Going Round in Circles

From Sigma Models to Vertex Algebras and Back

JOEL EKSTRAND

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Abstract
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In this thesis, we investigate sigma models and algebraic structures emerging from a Hamiltonian description of their dynamics, both in a classical and in a quantum setup. More specifically, we derive the phase space structures together with the Hamiltonians for the bosonic two-dimensional non-linear sigma model, and also for the N=1 and N=2 supersymmetric models.

A convenient framework for describing these structures are Lie conformal algebras and Poisson vertex algebras. We review these concepts, and show that a Lie conformal algebra gives a weak Courant–Dorfman algebra. We further show that a Poisson vertex algebra generated by fields of conformal weight one and zero are in a one-to-one relationship with Courant–Dorfman algebras.

Vertex algebras are shown to be appropriate for describing the quantum dynamics of supersymmetric sigma models. We give two definitions of a vertex algebra, and we show that these definitions are equivalent. The second definition is given in terms of a λ-bracket and a normal ordered product, which makes computations straightforward. We also review the manifestly supersymmetric N=1 SUSY vertex algebra.

We also construct sheaves of N=1 and N=2 vertex algebras. We are specifically interested in the sheaf of N=1 vertex algebras referred to as the chiral de Rham complex. We argue that this sheaf can be interpreted as a formal quantization of the N=1 supersymmetric non-linear sigma model. We review different algebras of the chiral de Rham complex that one can associate to different manifolds. In particular, we investigate the case when the manifold is a six-dimensional Calabi–Yau manifold. The chiral de Rham complex then carries two commuting copies of the N=2 superconformal algebra with central charge c=9, as well as the Odake algebra, associated to the holomorphic volume form.

Keywords: Chiral de Rham complex, Conformal field theory, Poisson vertex algebra, Sigma model, String theory, Vertex algebra

Joel Ekstrand, Uppsala University, Department of Physics and Astronomy, Theoretical Physics, Box 516, SE-751 20 Uppsala, Sweden.

Till Åsa, Signe, Astrid och Limpan

Some plans were made and rice was thrown
A house was built, a baby born
How time can move both fast and slow amazes me
And so I raise my glass to symmetry
To the second hand and its accuracy
To the actual size of everything
The desert is the sand

Conor Oberst, I believe in symmetry

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I J. Ekstrand and M. Zabzine, *Courant-like brackets and loop spaces*, Journal of High Energy Physics 2011, 20 (2011).

II J. Ekstrand, R. Heluani, J. Källén and M. Zabzine, *Non-linear sigma models via the chiral de Rham complex*, Advances in Theoretical and Mathematical Physics 13, 1221 (2009).

III J. Ekstrand, R. Heluani, J. Källén and M. Zabzine, *Chiral de Rham complex on special holonomy manifolds*, `arXiv:1003.4388 [hep-th]`. Under review in Communications in Mathematical Physics.

IV J. Ekstrand, *Lambda: A Mathematica package for operator product expansions in vertex algebras*, Computer Physics Communications 182, 409 (2011).

V J. Ekstrand, R. Heluani and M. Zabzine, *Sheaves of N=2 supersymmetric vertex algebras on Poisson manifolds*, `arXiv:1108.4943 [hep-th]`. Under review in Journal of Geometry and Physics.

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Contents

1. Introduction

Be patient, for the world is broad and wide. This advice is the first the reader of the novel Flatland encounter. The novel, written by E. A. Abbot, and first published in 1884, is an entertaining science fiction classic, describing the lives of the inhabitants of the two-dimensional Flatland [1].

> I call our world Flatland, not because we call it so, but to make its nature clearer to you, my happy readers, who are privileged to live in Space.
>
> Imagine a vast sheet of paper on which straight Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, on or in the surface, but without the power of rising above or sinking below it, very much like shadows — only hard and without luminous edges — and you will then have a pretty correct notion of my country and countrymen. Alas, a few years ago, I should have said "my universe": but now my mind has been opened to higher views of things.

The narrator of the novel is A. Square, and the reader gets acquainted with his journeys into Pointland, Lineland and Spaceland. The novel is, in addition to being mathematical fiction, a satire of the society. The shapes and the social statuses of the inhabitants are directly related — the more sides a polygon has, the higher its status in society, with the polygons approximating circles being the priest class.

In this thesis, we shall in a way investigate how the world is for an inhabitant of Lineland, living on a circle. We will see that the inhabitant can learn quite a lot about the ambient space by going round the circle and observe his own one-dimensional world. It turns out there is a rich interplay between the symmetries on the circle and the geometries of the surrounding Spaceland.

Observations. Physics. Mathematics. In some sense, this is the trinity of our scientific understanding of the world. From observations, we try to make models that captures and describes aspects of what we see. This process is called physics, and to formulate, develop, and understand the models, we need mathematics. It is a line, or rather a circle, of thought that we traverse over and over again. Our understanding of our world gradually evolves, changes, and sometimes get deeper. It may be fruitful to let the observations be of an imagined kind — so-called *gedanken experiments*. Sometimes, when the shackle of the experiments is loosened, the physics and the mathematics can develop a

fruitful symbioses that offers new perspectives on what we already thought we knew. String theory is of this kind. Regardless of whether it offers models that can pass experimental tests, it has created an immense input to mathematics and offered new insights in the already established physical theories, including the ones that are based on real experimental observations of the real world around us. The mathematical physics investigated in this thesis is closely related to string theory. We will investigate structures, and make connections, that are present regardless of the exact details of the physical model. Perhaps, we can, like A. Square of the Flatland, get a glimpse of the enclosing reality by going round the circle once more. *Be patient, for the world is broad and wide.*

Outline of the thesis

The thesis is organized as follows. In chapter 2, some of the basic notation is set. The geometry associated to a Hamiltonian treatment of mechanics is discussed. We conclude with a brief discussion about different approaches to quantization. In chapter 3, we review sigma models, starting with a sigma model formulation of a classical point-particle, followed by the two-dimensional bosonic non-linear sigma model, and the $N = 1$ and $N = 2$ supersymmetric versions thereof. The phase space structures of these models are described, and the Hamiltonians of the models are derived. Chapter 4 contains definitions of Poisson vertex algebras and Lie conformal algebras, in order to give an algebraic description of the phase space of the bosonic sigma model. We show that a Lie conformal algebra gives a weak Courant–Dorfman algebra, and we also show relations between Poisson vertex algebras and Courant–Dorfman algebras. Chapter 5 starts with a review of formal distributions, in order to describe vertex algebras in the forthcoming subsections. The main objective is to introduce the λ-bracket and the normal ordered product, and to show how the definition of a vertex algebra can be expressed using these operations. In chapter 6 we discuss sheaves of different vertex algebras. The type of vertex algebras under investigation are the bosonic $\beta - \gamma$ vertex algebra, the $N = 1$ SUSY vertex algebra, and finally the $N = 2$ SUSY vertex algebra. We briefly discuss global sections of the sheaf of the latter. Finally, in chapter 7, the sheaf of $N = 1$ SUSY vertex algebras called the chiral de Rham complex is discussed in a bit more detail. We argue that this sheaf can be interpreted as a formal quantization of the $N = 1$ supersymmetric non-linear sigma model, and discuss symmetry algebras present in the chiral de Rham complex on manifolds of special holonomy.

2. Poisson geometry and quantization

In this chapter, we start to explore relations between physics and geometry. Eventually, we want to see how these relations are affected when the physical system is quantized. We are mainly interested in the Hamiltonian approach to classical mechanics, and in this chapter, we consider point particles. In later chapters, when considering extended objects, we will encounter infinite-dimensional analogues of the structures described in this chapter.

We takeoff by describing the basic scenes where the physics are played, namely manifolds.

2.1 Manifolds

The classical objects to study in differential geometry are manifolds. One- and two-dimensional manifolds are mathematical descriptions of objects that are familiar to us: curves and surfaces. The description of these objects can be generalized to arbitrary dimension. Locally, in what is called a *patch*, a d-dimensional manifold looks like the vector space $\mathbb{R}^d$. To describe the full manifold, these patches are then sewn together, or *glued*, and in total, one can have arbitrarily complicated shapes, built up by the patches.

