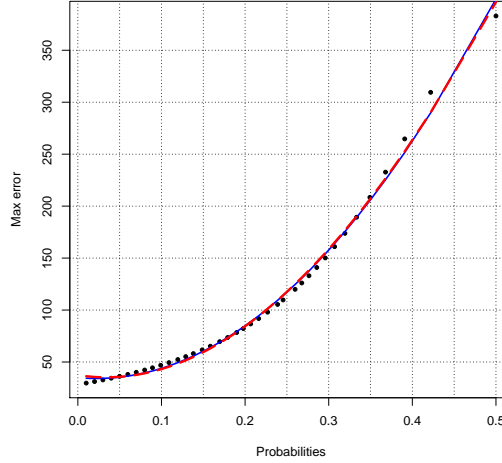
Figure 5: Maximum relative error for $npq = 10$ *with* continuity correction. The solid line is the regression curve, $M_{F_{rel}} = 34.9 - 69.8p + 1597.1p^2$ and the dashed line, $M_{F_{rel}} = 37.4 - 127.3p + 1891.8p^2 - 403.2p^3$.

Case 3: $npq = 3$ with continuity correction

Here, in the last case $npq = 3$ and continuity correction is used, see Figure 6. This gives the curves of regression, defined in (14) and (15),

$$M_{F_{rel}} = 0.473 + 2.204p + 2.123p^2 + \epsilon_l$$

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and

$$M_{F_{rel}} = 0.514 + 1.155p + 7.858p^2 - 7.885p^3 + \epsilon_l,$$

respectively. As we see $M_{F_{rel}}$ actually get the smallest value here, where $npq = 3$ and continuity correction is used. As well as in the two other cases regarding the relative error the difference between the quadratic and cubic regression model is minimal.
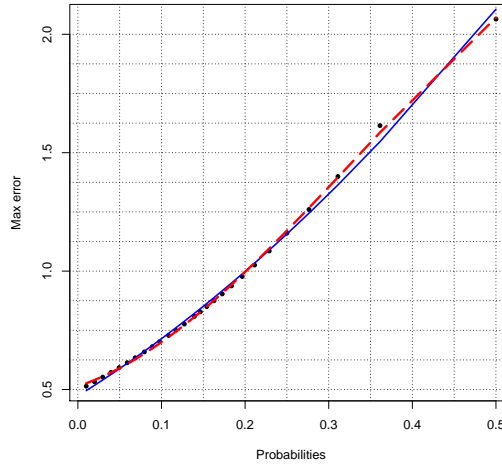
Figure 6: Maximum relative error for $npq = 3$ *with* continuity correction. The solid line is the regression curve, $M_{F_{rel}} = 0.473 + 2.204p + 2.123p^2$ and the dashed line, $M_{F_{rel}} = 0.514 + 1.155p + 7.858p^2 - 7.885p^3$.

5 Summary and conclusions

The three different rules of thumbs that are focused on turned out to give approximation errors of different sizes. Regarding the absolute errors, the largest difference is found between the case where $L = 10$ *without* continuity correction and $L = 10$ *with* continuity correction. The largest error decreases from ~ 0.08 to about ~ 0.02, which is approximately four times smaller, a relatively large difference. Letting $L = 3$ and using continuity correction we end up with the largest error ~ 0.035, closer to the latter case, but still between them. When using this common and simple way of approximating, depending on the problem, different levels of tolerance usually are accepted. A common level in many cases may be 0.01. If we look deeper, we see that the probabilities for getting such a small $M_{F_{abs}}$ differs from between the rules of thumb. Using $npq = 10$ *without* continuity correction does not even reach to the 0.01 level of accepted accuracy. Comparing this to the other

two cases which in contrast reach the 0.01 level for probabilities ~ 0.25 in the same case as above but in addition *with* continuity correction, and for probabilities ~ 0.35 in the case where $npq = 3$. Further, it would be interesting to investigate how the relationship between k and n affects the error. In addition, another interesting part would be some tables indicating how large n should be in order to get sufficiently small errors, for different probabilities.

Concerning the relative errors I would say that the applicability may be somewhat uncertain, due to the fact that $M_{F_{rel}}$ is very large for small values of k but rapidly decrease. This fact, I may say, make the plots look a bit extreme and there are other values of k that give much better approximations. Judging by Tables 4, 5 and 6 indeed this seems to be the case. We know that the approximation is motivated by the central limit theorem, however, what we also know, is that it does not hold the same accuracy for small probabilities, that is, the tails of the distributions. This is also the direct reason why the accuracy gets worse when using continuity correction, it puts extra mass on the already too large approximated value. In a similar way we get the explanation why the relative error increases when the value of npq changes from 10 to 3, (as one maybe would expect the opposite), the mean value of the normal distribution, np, gets closer to 0 which in turn gives additional mass. The conclusion is, one should remember that due to the fluctuations depending on k, of the relative errors, what we also can see in Tables 4, 5 and 6, that the regression model also provides conservative estimates of the errors. As a natural alternative, and most likely better, Poisson approximation is recommended for small probabilities. Like in the previous case concerning the absolute errors, some more exhaustive examination of the relative error would be interesting. How large should n be to get acceptable levels of the error, for instance 10% or 5% and so on.

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Tables

Regarding the plotted probabilities, that is the set **P**, only the maximum error is plotted. One can not tell from which k the error comes from, neither can one tell if the error is of similar size for other values of k. To get a more detailed picture this section contains tables both for the absolute errors and the relative errors. It would have been possible to table all the errors for all values of k, but due to the fact that the cardinality of **N** at times, that is for small probabilities, is relatively large, it would have taken too much place. This made me table only the 10 values of k which resulted in the largest errors. The columns in the tables, that contains the values of k is in descending order. What this means is that the first value of k in each column is the maximum error that is plotted. On the side of every column of k, there is a column where the corresponding error is written. These two sub columns, got a common header which tells the value of p in the specific case.