Definition 2.1 (Manifold). A d-dimensional smooth (or C^∞) manifold is a topological space M, with a collection of open sets $\{U_\alpha\}$, such that they cover M, i.e., $\bigcup_\alpha U_\alpha = M$, and to each such open set U_α, there is a homeomorphism $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^d$, where V_α is an open subset of $\mathbb{R}^d$. That ϕ_α is a homeomorphism means that it is a continuous map that are invertible, and the inverse map, $\phi_\alpha^{-1} : V_\alpha \to U_\alpha$, is also continuous. Using this homeomorphism, we can describe the coordinates on U_α by coordinates in V_α. Given a pair, U_α and U_β say, of open subsets of M that overlap, i.e., $U_\alpha \cap U_\beta \neq \emptyset$, we can construct a map $\psi_{\alpha\beta} \equiv \phi_\alpha \circ \phi_\beta^{-1}$ from $\phi_\beta(U_\alpha \cap U_\beta)$ to $\phi_\alpha(U_\alpha \cap U_\beta)$. In order for the manifold to be smooth, we require that these maps are smooth, i.e., all derivatives with respect to the coordinates on $U_\alpha \cap U_\beta$ exists: $\psi_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)$. $\qquad\bullet$

In addition to manifolds, we are also going to use supermanifolds, where some of the coordinates are "numbers" that do not commute.

2.2 Supermanifolds

We are here going to give a very brief description of supergeometry to remind the reader about the basic concepts and to set the notation. For a more proper introduction to the subject, see, e.g., [48], and the references therein.

In the definition of a manifold (definition 2.1) the coordinates used on each local patch took values in $\mathbb{R}^n$. They are therefore described by ordinary numbers. In particular, these numbers commute. By introducing anti-commuting "numbers", *Grassman numbers*, we can extend the concept of a manifold to also include anti-commuting coordinates. This is the basic idea of supermanifolds.

The interest in supermanifolds, and in the supergeometry that describes them, grew out of the concept of supersymmetry. Supersymmetry, at least in the original incarnation, is a symmetry between bosonic and fermionic fields in a field theory. With more than two space-time dimensions, the spin-statistic theorem demands bosons to transform in a representation of $\mathrm{SO}(n)$ under space-time rotations – they are integer spin-particles. The fermions, on the other hand, transforms in a $\mathrm{Spin}(n)$-representation – they are half-integer spin particles. In quantum mechanics, if we exchange the position of two equal fermions, the wave function that describes them changes by a minus sign. By introducing Grassman numbers, this behavior can be described and captured by classical functions. In supergeometry, such functions are given a geometrical meaning by interpreting them as "coordinates" on a supermanifold.

A *super vector space V* is a vector space that can be decomposed as $V = V_0 \oplus V_1$. Such a vector space is also called a $\mathbb{Z}_2$-graded vector space. The elements in V_0 are called even, the elements in V_1 are called odd. We denote the grading of an element $a \in V_m$ by $|a| = m$. We also define Π, the *parity reversion functor*. This functor reverses the parity of the elements, e.g., if we think of $\mathbb{R}^m$ as the ordinary m-dimensional even vector space, $\Pi\mathbb{R}^m$ is a purely odd vector space. Let θ^i, $i = 1, \ldots, m$ be a basis of this vector space. We endow this vector space with a product, so we have an algebra. We define the multiplication between the base vectors to fulfill $\theta^i \theta^j = -\theta^j \theta^i$. Such an algebra is called a supercommutative superalgebra. The coefficients of a vector are multiplied together, and they commute with the base vectors, $x_i \theta^i = \theta^i x_i, x_i \in \mathbb{R}$. The base vectors in this algebra are often called *Grassman numbers*. From the rule of how the base vectors are multiplied, we see that $(\theta^i)^2 = 0$.

The elements of the algebra of functions defined on $\Pi\mathbb{R}^m$ will be of the form

$$f_i \theta^i + \ldots + f_{i_1 \ldots i_m} \theta^{i_1} \ldots \theta^{i_m} \,, \tag{2.1}$$

where $f_{i_1 \ldots i_k} \in \mathbb{R}$. Denote this algebra by $\wedge^\bullet(\mathbb{R}^m)$. We are now ready to give the definition of a supermanifold.

Definition 2.2 (Supermanifold). A supermanifold of n even dimensions and m odd dimensions, which we denote by $\mathcal{M}^{n|m}$, is defined as an n-dimensional manifold M, with a sheaf of supercommutative superalgebras defined over it. Locally, the manifold should look like $C^\infty(U_\alpha) \otimes \wedge^\bullet(\mathbb{R}^m)$ where U_α is a local patch of M, isomorphic to a subset of $\mathbb{R}^n$. For a definition of sheaves, see section 6.1.   ●

As an example, we can consider functions on $\mathbb{R}^{1|m}$. These will be functions of m Grassman numbers. We can expand the function in the generators θ^i. Since the θ's square to zero, the Taylor expansion of the function terminates and we get:

$$f(x, \theta^1, \ldots, \theta^m) = f_0(x) + f_i(x)\theta^i + \ldots + f_{i_1 \ldots i_m}(x)\theta^{i_1} \ldots \theta^{i_m} , \qquad (2.2)$$

where x is the coordinate on the even part of $\mathbb{R}^{1|m}$.

We want to define integration of functions on a supermanifold. We define integration of Grassman numbers as follows:

$$\int \mathrm{d}\theta \, 1 = 0 , \qquad\qquad \int \mathrm{d}\theta \, \theta = 1 . \qquad (2.3)$$

These are called *Berezin* integrals. Note that integration equals derivation, $\int \mathrm{d}\theta f = \frac{\partial}{\partial \theta} f$. With this definition, the integration is linear and the formula of partial integration holds, i.e., $\int d\theta \frac{\partial}{\partial \theta} f(x, \theta) = 0$.

Let $\theta \to \alpha\theta$. We see that the integration measure must transform as $\alpha^{-1} \mathrm{d}\theta$, in order for the definition (2.3) to hold under this rescaling.

2.3 Poisson geometry

The dynamics of a physical system in the Hamiltonian formulation of classical mechanics is given by the Hamilton equations. This concept has been given a geometrical meaning and is the starting point for symplectic and Poisson geometry. It is also the cornerstone of the canonical quantization of the system.

To be more precise, let us consider a system described by classical mechanics. The state of the system at a given time, $t = t_0$, is described by a set of n generalized coordinates, q^i, where $i = 1, \ldots, n$, together with their conjugate momenta, p_i. In its simplest form, these can be given the interpretation of coordinates on the manifold $\mathbb{R}^{2n}$, called *the phase space*. Given a Hamiltonian $H(q, p)$, the time evolution of the system is given by the first order equations

$$\dot{q}^i = \partial H/\partial p_i, \qquad\qquad \dot{p}_i = -\partial H/\partial q^i . \qquad (2.4)$$

These equations are the Hamilton equations. Let us define a bracket, the *Poisson bracket*, between functions of the $2n$ variables (q^i, p_i) by

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} . \tag{2.5}$$

The Hamilton equations can now be written in a compact form, as $\dot{g} = \{H, g\}$, where $g = g(p, q)$. The time evolution of the system is thus given by the Poisson bracket $\{,\}$, and the Hamiltonian H.

From the definition (2.5), we see that the Poisson bracket is antisymmetric,

$$\{f, g\} = -\{g, f\} . \tag{2.6a}$$

It is bilinear, i.e., for three functions f, g, and h,

$$\{f, \alpha g + \beta h\} = \alpha\{f, g\} + \beta\{f, h\} , \tag{2.6b}$$

where α and β are constants. It also fulfills the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 . \tag{2.6c}$$

The properties (2.6) makes the bracket a *Lie bracket* on the functions on $\mathbb{R}^{2n}$. We can multiply functions together pointwise and get new functions. The partial derivates with respect to p_i and q^i in the definition of the Poisson bracket, (2.5), fulfill the Leibniz rule of derivation. When these derivates acts on products of functions, we get the property

$$\{f, gh\} = \{f, g\}h + g\{f, h\} . \tag{2.7}$$

These properties are crucial for the role the bracket (2.5) plays in classical mechanics. It is therefore natural to define an algebra, i.e., a way of combining objects, with a bracket that fulfills the properties (2.6) and (2.7) as a *Poisson algebra*. In this example, we considered the manifold $\mathbb{R}^{2n}$. In general, if we can define a Poisson algebra on the functions of a manifold M, this manifold is a *Poisson manifold*. A general phase space is a Poisson manifold.

Definition 2.3 (Poisson manifold). A *Poisson manifold* is a manifold M, with a Lie bracket $\{,\}$, see (2.6), defined on $C^\infty(M)$, such that (2.7) is fulfilled. The bracket is then a *Poisson bracket*. $\bullet$

A Poisson manifold has a bivector, i.e., a rank-two, contravariant and antisymmetric tensor, $\Pi \in \wedge^2 TM$, called the *Poisson structure*. The Poisson bracket can be expressed using this structure. In local coordinates, $\{x^i\}$, on the Poisson manifold, we have

$$\{f, g\} = \frac{\partial f}{\partial x^i}\Pi^{ij}(x)\frac{\partial g}{\partial x^j} . \tag{2.8}$$

The Poisson bracket (2.5) corresponds to having the coordinates $x^i = q^i$ for $i = 1, \ldots, n$, and $x^i = p_{i-n}$ for $i = n+1, \ldots, 2n$, and the Poisson structure

$$\Pi^{ij} = \begin{pmatrix} 0 & -\mathbb{1} \\ +\mathbb{1} & 0 \end{pmatrix} . \tag{2.9}$$

The property (2.6c) is expressed in terms of Π as $\{\Pi, \Pi\}_S = 0$, where $\{,\}_S$ is the Schouten bracket (see Paper I, section 4.1, for a definition). In local coordinates, this means

$$\Pi^{il} \Pi^{jk}_{,l} + \Pi^{kl} \Pi^{ij}_{,l} + \Pi^{jl} \Pi^{ki}_{,l} = 0 . \tag{2.10}$$

The study of Poisson manifolds is the aim of Poisson geometry. In section 6.4, we will see that the Poisson structure naturally emerges when we investigate manifolds with a manifest $N = 2$ supersymmetric vertex algebra defined on them.

2.4   Symplectic manifolds

If the Poisson structure Π of a Poisson manifold is invertible, i.e., we can find an ω, such that $\omega_{ij} \Pi^{jk} = \delta_i^k$, then this ω is a closed nondegenerate two-form. A manifold (M, ω) with a closed nondegenerate two-form ω is called a symplectic manifold. Since ω is nondegenerate, we can always find an inverse, and all symplectic manifolds are Poisson manifolds. We can always locally choose *Darboux coordinates*, where the Poisson structure takes the form (2.9).

2.5   Quantization

We have so far described classical systems. The observables are functions on a phase space P. These functions can be pointwise multiplied, and $f \cdot g = g \cdot f$ and $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for $f, g, h \in C^\infty(P)$, so $C^\infty(P)$ has the structure of a commutative and associative algebra. The phase space is also endowed with a Poisson bracket. One particular function, the Hamiltonian, governs the dynamics of the system via the Poisson bracket.

The procedure referred to as *quantization* of a system is ambiguous, and can have very different meanings. Different aspects of quantization have been formalized and generalized within different mathematical "programs". The general idea is to find a system that possess some desired quantum properties, and that has a parameter $\hbar$, such that when $\hbar \to 0$, the original classical system is recovered. The physical constant $\hbar$, the (reduced) *Planck constant*, is dimensionful with the dimension Energy $\cdot$ Time. In a physical system, there often exists a typical scale of energy and of time given by the measurement apparatus used to observe the system. The meaning of letting the *constant $\hbar$*

be treated as a variable parameter is that one considers the ratio of the constant with these scales.

In canonical quantization the observables $f \in C^\infty(P)$ are mapped to operators $\mathcal{Q}(f)$ that acts on a Hilbert space $\mathcal{H}$, *the space of states*. In the schoolbook recipe of quantization, one requires that the commutator of these operators is related to the Poisson bracket between the corresponding classical observables by

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar \mathcal{Q}(\{f, g\}) \,. \tag{2.11}$$

In the limit $\hbar \to 0$, we get back the commutativity of the observables. The map $\mathcal{Q}$ does not need to preserve the structure of pointwise multiplication of observables, i.e., $\mathcal{Q}(fg) \neq \mathcal{Q}(f) \circ \mathcal{Q}(g)$ in general. The problem is to construct $\mathcal{Q}$ in such way that (2.11) is respected. It turns out that, given some additional requirements on $\mathcal{Q}$ and $\mathcal{H}$, this is in general not possible. Even for the simplest case, when $P = \mathbb{R}^{2d}$, with coordinates q^i and p_i, and with the canonical Poisson bracket (2.5), we can not construct such $\mathcal{Q}$ for functions that are more than quadratic in q or p [29].

We here briefly want to mention two approaches that are closely related to the canonical quantization: geometric and deformation quantization. We also want to mention another, somewhat different, approach: the path integral formulation of quantum mechanics.

Geometrical quantization aims, as the name suggest, at giving the canonical quantization a geometrical meaning. The operators $\mathcal{Q}(f)$ are first order differential operators on a *line bundle* over P, and the Hilbert space is related to the square-integrable functions on this line bundle, via a choice of *polarization*. For a review, see [51].

In *deformation quantization*, one modifies, or deforms, the associative and commutative pointwise product between functions on P, while the observables are still represented by classical functions or distributions on the phase space. The deformation parameter is $\hbar$. In general, the deformed product, called the *star product* $*$, is an infinite power series: $f * g = \sum_{n=0}^\infty (i\hbar)^n C_n(f, g)$. One requires that the star product is associative, and that C_0 is the ordinary pointwise product, so the classical behavior is recovered when $\hbar \to 0$. Also, the term linear in $\hbar$ should correspond to the Poisson bracket of the phase space: $\{f, g\} = C_1(f, g) - C_1(g, f)$. The star product gives a non-commutative algebra of observables. For more details, see [51], and the references therein.

Another approach to quantization is the *path integral*. In 1948, Feynman showed [19] that the canonical quantization of a system (with phase space $\mathbb{R}^{2d}$) could equivalently be formulated as an infinite-dimensional integral over all possible paths in the phase space connecting the initial and the final state. The integrand contains the exponentiated action-functional of the theory, this is the weight of the contribution of each path. The main difficulty in this approach is to define the correct integration measure on the infinite-dimensional

space of such paths. Even though this approach to quantization has been extremely successful when it comes to producing physical and mathematical results, it is disappointing that more than 60 years after it was formulated, the path-integral still lacks a rigorous mathematical formulation.

In this thesis we will, starting with chapter 5, be concerned with vertex algebras. Vertex algebras are basically a quantization of Poisson vertex algebras, as described in section 5.3. They are, in contrast to the path integral approach, mathematically rigorously defined, and they describe formal aspects of two-dimensional quantum field theory. The quantization is similar to the canonically one, with operators acting on a Hilbert space and so forth, but the phase space will be infinite-dimensional. The classical functions, or rather functionals, are mapped to *vertex operators*. The commutator between these operators will be of the form (2.11), but in general also higher orders in $\hbar$ appears.

But before that, we want to describe some classical systems that we later want to quantize.

3. Sigma models

The name "sigma model" originates from Gell-Mann and Lévy, and their article [24] from 1960. They wanted to model the behavior of pion decays and in the model they constructed, they introduced a new field: a scalar meson. This field was dubbed σ, hence the name. Today, in general, sigma models are theories of maps from one manifold, which we call the *worldsheet*, to another manifold, called the *target manifold*.[†]

The standard sigma model is given as follows. Let X be a map from a d-dimensional worldsheet Σ to a D-dimensional target M. Let M have a metric g. We want to create an action out of this data, in the form of an integral over the worldsheet, where the map X is interpreted as a field living on Σ. In local coordinates, we have

$$X: \Sigma \to M \,, \quad \xi^\alpha \mapsto X(\xi)^i \,, \tag{3.1}$$

where ξ^α is a coordinate on Σ, and, for a given $\xi = \xi_0$, $X(\xi_0)^i$ is a coordinate on M. The map X induces a metric h on Σ from the metric g on M :

$$h_{\alpha\beta}(\xi) \equiv g_{ij}\left(X(\xi)\right) \frac{\partial X(\xi)^i}{\partial \xi^\alpha} \frac{\partial X(\xi)^j}{\partial \xi^\beta} \,. \tag{3.2}$$

The action of the sigma model is now given by

$$S[X] = \int_\Sigma h_{\alpha\beta} \gamma^{\alpha\beta} \, \mathrm{d}\,\mathrm{Vol}_\Sigma \,, \tag{3.3}$$

where γ is a fixed metric on Σ, with the associated volume form $\mathrm{d}\,\mathrm{Vol}_\Sigma$. We usually choose the worldsheet to be a flat Minkowski or Euclidean manifold. We can alternatively write (3.3) as $S = \int_\Sigma g_{ij}(X) \, \mathrm{d}X^i \wedge *\mathrm{d}X^j$, where $*$ is the Hodge dual on the worldsheet.

If the target manifold is flat, with a constant metric g, then the model is a *linear sigma model*. With a general metric, the action (3.3) will have non-linear terms in the field X and the model is hence called a *non-linear sigma model*.