Top section (p from 0.01 to 0.1899)

p=0.01		0.02		0.03		0.0399		0.0499		0.0597		0.0698		0.0799		0.0893		0.0991		0.109		0.1196		0.129		0.1381		0.1487		0.1584		0.1696		0.1792		0.1899	
k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$
10	0.0831	10	0.083	10	0.0828	10	0.0824	10	0.0819	10	0.0813	10	0.0805	10	0.0795	11	0.0793	11	0.0794	11	0.0793	11	0.079	11	0.0786	11	0.078	11	0.0771	11	0.0761	12	0.0763	12	0.0763	12	0.0761
9	0.0808	9	0.0795	9	0.078	11	0.073	11	0.0776	11	0.0782	11	0.0787	11	0.0791	10	0.0784	10	0.0772	10	0.0758	10	0.0741	12	0.0743	12	0.075	12	0.0757	12	0.0761	11	0.0747	11	0.0733	11	0.0715
11	0.0738	11	0.0749	11	0.0759	9	0.0765	9	0.0748	9	0.073	9	0.071	9	0.0689	12	0.0698	12	0.0711	12	0.0723	12	0.0734	10	0.0724	10	0.0706	10	0.0684	13	0.0665	13	0.0683	13	0.0696	13	0.0709
8	0.0672	8	0.065	8	0.0628	12	0.0621	12	0.0638	12	0.0654	12	0.067	12	0.0685	9	0.0669	9	0.0647	9	0.0623	9	0.0597	13	0.0614	13	0.0631	13	0.0649	10	0.0662	10	0.0636	10	0.0611	10	0.0583
12	0.0569	12	0.0587	12	0.0604	8	0.0605	8	0.0581	8	0.0557	8	0.0532	8	0.0507	8	0.0536	13	0.0555	13	0.0575	13	0.0596	9	0.0573	9	0.0549	14	0.0488	9	0.051	14	0.0536	14	0.0557	14	0.058
7	0.0465	7	0.0442	7	0.0419	13	0.0435	13	0.0455	13	0.0475	13	0.0496	13	0.0516	13	0.0483	14	0.0458	14	0.0433	14	0.0422	14	0.0443	14	0.0464	9	0.0521	14	0.0494	9	0.0463	9	0.0436	15	0.0418
13	0.0377	13	0.0396	13	0.0416	7	0.0396	7	0.0373	7	0.035	7	0.0328	7	0.0337	14	0.0356	8	0.0377	8	0.0398	8	0.0406	8	0.0382	8	0.0359	8	0.0332	15	0.0342	15	0.0368	15	0.0391	9	0.0406
6	0.0253	6	0.0235	6	0.0243	14	0.026	14	0.0278	14	0.0297	14	0.0317	14	0.0305	7	0.0285	7	0.0264	7	0.0244	15	0.0258	15	0.0277	15	0.0296	15	0.032	8	0.0308	8	0.0281	8	0.0259	16	0.0263
14	0.021	14	0.0226	14	0.0217	6	0.02	6	0.0184	6	0.0168	15	0.017	15	0.0186	15	0.0202	15	0.0219	15	0.0237	7	0.0222	7	0.0204	7	0.0187	16	0.018	16	0.0198	16	0.0219	16	0.0239	8	0.0235
15	0.0092	15	0.0103	15	0.0115	15	0.0128	15	0.0141	15	0.0155	6	0.0152	6	0.0138	6	0.0125	6	0.0112	16	0.0118	16	0.0133	16	0.0147	16	0.0162	7	0.0169	7	0.0152	7	0.0134	7	0.0125	17	0.0142

Bottom section (p from 0.1979 to 0.5)

0.1979		0.2066		0.2163		0.2269		0.2389		0.2454		0.2598		0.2678		0.2764		0.2857		0.2959		0.307		0.3194		0.3333		0.3492		0.3679		0.3909		0.4219		0.5	
k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$
12	0.0757	12	0.0751	12	0.0742	13	0.0736	13	0.0737	13	0.0737	13	0.073	13	0.0724	13	0.0715	14	0.0714	14	0.0715	14	0.0711	14	0.0701	15	0.0695	15	0.0693	15	0.0676	16	0.0675	17	0.0662	20	0.0627
13	0.0717	13	0.0725	13	0.0731	12	0.073	12	0.0712	12	0.0701	14	0.0698	14	0.0705	14	0.0711	13	0.0702	13	0.0685	15	0.0675	15	0.0688	14	0.0682	16	0.0656	16	0.0675	17	0.0645	16	0.0629	19	0.0614
11	0.07	11	0.0681	11	0.066	14	0.0653	14	0.0672	14	0.0681	12	0.0672	12	0.0654	12	0.0633	15	0.0643	15	0.066	13	0.0662	13	0.0632	16	0.0629	14	0.0652	15	0.0632	18	0.0626	18	0.0626	21	0.058
14	0.0597	14	0.0615	14	0.0634	11	0.0634	11	0.0602	11	0.0584	15	0.059	15	0.0608	15	0.0626	12	0.0607	12	0.0578	16	0.057	16	0.06	13	0.0593	17	0.0554	18	0.0552	15	0.0533	15	0.0533	18	0.0544
10	0.0561	10	0.0536	10	0.0508	10	0.0476	15	0.0541	15	0.0557	11	0.0541	11	0.0516	11	0.0489	16	0.0514	16	0.0541	12	0.0543	12	0.0502	17	0.0508	13	0.0542	14	0.0526	19	0.0532	14	0.0526	22	0.0487
15	0.0438	15	0.046	15	0.0484	16	0.0353	10	0.044	10	0.042	16	0.0442	16	0.0464	16	0.0488	11	0.0459	11	0.0425	17	0.0428	17	0.0466	12	0.0453	18	0.0418	19	0.0424	14	0.0424	20	0.0408	17	0.0434
9	0.0384	9	0.036	9	0.0333	9	0.0304	16	0.0384	16	0.0402	10	0.0376	10	0.0352	10	0.0337	17	0.0364	17	0.0394	11	0.0388	11	0.0347	18	0.0396	12	0.0366	13	0.0387	20	0.0402	13	0.0387	23	0.0373
16	0.0281	16	0.0301	16	0.0325	17	0.0213	9	0.0273	9	0.0256	17	0.0292	17	0.0314	17	0.0326	10	0.0298	10	0.0269	18	0.0286	18	0.0322	11	0.0301	19	0.0282	20	0.0293	21	0.0282	21	0.0282	16	0.0311
8	0.0217	8	0.0199	17	0.0191	8	0.0159	17	0.024	17	0.0256	9	0.022	9	0.0202	18	0.0206	18	0.0229	18	0.0255	10	0.0238	10	0.0205	19	0.0252	11	0.0252	12	0.025	13	0.0269	12	0.025	24	0.026
17	0.0156	17	0.0172	8	0.0179			8	0.0138	18	0.0142	18	0.017	18	0.0187	9	0.0182	9	0.0163	19	0.0163	19	0.017	19	0.0199	10	0.0171	20	0.017	21	0.0183	22	0.0177	22	0.0177	15	0.02