[†] The name worldsheet is of course most appropriate in the case of a two-dimensional manifold, but we here use the name for arbitrary dimension.

By considering additional geometrical data from the target space, the action (3.3) can be supplemented by further terms. We here consider the simplest case, built solely out of the metric of the target.

Demanding the sigma model action to be invariant under certain transformations on the worldsheet can put interesting constraints on the possible geometries of the target manifolds. For instance, if we have a sigma model, using only the target space metric, $N = 2$ supersymmetry forces the target manifold to be a Kähler manifold, see, e.g., [42] for a review of the relation between supersymmetries and target geometries. A good overview of the sigma models discussed here is given in [35].

3.1 One-dimensional worldsheet: classical mechanics

Let us first consider the simplest case, when Σ is one-dimensional. Then, instead of worldsheet, *worldline* is a more proper name for the manifold Σ. Let $\Sigma = \mathbb{R}$, and let t be the coordinate on the worldline. We have $X : \mathbb{R} \to M$, $t \mapsto X^i(t)$. The action (3.3) is then given by

$$S = \tfrac{1}{2} \int dt \, g_{ij}(X)\dot{X}^i\dot{X}^j \, , \tag{3.4}$$

where $\dot{\ } \equiv \partial/\partial t$, and a convenient factor of one-half is introduced. When going to the Hamiltonial formalism, we first introduce a momenta, P_i, conjugate to the field X^i. The momenta is defined by

$$P_i \equiv \frac{\delta S}{\delta \dot{X}^i} = g_{ij}(X)\dot{X}^j \tag{3.5}$$

Since g_{ij} is a metric, it is an invertible matrix, and we can write the action (3.4) as

$$S = \int dt \, \left(P_i \dot{X}^i - \tfrac{1}{2}g^{ij}P_iP_j \right) \, , \tag{3.6}$$

From (3.6) one can read of two things. First, the *Liouville one-form* is given by $\theta = P_i \, dX^i$, and the symplectic structure on the phase space is then given by $\omega = d\theta = dP_i \wedge dX^i$. This means that the Poisson bracket of the theory is the canonical one, given by

$$\{X^i, P_j\} = \delta^i_j \, . \tag{3.7}$$

Next, the Hamiltonian is given by

$$H = H(X, P) = \frac{1}{2}g^{ij}(X)P_iP_j \, . \tag{3.8}$$

Note that in the Hamiltonian picture, X and P is independent of the "time" t. The t-dependence is given by H, from the flow equations

$$\dot{X}^i = \{H, X^i\} = -g^{ij}P_j, \qquad \dot{P}_i = \{H, P_i\} = \frac{1}{2}\frac{\partial g^{jk}}{\partial X^i}P_jP_k. \qquad (3.9)$$

The configuration space is given by the manifold M itself. The phase space is, as usual, the cotangent bundle of the configuration space, i.e., T^*M. This sigma model is thus described by using ordinary classical mechanics.

From (3.9) we see that the second time-derivate of X is given by

$$\frac{\mathrm{d}^2 X^i}{\mathrm{d}t^2} = -\Gamma^i_{jk}\dot{X}^j\dot{X}^k, \qquad (3.10)$$

where $\Gamma^i_{jk} \equiv \frac{1}{2}g^{il}\left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}\right)$ is the Christoffel symbol of the Levi-Civita connection. We here used (3.5), the definition of P, to get an expression only involving X, that is now considered to depend on time. This is the geodesic equation, and the action (3.4) is describing a free point particle moving along geodesic curves on M.

To do the theory more interesting, we can of course add more terms to the action (3.4). For instance, we can add a potential term $-V(X)$, where V is a function on M. The physics will then describe how a particle moves on M under the influence of a potential V.

3.2 Two-dimensional bosonic sigma model

Let us now move on and consider a two-dimensional worldsheet. When sigma models are discussed in the literature, it is often implicitly understood that one considers two dimensions.

We let the worldsheet have the topology of a cylinder, and in addition to the time coordinate t on $\mathbb{R}$, we introduce a coordinate σ on the (unit) circle S^1. We have $\sigma \sim \sigma + 2\pi$. The worldsheet is then $\Sigma = \mathbb{R} \times S^1$, and we use a flat Minkowski metric on this space.

The action of the two-dimensional sigma model is

$$S = \frac{1}{2}\int_\Sigma \mathrm{d}t\,\mathrm{d}\sigma\,g_{ij}\left(\frac{\partial X^i}{\partial t}\frac{\partial X^j}{\partial t} - \frac{\partial X^i}{\partial\sigma}\frac{\partial X^j}{\partial\sigma}\right). \qquad (3.11)$$

The momenta is given by

$$P_i(\sigma) = g_{ij}(X(\sigma))\frac{\partial X^j(\sigma)}{\partial t}. \qquad (3.12)$$

and we can rewrite (3.11) as

$$S = \int_\Sigma \mathrm{d}t\left(\int \mathrm{d}\sigma P_i \dot{X}^i - H\right), \qquad (3.13)$$

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with the Hamiltonian

$$H = \frac{1}{2} \oint_{S^1} \mathrm{d}\sigma \, \left(g^{ij} P_i P_j + g_{ij} \partial_\sigma X^i \partial_\sigma X^j \right) \, . \tag{3.14}$$

Comparing with the one-dimensional sigma model (3.8), we see that the σ-dependence of X generates a potential term.

The configuration space of this model is given by the *loop space* $\mathcal{L}M$, i.e., the space of all maps from the circle S^1, to the manifold M,

$$\mathcal{L}M = \{ X : S^1 \to M \} \, . \tag{3.15}$$

A given X, i.e., a given way to map the circle S^1 to M, corresponds to a point on the infinite-dimensional space $\mathcal{L}M$. We now want to construct the cotangent bundle of $\mathcal{L}M$. Consider the point on M corresponding to a given map X, at a fixed σ. Let $P_{\mu i}(\sigma)$ be a map from the fiber of $T_\sigma S^1$ to the fiber of $T^*_{X(\sigma)}M$. Given a vector $v \in T_\sigma S^1$, we then have $P_{\mu i}(\sigma) v^\mu \, \mathrm{d}X^i \in T^*_{X(\sigma)}M$. The tangent bundle TS^1 has one-dimensional fibers, so the index μ only takes one value, and we therefore drop it ahead. It is important to note, however, that $P_i(\sigma)$ transform as a one-form under coordinate changes on S^1. This makes the action S in (3.13) invariant under diffeomorphisms of S^1.

We can now consider $T^*\mathcal{L}M$, the cotangent bundle of $\mathcal{L}M$, as the space of morphism between TS^1 and T^*M,

$$T^*\mathcal{L}M = \left\{ (X,P) : \begin{array}{ccc} TS^1 & \xrightarrow{(X,P)} & T^*M \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ S^1 & \xrightarrow{\quad X \quad} & M \end{array} \right\} \, . \tag{3.16}$$

This space is the phase space of the two-dimensional sigma model. From (3.13) we see that the symplectic structure we should use for the model (3.11) is the canonical one, given by

$$\omega = \oint_{S^1} \mathrm{d}\sigma \, \delta P_i \wedge \delta X^i \, . \tag{3.17}$$

Here, δ is the de Rham-operator on $T^*\mathcal{L}M$, and $\delta P_i(\sigma)$ and $\delta X^i(\sigma)$ is a local basis of the one-forms of $T^*(T^*\mathcal{L}M)$. The integration can be thought of as an infinite-dimensional analogue to the summation over the index i. The transformation properties of X and P, discussed above, makes ω a well-defined two-form on $T^*\mathcal{L}M$. This gives us the Poisson brackets

$$\{ X^i(\sigma), P_j(\sigma') \} = \delta^i_j \, \delta(\sigma - \sigma') \, , \tag{3.18}$$

and

$$\{X^i(\sigma), X^j(\sigma')\} = 0 , \qquad \{P_i(\sigma), P_j(\sigma')\} = 0 . \qquad (3.19)$$

The time evolution of X is given by

$$\begin{aligned}
\dot{X}^i(\sigma) = \{H, X^i(\sigma)\} &= \oint d\sigma' g^{jk} P_k(\sigma') \{P_j(\sigma'), X^i(\sigma)\} \\
&= -g^{ij}(X(\sigma)) P_j(\sigma) .
\end{aligned} \qquad (3.20)$$

For P, we get

$$\dot{P}_i(\sigma) = -g_{ij}\partial^2 X^j + \frac{1}{2} g^{jk}_{,i} P_j P_k + \left(\frac{1}{2} g_{jk,i} - g_{ij,k} \right) \partial X^j \partial X^k . \qquad (3.21)$$

Combining this with the definition of momenta, we get the following equation of motion:

$$\frac{\partial^2 X^i}{\partial t^2} + \Gamma^i_{jk} \dot{X}^j \dot{X}^k = \frac{\partial^2 X^i}{\partial \sigma^2} + \Gamma^i_{jk} \partial_\sigma X^j \partial_\sigma X^k . \qquad (3.22)$$

This equation describes a two-dimensional geodesic flow, compare with the one-dimensional counterpart (3.10). If the target manifold is flat, (3.22) reduces to the wave equation,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \sigma} \right) X^i = 0 , \qquad (3.23)$$

which is solved by decomposing the map X in left- and right-going parts: $X^i(t, \sigma) = X^i_+(t + \sigma) + X^i_-(t - \sigma)$.

Let us do a Wick-rotation and consider an Euclidean worldsheet. Let $t = i\tau$. We can then introduce complex coordinates on the worldsheet, by $z = \sigma + i\tau$. The equation of motion will now be $\partial \bar{\partial} X^i = 0$, and the left- and right-going maps will now be represented by holomorphic respectively anti-holomorphic maps.

3.3   Conformal invariance

We here want to review symmetries of sigma models related to the choice of metric and to the choice of coordinates on the worldsheet. These considerations leads to the notion of conformal invariance. Field theories with this property are known as conformal field theories (CFTs). Good reviews about CFTs includes [50, 25], which we follow here. Also see [20] and the seminal paper by Belavin, Polyakov and Zamolodchikov [7].

Let us consider the sigma model (3.3), but now with an arbitrary metric γ. The volume form of the d-dimensional worldsheet can be written $\sqrt{-|\gamma|} \, d^d \xi$,

where ξ^α, as before, are the coordinates of the worldsheet, and $|\gamma|$ is the determinant of the metric. Let us do a local rescaling of the metric:

$$\gamma_{\alpha\beta} \to \tilde{\gamma}_{\alpha\beta} = e^{\Omega(\xi)}\gamma_{\alpha\beta} . \tag{3.24}$$

This rescaling amounts to choosing a new metric on the worldsheet. Lengths are rescaled when measured with the new metric, but angles are preserved. The determinant of the new metric is $|\tilde{\gamma}| = e^{d \cdot \Omega(\xi)}|\gamma|$. Under the rescaling we thus have

$$\gamma^{\alpha\beta} \, d\,\mathrm{Vol}_\Sigma \to e^{(\frac{d}{2}-1)\cdot\Omega(\xi)}\gamma^{\alpha\beta} \, d\,\mathrm{Vol}_\Sigma . \tag{3.25}$$

From this, we see that the case when $d = 2$ is special: the two-dimensional sigma model (with a non-fixed metric) is invariant under a local rescaling of the metric. This is called *Weyl invariance*. It is important to note that we do not transform the fields in our theory under this rescaling, it is only the metric that changes.

The action (3.3) is also invariant under diffeomorphisms of the worldsheet. If we regard the maps X^i as scalars, i.e., invariant under a change of coordinates on the worldsheet, the action is itself manifestly invariant under a coordinate change. Under such change, $\xi \to \tilde{\xi}$, the metric transforms as

$$\gamma_{\alpha\beta} \to \frac{\partial\tilde{\xi}^\epsilon}{\partial\tilde{\xi}^\alpha}\frac{\partial\tilde{\xi}^\delta}{\partial\tilde{\xi}^\beta}\gamma_{\epsilon\delta} . \tag{3.26}$$

For some particular changes of the coordinates, the transformation (3.26) is of the form (3.24). Such coordinate transformations are called *conformal transformations*. We can do such a coordinate change, followed by a Weyl transformation that absorbs the transformation of the metric. The result is that the fields in the theory transforms according to their transformation rules under reparametrization of the worldsheet — while the metric is left unchanged! We can thus regard such coordinate transformations even in theories with a fixed metric, as in the two-dimensional bosonic sigma model under consideration in the last section. The two-dimensional sigma model thus have a *conformal symmetry*, it is invariant under conformal transformations.

Now, consider an infinitesimal change of coordinates: $\tilde{\xi}^\alpha = \xi^\alpha + \epsilon v^\alpha(\xi)$, where ϵ is an infinitesimal parameter. To linear order in ϵ, we have $\xi^\alpha = \tilde{\xi}^\alpha - \epsilon v^\alpha(\xi)$, and the change in metric is $\delta\gamma_{\alpha\beta} = -\epsilon\partial_{(\alpha}v_{\beta)}$, where we used the metric to lower the index on v (we here consider the flat Minkovski metric). Requiring the change of the metric to be proportional to the metric itself, gives the equation

$$\partial_{(\alpha}v_{\beta)} \propto \gamma_{\alpha\beta}. \tag{3.27}$$

Choosing light-cone coordinates, $\sigma^\pm \equiv \frac{1}{2}(t \pm \sigma)$, this equation tells us that

$$\partial_+ v^- = 0 , \qquad\qquad \partial_- v^+ = 0 . \tag{3.28}$$

The allowed infinitesimal transformations thus are $\tilde{\sigma}^{\pm} = \sigma^{\pm} + \epsilon v^{\pm}(\sigma^{\pm})$. The finite version of these transformations is reparametrizations of $\sigma^{\pm}$, where $\sigma^{\pm} \to \tilde{\sigma}^{\pm}(\sigma_{\pm})$.

The infinitesimal changes of σ^{+} is determined by v^{+}, an arbitrary function of σ^{+}. We have infinitely many choices in choosing this function. Let us write it as $v^{+} = \sum_{n=-\infty}^{\infty} v^{n} \cdot (\sigma^{+})^{n+1}$. Each coefficient v^{n} can be chosen independently. The generators of these possible coordinate transformations are given by

$$L_n = -(\sigma^{+})^{n+1}\partial_{+} \,, \tag{3.29}$$

and $\delta\sigma^{+} = -\epsilon[v^{n}L_n, \sigma^{+}]$. These generators fulfill the commutation relations

$$[L_n, L_m] = (n-m)L_{m+n} \,. \tag{3.30}$$

This is *the Witt algebra*. For the transformations of σ^{-}, we have analogously the generators $\bar{L}_n$, fulfilling the same algebra, and $[L_n, \bar{L}_m] = 0$. When this algebra is quantized, it may get a *central extension*, i.e., an extra generator that commutes with all generators of the algebra. It is then called *the Virasoro algebra*, see section 5.4.2.

3.3.1 String theory

Let us conclude this section by briefly mention the relations between sigma models and string theory. In one sentence: String theory is a sigma model coupled to two-dimensional gravity. In bosonic string theory, one considers an action similar to (3.3), but the metric is considered to be a dynamical field,

$$S[X, \gamma] = \frac{1}{2} \int_{\Sigma} g_{ij}(X(\xi)) \, \partial_{\alpha}X(\xi)^{i}\partial_{\beta}X(\xi)^{j}\gamma^{\alpha\beta} \, \sqrt{-|\gamma|}\,\mathrm{d}^2\,\xi \,. \tag{3.31}$$

In the path integral quantization of string theory, the path integral is over all possible maps X, all possible metrics γ, and also over all possible topologies of the worldsheet. In addition to (3.31), the full action of the bosonic string theory contains a term involving the scalar curvature of the two-dimensional worldsheet. This term is analogue to the Einstein–Hilbert action of gravity. It respects diffeomorphisms and Weyl transformations. In two-dimensions, the term is proportional to the topology-dependent Euler characteristic of the worldsheet and will effectively give an expansion parameter, *the string coupling constant*. In the expansion over this coupling constant, each term will consider a worldsheat of fixed topology.

The string action (3.31) has a large set of gauge symmetries. We only want to consider inequivalent contributions, two field configurations related by a gauge symmetry should only be considered once. We can use the symmetries of the action to locally get the metric γ to be of the form $\gamma_{\alpha\beta} = \mathrm{diag}(-1, 1)_{\alpha\beta}$,

i.e., a flat Minkovski metric. The action (3.31) then reduces to the sigma model action (3.11). Although the metric now is fixed, we still have the symmetries of the form (3.27), the conformal symmetries. In order to handle this residual symmetry, the generators (3.29) should be treated as constraints.

It is not the aim of this thesis to give an introduction to string theory. For this, the reader is referred to, e.g., [26]. We want to point out, though, that the sigma model considerations in this thesis is relevant for aspects of string theory.