Table 1: Table of the 10 largest errors, $\varepsilon_{F_{abs}}$, and which k is comes from, for every p_i, under $npq = 10$ *without* continuity correction.

Table 2 (rotated on page). Data reproduced with each p-value column containing sub-columns k and $\varepsilon_{F_{abs}}$.

Block 1

p=0.01		0.02		0.03		0.0399		0.0499		0.0597		0.0698		0.0799		0.0893		0.0991		0.109		0.1196		0.129		0.1381		0.1487		0.1584		0.1696		0.1792		0.1899	
k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$
11	0.0201	11	0.02	11	0.0198	11	0.0195	11	0.0192	11	0.0187	11	0.0182	11	0.0177	11	0.0171	12	0.0165	12	0.0163	12	0.0161	12	0.0158	12	0.0154	12	0.0149	12	0.0144	12	0.0136	13	0.0133	13	0.0131
10	0.02	10	0.0193	10	0.0185	10	0.0176	10	0.0167	12	0.0164	12	0.0165	12	0.0166	12	0.0165	11	0.0164	11	0.0157	11	0.0148	11	0.0139	13	0.0133	13	0.0134	13	0.0135	13	0.0134	12	0.013	12	0.0121
12	0.0149	12	0.0153	12	0.0156	12	0.0159	12	0.0162	10	0.0157	10	0.0147	10	0.0137	10	0.0126	13	0.0121	13	0.0125	13	0.0129	13	0.0131	11	0.0131	11	0.0121	11	0.0111	11	0.0099	14	0.01	14	0.0104
9	0.0141	9	0.0129	9	0.0118	9	0.0106	6	0.0103	6	0.0102	13	0.0107	13	0.0112	13	0.0117	10	0.0116	10	0.0104	10	0.0092	7	0.0084	7	0.0083	14	0.0086	14	0.0091	14	0.0097	11	0.0088	11	0.007
5	0.0114	5	0.011	5	0.0107	6	0.0104	5	0.0098	13	0.0101	6	0.01	6	0.0098	6	0.0096	6	0.0093	6	0.0089	7	0.0084	6	0.0082	14	0.008	7	0.0081	7	0.0079	7	0.0076	7	0.0074	7	0.007
6	0.0104	6	0.0105	6	0.0105	5	0.0103	13	0.0095	5	0.0094	5	0.0089	5	0.0084	7	0.0082	7	0.0083	7	0.0084	6	0.0085	10	0.0082	6	0.0078	6	0.0073	6	0.0069	8	0.0066	8	0.0067	8	0.0068
4	0.0091	4	0.0086	13	0.0082	13	0.0089	9	0.0094	9	0.0083	7	0.0077	7	0.008	5	0.008	5	0.0075	5	0.0071	14	0.0069	14	0.0075	10	0.0071	8	0.0061	8	0.0064	6	0.0064	6	0.006	6	0.0055
16	0.0076	13	0.0075	4	0.0081	4	0.0076	4	0.0071	7	0.0074	17	0.0071	17	0.0066	17	0.0064	17	0.0062	14	0.0062	5	0.0065	5	0.0061	8	0.0058	10	0.0059	18	0.0053	18	0.0052	18	0.005	15	0.0055
17	0.0071	16	0.0073	16	0.0071	17	0.007	7	0.007	17	0.0068	9	0.0067	9	0.006	18	0.0058	18	0.0058	17	0.006	17	0.0058	18	0.0057	5	0.0057	18	0.0055	10	0.0048	19	0.0047	15	0.0048	18	0.0047
13	0.0068	17	0.0071	17	0.007	16	0.0069	17	0.0069	4	0.0066	4	0.0061	18	0.0058	4	0.0053	14	0.0055	18	0.0058	18	0.0057	17	0.0055	18	0.0056	5	0.0052	5	0.0047	5	0.0043	19	0.0047	19	0.0046