3.4 $N = 1$ supersymmetric sigma model

We can enlarge the model that the action (3.11) describes by adding more fields. An interesting option, that enlarge the symmetries of the action in a fundamental way, is to add fermionic fields. We then get a *supersymmetric sigma model*.

We want to consider the classical supersymmetric non-linear sigma model defined over $\mathbb{R} \times S^1$, with Minkowski signature. Let t and σ be coordinates on this manifold, as in the last section. We extend this worldsheet to a supermanifold, $\Sigma^{2|2}$, by adding two fermionic coordinates, θ^+ and θ^-. Under coordinate changes of the even part of the worldsheet, where $\sigma^\pm \equiv \frac{1}{2}(t \pm \sigma) \to \tilde{\sigma}^\pm(\sigma^\pm)$, the odd coordinates transforms as $\theta^\pm \to \sqrt{\frac{\partial \sigma^\pm}{\partial \tilde{\sigma}^\pm}} \theta^\pm$. We call this $(1,1)$ supersymmetry, since we have one left-going and one right-going supersymmetry. Note that this transformation of the odd coordinates is a choice. We could consistently assign different transformations to the odd coordinates.

The functions on $\Sigma^{2|2}$ are *superfields*, and they can be expanded as

$$\Phi(\sigma, t, \theta^+, \theta^-) = X(\sigma, t) + \theta^+ \psi_+(\sigma, t) + \theta^- \psi_-(\sigma, t) + \theta^+ \theta^- F(\sigma, t) . \quad (3.32)$$

We have odd derivatives $D_\pm$, acting on the functions, defined as:

$$D_\pm = \frac{\partial}{\partial \theta^\pm} + \theta^\pm (\partial_0 \pm \partial_1) , \qquad D_\pm^2 = \partial_0 \pm \partial_1 \equiv \partial_\pm . \quad (3.33)$$

where $\partial_0 \equiv \frac{\partial}{\partial t}$ and $\partial_1 \equiv \frac{\partial}{\partial \sigma}$.

The supersymmetric $N = (1,1)$ sigma model is now given by the following action:

$$S = \frac{1}{2} \int dt \, d\sigma \, d\theta^- \, d\theta^+ \, g_{\mu\nu}(\Phi) \, D_+ \Phi^\mu \, D_- \Phi^\nu . \quad (3.34)$$

This action is manifestly invariant under $N = (1,1)$ superconformal transformations, i.e. under a supersymmetric generalization of conformal transformations.

Integrating out the two odd coordinates, the action is

$$S = \frac{1}{2} \int dt\, d\sigma \left(g_{ij}\partial_+ X^i \partial_- X^j + g_{ij}\nabla_-\psi_+^i \psi_+^j \right.$$
$$\left. + g_{ij}\nabla_+\psi_-^i \psi_-^j + R_{ijkl}\psi_+^i \psi_+^k \psi_-^i \psi_-^l \right) \quad (3.35)$$

Here, we used the expansion (3.32), and that the component F is an auxiliary field, which can be eliminated using the equation of motion. The terms $\nabla_\pm \psi_\mp^i$ mean the covariant derivatives $\partial_\pm \psi_\mp^i + \Gamma^i_{jk}\partial_\pm X^j \psi_\mp^k$ and R is the Riemann curvature tensor of the target manifold. We see that the bosonic part of the action equals (3.11).

We want to go to the Hamiltonian formalism, keeping the "spatial" supersymmetry manifest, for the model (3.34). We do something that resembles of dimensional reduction, and get rid of one odd θ. This treatment of the sigma model was initiated in [53, 11]. We here follow Paper II.

Introduce new odd coordinates as follows:

$$\theta^0 = \frac{1}{\sqrt{2}}(\theta^+ + i\theta^-) , \qquad \theta^1 = \frac{1}{\sqrt{2}}(\theta^+ - i\theta^-) , \qquad (3.36)$$

together with the odd derivatives

$$D_0 = \frac{1}{\sqrt{2}}(D_+ - iD_-) , \qquad D_1 = \frac{1}{\sqrt{2}}(D_+ + iD_-) , \qquad (3.37)$$

which satisfy $D_0^2 = \partial_1$, $D_1^2 = \partial_1$ and $D_1 D_0 + D_0 D_1 = 2\partial_0$.

Introduce new $N = 1$ superfields, that are functions of one odd coordinate θ^1, by:

$$\phi^\mu = \Phi^\mu|_{\theta^0=0} , \qquad S_\mu = g_{\mu\nu}D_0\Phi^\nu|_{\theta^0=0} . \qquad (3.38)$$

From now on, we let $D_1 \equiv D_1|_{\theta^0=0}$.

After performing θ_0-integration, the action (3.34) becomes

$$S = \int dt\, d\sigma\, d\theta^1 \left(S_\mu \partial_0 \phi^\nu - \tfrac{1}{2}\mathcal{H} \right) , \qquad (3.39)$$

where

$$\mathcal{H} = \partial_1 \phi^\mu D_1 \phi^\nu g_{\mu\nu} + g^{\mu\nu}S_\mu D_1 S_\nu + S_\rho D_1 \phi^\gamma S_\lambda g^{\nu\lambda}\Gamma^\rho_{\gamma\nu} . \qquad (3.40)$$

We see that the configuration space of the model is the *superloop space*

$$\mathcal{L}^{|1}M = \{\phi : S^{1|1} \to M\} , \qquad (3.41)$$

the space of maps from the "supercircle" $S^{1|1}$ to the target M. The even coordinate on $S^{1|1}$ is given by σ, and θ^1 is the odd coordinate. Here, θ^1 transforms as

a section of the square root of the canonical bundle over S^1. Note that it is possible to assign different transformation properties of θ^1, leading to different supercircles.

The phase space corresponds to the cotangent bundle $T^*\mathcal{L}^{|1}M$ of the superloop space. The odd fields S_μ are the coordinates on the fiber of this bundle.

We see from (3.39) that we have the natural symplectic structure

$$\int d\sigma \, d\theta^1 \, \delta S_\mu \wedge \delta\phi^\mu \, . \tag{3.42}$$

The space of local functionals on $T^*\mathcal{L}^{|1}M$ is thus equipped with a (super) Poisson bracket $\{\,,\}$:

$$\{\phi^\mu(\sigma,\theta^1)\,,\,S_\nu(\tilde{\sigma},\tilde{\theta}^1)\} = \delta^\mu_\nu \delta(\sigma-\tilde{\sigma})\delta(\theta^1-\tilde{\theta}^1) \, . \tag{3.43}$$

From (3.39) and (3.40), we read of the Hamiltonian:

$$H = \frac{1}{2}\int d\sigma \, d\theta^1 \, \mathcal{H} \, . \tag{3.44}$$

As usual, this Hamiltonian generates the time behavior of our fields, using the Poisson bracket (3.43), through the flow equations

$$\dot{\phi}^\mu = \{H\,,\phi^\mu\}\,, \qquad\qquad \dot{S}_\mu = \{H\,,S_\mu\}\, . \tag{3.45}$$

If the target manifold is a Kähler manifold, the action (3.34) is invariant under additional supersymmetry transformations, in addition to the manifest $(1,1)$-supersymmetry. The model gets $(2,2)$-superconformal invariance [54, 4]. In the Hamiltonian treatment, these symmetries are generated by functions acting with the Poisson bracket, see Paper II for explicit expressions. We here just point out, a bit ahead, that these generators are, modulo $\hbar$-terms, identical to the operators defined in section 7.2, when discussing the chiral de Rham complex.

3.5 $N = 2$ supersymmetric sigma model

We now want to extend the number of manifest supersymmetries, and discuss the $N = (2,2)$ supersymmetric sigma model.

The worldsheet $\Sigma^{2|4}$ now has four odd coordinates: $\theta^1_+, \theta^1_-, \theta^2_+$ and θ^2_-, where each pair $\theta^i_\pm$ transforms as $\theta^\pm$ did in the previous section. The even part of the worldsheet is given by $\Sigma = \mathbb{R}\times S^1$, with coordinates t and σ as before.

We have two copies of the $N = (1,1)$ algebra.[†] The odd derivatives are defined by

$$D^i_\pm = \frac{\partial}{\partial\theta^i_\pm} + i\theta^i_\pm\partial_\pm, \quad i,j = 1,2 \, , \tag{3.46}$$

[†] We changed convention by a factor of i compared to the previous section.

where $\partial_\pm = \partial_0 \pm \partial_1$. The derivatives fulfill the algebra

$$\left(D_\pm^i\right)^2 = i\partial_\pm \,, \qquad [D_+^i, D_-^j] = 0 \,, \qquad [D_\pm^1, D_\pm^2] = 0 \,. \qquad (3.47)$$

Here, $[\,,\,]$ is the graded commutator, $[A, B] = AB - (-1)^{|A||B|}BA$.

The unconstraint $(2,2)$ superfields has $2^4 = 16$ independent components. It turns out that this is to many. When we integrate out one supersymmetry, and write our model in $N = (1,1)$ superfields, some of these components needs to be related in order to only get physical degrees of freedom. The $(2,2)$ superfields thus needs to be constrained. The constraint equations should be linear in the odd derivatives, and respected by the algebra. This leads to the possibilities of so called chiral, twisted chiral and semi-chiral fields. These different fields are needed, and sufficient, to describe different possible target manifolds of a general $N = (2,2)$ supersymmetric sigma model, see [41]. We are here going to restrict ourselfs to the $N = (2,2)$ supersymmetric sigma model with the target manifold M being a Kähler manifold. It then suffices to consider chiral, and anti-chiral, fields, fulfilling the constraints

$$\left(D_\pm^1 - iD_\pm^2\right)\Phi^\alpha = 0 \,, \qquad \left(D_\pm^1 + iD_\pm^2\right)\bar\Phi^{\bar\alpha} = 0 \,, \qquad (3.48)$$

respectively. Here, the greek indices indicates complex coordinates. The action functional for a classical $N = (2,2)$ supersymmetric sigma model with a Kähler target manifold is given by

$$S = \int_{\Sigma^{2|4}} dt \, d\sigma \, d\theta_+^1 \, d\theta_-^1 \, d\theta_+^2 \, d\theta_-^2 \, K(\Phi, \bar\Phi) \,, \qquad (3.49)$$

where K is the Kähler potential, which is only defined locally. Nevertheless, the action functional (3.49) is well-defined.

In Paper V, we observe the outcome of the following manipulations of the action (3.49). By doing a change of the odd coordinates, similar to (3.36), and integrating out two of them, the action (3.49) can be written as

$$S = \int dt \, d\sigma \, d\theta^2 \, d\theta^1 \left(iK_{,\alpha}\partial_0\phi^\alpha - \frac{1}{2}\mathcal{H} \right) \,, \qquad (3.50)$$

with

$$\mathcal{H} = g_{\alpha\bar\beta}D_2\phi^\alpha D_1\phi^{\bar\beta} - g_{\alpha\bar\beta}D_1\phi^\alpha D_2\phi^{\bar\beta} \,. \qquad (3.51)$$

Here, θ^1 and θ^2 are the remaining two odd coordinates, with the corresponding odd derivatives, $D_i = \frac{\partial}{\partial\theta^i} + \theta^i\partial_1$. Also, recall that for a Kähler manifold, the metric can be expressed by derivatives on the Kähler potential: $g_{\alpha\bar\beta} = K_{,\alpha\bar\beta}$.

We see that the Hamiltonian of the $N = (2,2)$ supersymmetric sigma model is given by

$$H = \int d\sigma \, d\theta^2 \, d\theta^1 \, \frac{1}{2}\mathcal{H} \,, \qquad (3.52)$$

and from (3.50), we also see that the Poisson bracket is given by

$$\{\phi^\alpha, \phi^{\bar\beta}\} = \omega^{\alpha\bar\beta}(\phi) \,, \tag{3.53}$$

where $\omega^{\alpha\bar\beta}$ is the inverse of the Kähler form. This allows for a Hamiltonian treatment of the sigma model, with two supersymmetries manifest. Locally, we can choose Darboux coordinates where half of the fields ϕ will be interpreted as momenta. The phase space is given by the $\mathcal{L}^{|2}M$, the space of maps from the supercircle $S^{1|2}$ to M.

In section 6.4, a vertex algebra expression, in the same form as the Hamiltonian density (3.51), will be investigated and related to the existence of $N = (2,2)$ superconformal symmetry of the model.

It is unclear how to generalize this setting to the most general $N = (2,2)$ sigma model, where also twisted chiral and semi-chiral fields are present.

4. Currents on the phase space

In the last chapter we derived the phase space structures, together with the Hamiltonians, for the bosonic and the $N = 1$ and $N = 2$ supersymmetric sigma models. We now want to investigate functions, or rather functionals, defined on these phase spaces. We concentrate on the bosonic setting.

Recall that the phase space of the bosonic sigma model, $T^*\mathcal{L}M$, is an infinite-dimensional manifold and that each point on the manifold corresponds to two given mappings, $X^i(\sigma)$ and $P_i(\sigma)$, as described in (3.16). On this space, we have the canonical Poisson bracket

$$\{X^i(\sigma), P_j(\sigma')\} = \delta^i_j\, \delta(\sigma - \sigma') \, . \tag{4.1}$$

Note that this is a local Poisson bracket which is only non-zero when the two coordinates σ and σ' on the worldsheet coincide. We can construct local functionals defined on $T^*\mathcal{L}M$ out of the coordinates X and P, and the derivation with respect to the worldsheet coordinate σ, $\partial \equiv \partial_\sigma$. We want these expressions to stay local, which means that we only allow a finite number of derivatives hitting the coordinates. The allowed expressions thus are of the form

$$A(X, \partial X, \ldots, \partial^k X, P, \partial P, \ldots, \partial^l P) \, , \tag{4.2}$$

where k and l are finite. We can create a distribution out of A, that sends a test function $\epsilon(\sigma)$ defined on S^1 to a number, by multiplying A with ϵ and integrating over σ. We call such functional a "current", and denote it by $J_\epsilon(A) \equiv \int_{S^1} \epsilon(\sigma) A(X(\sigma), \ldots)\, \mathrm{d}\sigma$. We have not specified how ϵ, or A, changes under a coordinate change on S^1. We do not necessary require that $J_\epsilon(A)$ is a scalar, i.e., invariant under a reparametrization of σ. However, we require that the expression is invariant when we perform a change of coordinates on the target space M. To achieve this, the integrand A will necessarily need geometrical objects from M: tensors and connections. By calculating the Poisson bracket between currents, and extracting algebraic properties from the bracket, interesting operations between the geometrical objects on M can be derived. This is one of the aims of Paper I.

Paper I takes as its starting point the observation made in [3]: the Poisson brackets between currents parametrized by a vector and a one-form can be

written in terms of the *Dorfman bracket*, and the *natural pairing*, which are the basic objects in *generalized geometry*, see the second example in section 4.4.1. In [3], only the bosonic case, with currents of the form just mentioned, were considered. In Paper I, the analysis is extended to the $N = 1$ supersymmetric case. We then also have odd objects and can construct currents parametrized by antisymmetric tensors: forms and multivectors. In Paper I, we derive some algebraic properties that all these examples share. We call these properties *a weak Courant–Dorfman algebra*, with the definition given as follows.

Definition 4.1 (Weak Courant–Dorfman algebra). A *weak Courant–Dorfman algebra* $(\mathcal{E}, \mathcal{R}, \partial, \langle\ ,\ \rangle, *)$ is defined by the following data:

○ a vector space $\mathcal{R}$,
○ a vector space $\mathcal{E}$,
○ a symmetric bilinear form $\langle\ ,\ \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{R}$,
○ a map $\partial : \mathcal{R} \to \mathcal{E}$,
○ a *Dorfman bracket* $* : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$,

which satisfy the following axioms:

Axiom 1: $\quad A * (B * C) = (A * B) * C + B * (A * C)$

Axiom 2: $\quad A * B + B * A = \partial \langle A, B \rangle$

Axiom 3: $\quad (\partial f) * A = 0$

where $A, B, C \in \mathcal{E}$ and $f, g \in \mathcal{R}$.   ●

In a way, the sigma model can be used as a tool for generating such algebras. On the other hand, these algebras can help in the understanding of the sigma model, see for example the discussion about first class constraints in [3].

In Paper I, we derive these properties by using variational calculus, on the currents and the Poisson bracket. We also present a number of examples of weak Courant–Dorfman algebras derived this way. We effectively do a formal variational calculus, and we ignore any analytical problems that might be present. For example, T^*LM may not be simply connected, even if M is simply connected.

We are here going to rederive these algebraic properties but instead using the language of Poisson vertex algebras, which seems to be a more powerful and suitable language for these considerations.

4.1   Poisson vertex algebra

A Poisson vertex algebra is an efficient description of a system with a local Poisson bracket and with functionals of the type (4.2). We here closely follow [6].