Block 2

0.1979		0.2066		0.2163		0.2269		0.2389		0.2454		0.2598		0.2678		0.2764		0.2857		0.2959		0.307		0.3194		0.3333		0.3492		0.3679		0.3909		0.4219		0.5	
k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$
13	0.0128	13	0.0125	13	0.012	13	0.0114	14	0.0108	14	0.0106	14	0.0102	14	0.0099	14	0.0095	14	0.0089	15	0.0085	15	0.0082	15	0.0077	16	0.007	16	0.0064	17	0.0046	18	0.0046	18	0.0032	14	7e-04
12	0.0114	14	0.0108	14	0.0109	14	0.0109	13	0.0107	13	0.0102	13	0.009	15	0.0086	15	0.0087	15	0.0086	14	0.0083	14	0.0074	16	0.0067	15	0.0067	17	0.0059	16	0.0043	17	0.0043	17	0.0032	27	7e-04
14	0.0106	12	0.0106	12	0.0097	12	0.0087	15	0.008	15	0.0082	15	0.0085	13	0.0083	13	0.0075	13	0.0066	16	0.0064	16	0.0066	14	0.0064	17	0.005	15	0.005	18	0.0036	19	0.0036	19	0.0024	15	6e-04
11	0.0068	8	0.0067	15	0.007	15	0.0075	12	0.0074	12	0.0068	9	0.0054	9	0.0053	16	0.0057	16	0.006	13	0.0057	13	0.0046	10	0.0042	14	0.0046	11	0.0038	11	0.0028	16	0.0028	16	0.0022	26	6e-04
8	0.0067	15	0.0065	8	0.0066	8	0.0063	8	0.0061	9	0.0059	8	0.0054	16	0.0053	9	0.0052	9	0.005	9	0.0048	10	0.0045	17	0.0041	10	0.004	18	0.0036	15	0.0027	12	0.0022	12	0.0022	28	6e-04
7	0.0067	7	0.0064	7	0.006	7	0.0056	9	0.0054	8	0.0054	12	0.0053	8	0.0051	8	0.0048	8	0.0044	10	0.0043	9	0.0043	9	0.0041	11	0.0036	10	0.0034	10	0.0023	13	0.0019	13	0.0019	13	6e-04
15	0.006	11	0.0058	9	0.0049	9	0.0052	7	0.005	7	0.0048	16	0.0049	12	0.0044	10	0.004	10	0.0042	8	0.004	17	0.0036	13	0.0034	9	0.0033	9	0.003	12	0.0022	11	0.0023	11	0.0019	25	4e-04
6	0.0051	6	0.0047	11	0.0048	19	0.0042	19	0.004	16	0.004	7	0.0041	7	0.0038	12	0.0036	20	0.0033	17	0.003	8	0.0035	8	0.0031	8	0.0026	12	0.0026	19	0.0019	10	0.0013	10	0.0013	16	4e-04
19	0.0046	9	0.0046	19	0.0044	6	0.0038	20	0.0037	19	0.0039	20	0.0036	10	0.0037	7	0.0034	7	0.0031	7	0.003	20	0.0028	11	0.003	21	0.0024	14	0.0021	9	0.0015	20	0.0011	20	0.0011	12	4e-04
18	0.0045	19	0.0045	6	0.0043	20	0.0037	16	0.0036	20	0.0037	19	0.0035	20	0.0035	20	0.0034	21	0.0028	21	0.0028	21	0.0027	21	0.0026	22	0.0021	22	0.002	13	0.0015	14	0.001	14	0.001	29	4e-04

Table 2: Table of the 10 largest errors, $\varepsilon_{F_{abs}}$ and which k is comes from, for every p_i, under $npq = 10$ *with* continuity correction.

Upper block:

p = 0.01		0.0199		0.0297		0.0395		0.0493		0.059		0.0685		0.0795		0.089		0.0978		0.1086		0.1172		0.1273		0.1394	
k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$
2	0.0356	3	0.0343	3	0.0332	3	0.033	3	0.0328	3	0.0325	3	0.0322	3	0.0318	3	0.0314	3	0.031	3	0.0304	3	0.03	3	0.0294	3	0.0286
3	0.0335	3	0.0334	3	0.0331	3	0.0318	2	0.0305	2	0.0292	2	0.0279	2	0.0265	2	0.0252	2	0.0239	2	0.0224	2	0.0212	2	0.0198	0	0.0189
0	0.0245	0	0.0242	0	0.0239	0	0.0236	0	0.0233	0	0.0229	0	0.0225	0	0.0221	0	0.0216	0	0.0212	0	0.0207	0	0.0202	0	0.0196	2	0.0181
6	0.0117	6	0.0116	6	0.0114	6	0.0112	4	0.0111	4	0.0116	4	0.0121	4	0.0126	4	0.0131	4	0.0135	4	0.014	6	0.0143	4	0.0147	4	0.0152
4	0.009	4	0.0096	4	0.0101	4	0.0106	6	0.0111	6	0.0109	6	0.0107	6	0.0105	6	0.0103	6	0.0101	6	0.0098	6	0.0096	6	0.0094	6	0.009
5	0.009	5	0.0086	5	0.0082	5	0.0077	5	0.0072	7	0.0071	7	0.007	7	0.007	7	0.007	7	0.0069	7	0.0069	7	0.0068	1	0.0068	1	0.0075
7	0.0072	7	0.0072	7	0.0071	7	0.0071	7	0.0071	5	0.0068	5	0.0063	5	0.0058	5	0.0053	5	0.0048	1	0.0053	1	0.0059	7	0.0067	7	0.0067
1	0.0047	1	0.0035	1	0.0024	8	0.003	8	0.003	8	0.003	8	0.003	8	0.003	1	0.0036	1	0.0044	5	0.0042	5	0.0038	5	0.0032	8	0.003
8	0.0031	8	0.003	8	0.003	1	0.0013	9	0.001	9	0.001	1	0.0017	1	0.0028	8	0.003	8	0.003	8	0.003	8	0.003	8	0.003	5	0.0025
9	0.001	9	0.001	9	0.001	9	0.001	10	3e-04	1	8e-04	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001

Lower block:

0.1464		0.1542		0.1629		0.1727		0.1838		0.1965		0.2113		0.2288		0.25		0.2764		0.311		0.3613		0.5	
k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$	k	$\varepsilon_{F_{abs}}$
3	0.0281	3	0.0276	3	0.0269	3	0.0262	3	0.0252	3	0.0241	3	0.0227	3	0.0209	3	0.0186	4	0.016	4	0.0146	4	0.0112	9	0.0024
0	0.0185	0	0.018	0	0.0175	0	0.0169	4	0.0164	4	0.0166	4	0.0167	4	0.0168	4	0.0166	3	0.0155	3	0.0114	1	0.0079	2	0.0024
2	0.0172	2	0.0161	4	0.0159	4	0.0161	0	0.0161	0	0.0153	0	0.0143	0	0.0131	0	0.0116	1	0.0108	1	0.0099	5	0.0068	10	0.0015
4	0.0154	4	0.0156	2	0.0149	2	0.0135	2	0.012	2	0.0104	1	0.0106	1	0.0109	1	0.011	0	0.0098	0	0.0076	3	0.0057	1	0.0015
6	0.0088	6	0.0086	6	0.0088	6	0.008	1	0.0098	1	0.0102	2	0.0085	2	0.0063	7	0.0055	5	0.005	5	0.0061	0	0.0047	3	0.0015
1	0.0079	1	0.0084	1	0.0083	7	0.0064	6	0.0077	6	0.0072	6	0.0067	6	0.006	6	0.0051	7	0.0048	7	0.0042	2	0.0043	8	0.0015
7	0.0066	7	0.0066	7	0.0065	8	0.003	7	0.0063	7	0.0062	7	0.006	7	0.0058	2	0.0039	6	0.0039	8	0.0024	7	0.0027	6	8e-04
8	0.003	8	0.003	8	0.003	9	9e-04	8	0.003	8	0.0029	8	0.0029	8	0.0029	5	0.0036	2	0.0026	6	0.0024	8	0.0017	5	8e-04
5	0.0021	5	0.0017	5	0.0012	5	7e-04	9	9e-04	5	9e-04	5	0.0015	5	0.0025	8	0.0028	8	0.0012	2	0.0017	9	5e-04	7	6e-04
9	0.001	9	0.001	9	9e-04	10	2e-04	10	2e-04	9	7e-04	9	9e-04	9	9e-04	9	9e-04	9	9e-04	9	8e-04	6	2e-04	4	6e-04

Table 3: Table of the 10 largest errors, $\varepsilon_{F_{abs}}$ and which k is comes from, for every p_i, under $npq = 3$ *with* continuity correction.

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Upper half — $p = 0.01$ to 0.1899 (each cell: k / $\varepsilon_{F_{rel}}$)

p	1	2	3	4	5	6	7	8	9	10	11
0.01	0 / 16.9732	1 / 3.5794	2 / 1.1123	3 / 0.3227	7 / 0.2216	8 / 0.2097	6 / 0.2064	9 / 0.1817	10 / 0.1457	5 / 0.1454	
0.02	0 / 17.7468	1 / 3.7342	2 / 1.1665	3 / 0.3473	7 / 0.2213	8 / 0.2111	6 / 0.2036	9 / 0.1843	10 / 0.1489	5 / 0.139	
0.03	0 / 18.5666	1 / 3.8973	2 / 1.2233	3 / 0.373	7 / 0.2209	8 / 0.2125	6 / 0.2007	9 / 0.1869	10 / 0.1522	5 / 0.1322	
0.0399	0 / 19.4284	1 / 4.0678	2 / 1.2825	3 / 0.3998	7 / 0.2203	8 / 0.2137	6 / 0.1975	9 / 0.1895	10 / 0.1555	5 / 0.1251	
0.0499	0 / 20.3435	1 / 4.2478	2 / 1.3447	3 / 0.4278	7 / 0.2195	8 / 0.2149	6 / 0.194	9 / 0.1919	10 / 0.1588	5 / 0.1217	
0.0597	0 / 21.3015	1 / 4.435	2 / 1.4092	3 / 0.4568	7 / 0.2186	8 / 0.216	9 / 0.1944	6 / 0.1904	10 / 0.162	11 / 0.1253	
0.0698	0 / 22.3354	1 / 4.636	2 / 1.4781	3 / 0.4877	7 / 0.2176	8 / 0.2169	9 / 0.1968	6 / 0.1864	10 / 0.1654	11 / 0.129	
0.0799	0 / 23.4367	1 / 4.8487	2 / 1.5507	3 / 0.5202	8 / 0.2178	7 / 0.2163	9 / 0.1991	6 / 0.1821	10 / 0.1687	11 / 0.1328	
0.0893	0 / 24.515	1 / 5.0558	2 / 1.621	3 / 0.5516	8 / 0.2184	7 / 0.2149	9 / 0.2012	6 / 0.1779	10 / 0.1718	11 / 0.1364	
0.0991	0 / 25.7144	1 / 5.2848	2 / 1.6985	3 / 0.5861	8 / 0.219	7 / 0.2133	9 / 0.2034	10 / 0.175	6 / 0.1731	11 / 0.1401	
0.109	0 / 26.9868	1 / 5.5263	2 / 1.7799	3 / 0.6222	8 / 0.2194	7 / 0.2114	9 / 0.2055	10 / 0.1782	6 / 0.1681	11 / 0.1528	
0.1196	0 / 28.4404	1 / 5.8005	2 / 1.8718	3 / 0.6629	8 / 0.2197	7 / 0.2092	9 / 0.2076	10 / 0.1816	6 / 0.1737	11 / 0.1623	
0.129	0 / 29.8146	1 / 6.0582	2 / 1.9578	3 / 0.7009	8 / 0.2199	9 / 0.2094	7 / 0.2076	6 / 0.207	10 / 0.1931	4 / 0.1846	11 / 0.1568
0.1381	0 / 31.2175	1 / 6.3197	2 / 2.0447	3 / 0.7391	9 / 0.2127	8 / 0.2199	7 / 0.2111	6 / 0.2047	10 / 0.1874	11 / 0.1553	
0.1487	0 / 32.9391	1 / 6.6388	2 / 2.1502	3 / 0.7855	8 / 0.2364	9 / 0.2197	7 / 0.2129	6 / 0.2018	10 / 0.1907	11 / 0.1595	
0.1584	0 / 34.6254	1 / 6.9494	2 / 2.2524	3 / 0.8302	4 / 0.2592	8 / 0.2193	9 / 0.2145	10 / 0.1988	7 / 0.1936	11 / 0.1633	
0.1696	0 / 36.684	1 / 7.3262	2 / 2.3758	3 / 0.884	4 / 0.2867	8 / 0.2187	9 / 0.2161	10 / 0.1969	7 / 0.1951	11 / 0.1677	
0.1792	0 / 38.5519	1 / 7.6659	2 / 2.4865	3 / 0.9322	4 / 0.3112	8 / 0.2179	9 / 0.2174	10 / 0.1997	7 / 0.1916	11 / 0.1715	
0.1899	0 / 40.7878	1 / 8.0701	2 / 2.6176	3 / 0.989	4 / 0.3401	8 / 0.2187	9 / 0.2169	10 / 0.2027	7 / 0.1874	11 / 0.1757	

Lower half — $p = 0.1979$ to 0.5 (each cell: k / $\varepsilon_{F_{rel}}$)

p	1	2	3	4	5	6	7	8	9
0.1979	0 / 42.5395	1 / 8.385	2 / 2.7193	3 / 1.0329	4 / 0.3625	9 / 0.2196	8 / 0.216	10 / 0.2049	7 / 0.1841
0.2066	0 / 44.5543	1 / 8.7454	2 / 2.8352	3 / 1.0828	4 / 0.3878	9 / 0.2204	8 / 0.2148	10 / 0.2072	11 / 0.1822
0.2163	0 / 46.8956	1 / 9.1618	2 / 2.9685	3 / 1.14	4 / 0.4168	9 / 0.2212	8 / 0.2133	10 / 0.2097	11 / 0.1859
0.2269	0 / 49.6484	1 / 9.6485	2 / 3.1235	3 / 1.2063	4 / 0.4504	9 / 0.2218	10 / 0.2124	8 / 0.2114	11 / 0.1899
0.2389	0 / 52.9297	1 / 10.2247	2 / 3.306	3 / 1.2841	4 / 0.4897	9 / 0.2224	10 / 0.2152	8 / 0.209	11 / 0.1944
0.2454	0 / 54.8179	1 / 10.5545	2 / 3.41	3 / 1.3283	4 / 0.5119	10 / 0.2226	9 / 0.2216	8 / 0.2075	11 / 0.1968
0.2598	0 / 59.2257	1 / 11.3197	2 / 3.6501	3 / 1.4299	4 / 0.5631	9 / 0.2227	10 / 0.2197	8 / 0.2039	11 / 0.202
0.2678	0 / 61.819	1 / 11.767	2 / 3.7897	3 / 1.4887	4 / 0.5926	9 / 0.2227	10 / 0.2212	11 / 0.204	8 / 0.2017
0.2764	0 / 64.7361	1 / 12.2677	2 / 3.9454	3 / 1.5541	4 / 0.6253	10 / 0.2229	9 / 0.2225	11 / 0.2079	8 / 0.1992
0.2857	0 / 68.041	1 / 12.8322	2 / 4.1201	3 / 1.6273	4 / 0.6619	9 / 0.2262	10 / 0.2245	5 / 0.2381	11 / 0.2154
0.2959	0 / 71.8159	1 / 13.4733	2 / 4.3176	3 / 1.7097	4 / 0.703	5 / 0.2381	10 / 0.2279	9 / 0.2222	11 / 0.2217
0.307	0 / 76.168	1 / 14.2081	2 / 4.5428	3 / 1.8032	4 / 0.7495	5 / 0.2637	10 / 0.2277	9 / 0.2209	11 / 0.2182
0.3194	0 / 81.2398	1 / 15.0589	2 / 4.8021	3 / 1.9105	4 / 0.8026	5 / 0.293	10 / 0.2299	9 / 0.2222	11 / 0.2199
0.3333	0 / 87.2252	1 / 16.0558	2 / 5.1041	3 / 2.0347	4 / 0.8639	5 / 0.3267	10 / 0.2315	11 / 0.2265	9 / 0.2185
0.3492	0 / 94.3951	1 / 17.2405	2 / 5.4604	3 / 2.1804	4 / 0.9355	5 / 0.3659	10 / 0.2334	11 / 0.2314	9 / 0.2168
0.3679	0 / 103.1377	1 / 18.6718	2 / 5.8873	3 / 2.3537	4 / 1.0201	5 / 0.412	10 / 0.2371	11 / 0.2357	12 / 0.2249
0.3909	0 / 114.019	1 / 20.4335	2 / 6.4072	3 / 2.5626	4 / 1.1211	5 / 0.4665	11 / 0.2443	10 / 0.2388	12 / 0.2355
0.4219	0 / 127.794	1 / 22.6286	2 / 7.0446	3 / 2.8144	4 / 1.2405	5 / 0.5294	11 / 0.2554	10 / 0.251	12 / 0.2449
0.5	0 / 138.6176	1 / 24.1341	2 / 7.4028	3 / 2.9154	4 / 1.2619	5 / 0.5199	11 / 0.3121	10 / 0.3112	12 / 0.302

Table 4: Table of the 10 largest errors, $\varepsilon_{F_{rel}}$ and which k is comes from, for every p_i, under $npq = 10$ *without* continuity correction.

Table 5 — Table of the 10 largest errors, $\varepsilon_{F_{rel}}$, and which k is comes from, for every p_i, under $npq = 10$ *with* continuity correction.

Top block

$p=$0.01		0.02		0.03		0.0399		0.0499		0.0597		0.0698		0.0799		0.0893		0.0991		0.109		0.1196		0.129		0.1381		0.1487		0.1584		0.1696		0.1792		0.1899	
k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$
0	29.7208	0	31.1973	0	32.7709	0	34.4347	0	36.212	0	38.084	0	40.1172	0	42.297	0	44.4454	0	46.851	0	49.4206	0	52.3782	0	55.1954	0	58.0921	0	61.6752	0	65.2141	0	69.5729	0	73.5645	0	78.3871
1	6.4725	1	6.7619	1	7.0685	1	7.3907	1	7.7327	1	8.0908	1	8.4774	1	8.8893	1	9.2928	1	9.7419	1	10.2188	1	10.7642	1	11.2805	1	11.8083	1	12.4572	1	13.0941	1	13.8735	1	14.5828	1	15.4344
2	2.2923	2	2.3926	2	2.4984	2	2.6091	2	2.7261	2	2.848	2	2.9789	2	3.1178	2	3.2532	2	3.4032	2	3.5617	2	3.7421	2	3.912	2	4.0849	2	4.2964	2	4.503	2	4.7545	2	4.9822	2	5.2542
3	0.9707	3	1.0166	3	1.0648	3	1.1151	3	1.1681	3	1.2232	3	1.2822	3	1.3445	3	1.4051	3	1.4721	3	1.5425	3	1.6224	3	1.6975	3	1.7735	3	1.8663	3	1.9566	3	2.066	3	2.1647	3	2.2822
4	0.4247	4	0.4491	4	0.4747	4	0.5014	4	0.5295	4	0.5586	4	0.5898	4	0.6227	4	0.6547	4	0.6899	4	0.7269	4	0.7689	4	0.8081	4	0.8479	4	0.8963	4	0.9433	4	1.0002	4	1.0514	4	1.1121
5	0.1665	5	0.1804	5	0.1951	5	0.2104	5	0.2265	5	0.2433	5	0.2612	5	0.2802	5	0.2986	5	0.3189	5	0.3403	5	0.3645	5	0.3873	5	0.4102	5	0.4382	5	0.4654	5	0.4982	5	0.5278	5	0.5629
6	0.0451	6	0.0468	6	0.0553	6	0.0642	6	0.0736	6	0.0835	6	0.0941	6	0.1054	6	0.1164	6	0.1286	6	0.1414	6	0.156	6	0.1698	6	0.1837	6	0.2008	6	0.2173	6	0.2374	6	0.2555	6	0.2771
7	0.0439	7	0.0447	7	0.0442	7	0.0436	7	0.0428	7	0.0419	7	0.0408	7	0.0394	7	0.038	7	0.0372	7	0.0373	7	0.0461	7	0.0544	7	0.0629	7	0.0733	7	0.0835	7	0.096	7	0.1073	7	0.1208
8	0.0387	8	0.042	8	0.0398	8	0.0375	8	0.0371	8	0.0373	8	0.0375	8	0.0375	8	0.0374	8	0.0363	8	0.0361	8	0.0362	8	0.0356	8	0.0348	8	0.0336	8	0.0324	8	0.0306	8	0.0303	8	0.0354
9	0.0353	9	0.0359	9	0.0375	9	0.0368	9	0.0349	9	0.0321	9	0.0291	9	0.0278	9	0.0284	9	0.0297	9	0.0344	9	0.0321	9	0.0304	9	0.0307	9	0.0308	9	0.0308	9	0.0306	9	0.0289	9	0.0298
10		10		10	0.0364	10		10		10		10		10		10		10		10		10		10		10		10		10		10		10		11	

Bottom block

$p=$0.1979		0.2066		0.2163		0.2269		0.2389		0.2454		0.2598		0.2678		0.2764		0.2857		0.2959		0.307		0.3194		0.3333		0.3492		0.3679		0.3909		0.4219		0.5	
k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$
0	82.1989	0	86.6195	0	91.8044	0	97.9661	0	105.4031	0	109.7278	0	119.9526	0	126.053	0	132.9907	0	140.9488	0	150.1682	0	160.9731	0	173.8113	0	189.3227	0	208.4603	0	232.7278	0	264.7054	0	309.5134	0	382.9711
1	16.1038	1	16.8761	1	17.7777	1	18.8412	1	20.1171	1	20.8552	1	22.5896	1	23.6179	1	24.7821	1	26.1109	1	27.6424	1	29.4274	1	31.5357	1	34.0669	1	37.1685	1	41.0727	1	46.177	1	53.271	1	64.8288
2	5.4669	2	5.7115	2	5.9954	2	6.3291	2	6.727	2	6.9561	2	7.4919	2	7.8079	2	8.1642	2	8.5694	2	9.0342	2	9.5734	2	10.2072	2	10.964	2	11.886	2	13.0394	2	14.5375	2	16.6055	2	19.9613
3	2.3738	3	2.4787	3	2.6001	3	2.7423	3	2.9111	3	3.0079	3	3.2335	3	3.366	3	3.515	3	3.6839	3	3.8769	3	4.1001	3	4.3612	3	4.6718	3	5.0485	3	5.5174	3	6.1233	3	6.9555	3	8.304
4	1.1594	4	1.2134	4	1.2758	4	1.3487	4	1.435	4	1.4844	4	1.5992	4	1.6665	4	1.7419	4	1.8273	4	1.9246	4	2.0368	4	2.1677	4	2.323	4	2.5107	4	2.7436	4	3.0436	4	3.4543	4	4.1207
5	0.5902	5	0.6214	5	0.6573	5	0.6993	5	0.749	5	0.7774	5	0.8433	5	0.8819	5	0.9252	5	0.974	5	1.0297	5	1.0938	5	1.1686	5	1.257	5	1.3638	5	1.4962	5	1.6663	5	1.8991	5	2.2788
6	0.2939	6	0.3131	6	0.3352	6	0.3612	6	0.3919	6	0.4094	6	0.4503	6	0.4742	6	0.501	6	0.5314	6	0.5659	6	0.6057	6	0.6522	6	0.7072	6	0.7736	6	0.856	6	0.962	6	1.1074	6	1.3466
7	0.1314	7	0.1435	7	0.1576	7	0.1741	7	0.1938	7	0.205	7	0.2314	7	0.2468	7	0.2642	7	0.2839	7	0.3065	7	0.3325	7	0.3629	7	0.399	7	0.4428	7	0.4973	7	0.5676	7	0.6647	7	0.8267
8	0.0419	8	0.0495	8	0.0584	8	0.0689	8	0.0815	8	0.0888	8	0.1059	8	0.116	8	0.1274	8	0.1405	8	0.1554	8	0.1728	8	0.1933	8	0.2177	8	0.2475	8	0.2848	8	0.3334	8	0.401	8	0.5164
11	0.0292	11	0.0285	11	0.0274	12	0.026	12	0.0247	12	0.0244	9	0.0348	9	0.0412	9	0.0486	9	0.0571	9	0.067	9	0.0786	9	0.0923	9	0.1089	9	0.1294	9	0.1553	9	0.1895	9	0.2378	9	0.3228

Table 5: Table of the 10 largest errors, $\varepsilon_{F_{rel}}$, and which k is comes from, for every p_i, under $npq = 10$ *with* continuity correction.

First block

$p=0.01$		0.0199		0.0297		0.0395		0.0493		0.059		0.0685		0.0795		0.089		0.0978		0.1086		0.1172		0.1273		0.1394	
k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$
0	0.514	0	0.533	0	0.5523	0	0.5721	0	0.5924	0	0.6132	0	0.634	0	0.6588	0	0.6809	0	0.7019	0	0.7283	0	0.7501	0	0.7764	0	0.8088
2	0.0856	2	0.0842	2	0.0828	2	0.0813	2	0.0797	2	0.078	2	0.0762	2	0.0741	2	0.0721	2	0.0702	2	0.0678	2	0.0657	2	0.0631	1	0.0634
3	0.0524	3	0.0527	3	0.053	3	0.0533	3	0.0536	3	0.0538	3	0.054	3	0.0541	3	0.0542	3	0.0542	3	0.0542	3	0.0542	3	0.0535	2	0.0599
1	0.0243	1	0.0189	4	0.0132	4	0.0133	4	0.0141	4	0.0148	4	0.0155	1	0.0181	1	0.0247	1	0.031	1	0.039	1	0.0455	1	0.0535	3	0.0538
6	0.0121	6	0.012	6	0.0126	6	0.0117	6	0.0115	6	0.0113	6	0.0111	4	0.0163	4	0.017	4	0.0177	4	0.0184	4	0.0191	4	0.0198	4	0.0206
4	0.0112	4	0.0119	5	0.0118	5	0.0085	5	0.008	5	0.0075	5	0.0108	6	0.0109	6	0.0107	6	0.0105	6	0.0103	6	0.0101	6	0.0098	6	0.0095
5	0.0099	5	0.0094	1	0.009	1	0.0075	7	0.0072	7	0.0072	7	0.0071	7	0.0071	7	0.0071	7	0.007	7	0.007	7	0.0069	7	0.0069	7	0.0068
7	0.0073	7	0.0073	7	0.0072	7	0.0072	8	0.003	1	0.0046	1	0.007	5	0.0064	5	0.0059	5	0.0054	5	0.0048	5	0.0043	5	0.0037	8	0.003
8	0.0031	8	0.0031	8	0.0031	8	0.0031	1	0.0015	8	0.003	8	0.0029	8	0.003	8	0.003	8	0.003	8	0.003	8	0.003	8	0.003	5	0.0029
9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001	9	0.001

Second block

0.1464		0.1542		0.1629		0.1727		0.1838		0.1965		0.2113		0.2288		0.25		0.2764		0.311		0.3613		0.5	
k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$	k	$\varepsilon_{F_{rel}}$
0	0.8281	0	0.8499	0	0.8748	0	0.9036	0	0.9373	0	0.9773	0	1.0255	0	1.085	0	1.1605	0	1.2602	0	1.3994	0	1.6142	0	2.0641
1	0.0692	1	0.0759	1	0.0836	1	0.0925	1	0.1029	1	0.1153	1	0.1304	1	0.1491	1	0.173	1	0.2048	1	0.2498	1	0.3203	1	0.4769
2	0.0579	2	0.0557	2	0.0531	3	0.0526	3	0.052	3	0.0512	3	0.05	3	0.0484	3	0.0459	3	0.042	3	0.0354	2	0.0436	2	0.1227
3	0.0536	3	0.0534	3	0.053	2	0.0499	2	0.0462	2	0.0417	2	0.0361	2	0.029	4	0.0263	4	0.0268	4	0.0266	4	0.024	3	0.02
4	0.0211	4	0.0216	4	0.0222	4	0.0228	4	0.0234	4	0.0241	4	0.0249	4	0.0256	2	0.0196	2	0.0122	2	0.0122	3	0.0226	4	0.0031
6	0.0093	6	0.0091	6	0.0088	6	0.0085	6	0.0081	6	0.0077	6	0.0071	6	0.0064	6	0.0056	5	0.0081	5	0.0099	9	0.0099	9	0.0024
7	0.0068	7	0.0067	7	0.0066	7	0.0066	7	0.0065	7	0.0063	7	0.0062	7	0.006	7	0.0055	7	0.0051	7	0.0043	5	0.002	5	0.002
8	0.003	8	0.003	8	0.003	8	0.003	8	0.003	8	0.003	8	0.0029	8	0.003	5	0.0044	6	0.0043	6	0.0026	8	0.0018	8	0.0016
5	0.0025	5	0.002	5	0.0014	9	9e-04	9	9e-04	9	9e-04	5	0.0018	5	0.0029	8	0.0028	8	0.0027	8	0.0024	9	5e-04	10	0.0015

Table 6: Table of the 10 largest errors, $\varepsilon_{F_{rel}}$ and which k is comes from, for every p_i, under $npq = 3$ *with* continuity correction.

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