Let us denote the coordinates on the phase space collectively by $u^\alpha(\sigma) = \{X^i(\sigma), P_{i-d}(\sigma)\}^\alpha$, where $\alpha = 1, \ldots, 2d$. Furthermore, let $u^{\alpha(n)} \equiv \partial^n u^\alpha$. We have $\partial u^{\alpha(n)} = u^{\alpha(n+1)}$. Expressions like (4.2) can now be written as polynomials in the variables $u^{\alpha(n)}$, with $\alpha = 1, \ldots, 2d$, and $n = 1, \ldots, N \ll \infty$,

$$a(u^\alpha, u^{\alpha(1)}, \ldots, u^{\alpha(N)}) \,. \tag{4.3}$$

We have a total derivative operator that acts on the polynomials like

$$\partial = u^{\alpha(1)} \frac{\partial}{\partial u^\alpha} + u^{\alpha(2)} \frac{\partial}{\partial u^{\alpha(1)}} + \ldots + u^{\alpha(N+1)} \frac{\partial}{\partial u^{\alpha(N)}} \,. \tag{4.4}$$

We can multiply two polynomials together and get a new polynomial. The algebra of polynomials like (4.3), together with the derivative (4.4), is called an *algebra of differential functions* $\mathcal{V}$. When integrating functions over S^1, total derivatives in ∂_σ are not contributing to the result. Likewise, we can consider the space $\mathcal{V}/\partial\mathcal{V}$, where expressions are considered the same if they differ by a total derivative. On this space, we can integrate by parts. We denote the image of a polynomial $a \in \mathcal{V}$ in $\mathcal{V}/\partial\mathcal{V}$ by $\int a$.

A general local Poisson bracket on the phase space can now be described by

$$\{u^\alpha(\sigma), u^\beta(\sigma')\} = H_0^{\alpha\beta}\delta + H_1^{\alpha\beta}\partial_{\sigma'}\delta + \ldots + H_N^{\alpha\beta}\partial_{\sigma'}^N\delta \,, \tag{4.5}$$

where $H_k^{\alpha\beta} \in \mathcal{V}$ and evaluated at σ', and δ are short for the δ-function $\delta(\sigma - \sigma')$. For polynomials $a, b \in \mathcal{V}$, we have the following bracket:

$$\{a(\sigma), b(\sigma')\} = \sum_{m,n} \frac{\partial a(\sigma)}{\partial u^{\alpha(m)}} \frac{\partial b(\sigma')}{\partial u^{\beta(n)}} \partial_\sigma^m \partial_{\sigma'}^n \{u^\alpha(\sigma), u^\beta(\sigma')\} \,. \tag{4.6}$$

The key idea is to do a Fourier transformation of this Poisson bracket. We define the Fourier transformed bracket by

$$\{a_\lambda b\} \equiv \int_{S^1} e^{\lambda(\sigma - \sigma')} \{a(\sigma), b(\sigma')\} \, d\sigma \,. \tag{4.7}$$

Instead of working with the bracket (4.6), we work with (4.7), called the λ-bracket. The fact that (4.6) is a Poisson bracket, see def. 2.3, translates into certain algebraic properties of the λ-bracket, and this bracket, together with $\mathcal{V}$ and ∂, is a *Lie conformal algebra* [15].

Definition 4.2 (Lie conformal algebra [39])**.** A *Lie conformal algebra* $\mathcal{W}$ is a $\mathbb{C}[\partial]$-module, i.e. ∂ can act on elements of $\mathcal{W}$, with complex coefficients. It has a bracket, called a λ-bracket: $\mathcal{W} \otimes \mathcal{W} \to \mathcal{W}[\lambda]$. This bracket must fulfill three axioms:

Axiom 1 (Sesquilinearity): $\quad \{\partial a_\lambda b\} = -\lambda\{a_\lambda b\},$
$$\{a_\lambda \partial b\} = (\partial + \lambda)\{a_\lambda b\} \,.$$

Axiom 2 (Skew symmetry): $\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}$.

Axiom 3 (Jacobi identity): $\{a_\lambda \{b_\mu c\}\} = \{\{a_\lambda b\}_{\mu+\lambda} c\} + \{b_\mu \{a_\lambda c\}\}$.

●

We can always transform our expressions back to the original Poisson bracket (4.6), but many calculations are more efficiently performed in the λ-bracket setting.

The algebra of differential functions $\mathcal{V}$, together with ∂, the multiplication of polynomials and the λ-bracket (4.7) is furthermore a Poisson vertex algebra, defined as follows [15, 21].

Definition 4.3 (Poisson vertex algebra). Let $\mathcal{W}$ be an algebra, where the product $\cdot$ of the algebra is commutative, i.e., $a \cdot b = b \cdot a$, and associative, i.e., $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. Let there be a derivation ∂ that acts on elements of $\mathcal{V}$. That ∂ is a derivation means that $\partial(a \cdot b) = \partial(a) \cdot b + a \cdot \partial(b)$. $\mathcal{W}$ should have a λ-bracket, so that $\mathcal{W}$ is a Lie conformal algebra (def. 4.2). The λ-bracket should fulfill Leibniz rule, i.e., be a derivation with respect to the product $\cdot$,

$$\{a_\lambda b \cdot c\} = \{a_\lambda b\} \cdot c + b \cdot \{a_\lambda c\} .$$

Then $\mathcal{W}$ is a *Poisson vertex algebra*. ●

As we will see in section 5.3 (see definition 5.4), the axioms of a Poisson vertex algebra is equivalent to the axioms of a vertex algebra in a certain classical limit. This motivates the name. Compare with how Poisson algebras appear in classical limits of operator algebras in quantum mechanics.

We now want to use these properties to show that a Lie conformal algebra implies a weak Courant–Dorfman algebra.

But first, we want to have a short discussion about the behavior of the coordinates u^i under a change of coordinates on S^1.

4.2 Scaling and conformal weight

Under a change of coordinates on the worldsheet, the coordinates on the phase space typically transforms. Let us consider S^1, and diffeomorphisms of the type $\sigma \to \alpha\sigma$, where σ is scaled by a factor α. In the phase space under consideration, $T^*\mathcal{L}M$, the map X is assumed to be invariant, but P transforms as a one form: $P \to \alpha^{-1}P$. By a slight abuse of terminology, we say that X has conformal weight zero, and P has conformal weight one.

It is worth pointing out that an element of the differential functions $\mathcal{V}$ might not scale homogeneously. It some cases, however, it might be possible to find a basis of $\mathcal{V}$, so that each element has a definite scaling. We can then write

$\mathcal{V} = \bigoplus \mathcal{V}_n$, where elements from $\mathcal{V}_n$ has conformal weight Δ_n. We note that $\partial : \mathcal{V}_n \rightarrow \mathcal{V}_{n+1}$.

We want our formulas to be covariant under reparametrisation of S^1. In particular, from the definition of the λ-bracket (4.7), we see that if a has conformal weight Δ_a and b has conformal weigh Δ_b, then the right hand side should scale as $\Delta_a + \Delta_b$.

Because of the integration, the integrand scale as $\Delta_a + \Delta_b - 1$. Since each power of λ is accompanied by the corresponding power of $(\sigma - \sigma')$, the scaling of the term with λ^k will be $\Delta_a + \Delta_b - 1 - k$. If all elements in $\mathcal{V}$ has positive, or zero, conformal weight, then the highest possible power of λ will be $\Delta_a + \Delta_b - 1$. In many calculations, the conformal weight is a good bookkeeping device.

4.3 Weak Courant–Dorfman algebra from Lie conformal algebra

We now want to derive the properties of a weak Courant–Dorfman algebra from a Lie conformal algebra $\mathcal{W}$. In the following, $a, b, c \in \mathcal{W}$.

The λ-bracket of a Lie conformal algebra is a polynomial in λ, $\{a_\lambda b\} = \sum_{j=0} c_j \lambda^j$. Let us define two binary operations, $*$ and $\langle\,_\lambda\,\rangle$, by writing the λ-bracket as

$$\{a_\lambda b\} \equiv a * b + \lambda \langle a_\lambda b \rangle . \tag{4.8}$$

Sesquilinearity gives that $\{\partial a_\lambda b\} = -\lambda\, a * b - \lambda^2 \langle a_\lambda b \rangle$. On the other hand, from the definition, $\{\partial a_\lambda b\} = (\partial a) * b + \lambda \langle \partial a_\lambda b \rangle$. Setting $\lambda = 0$, we see that

$$\boxed{(\partial a) * b = 0} . \tag{4.9}$$

From the Jacobi identity, with $\lambda = \mu = 0$, we find that $*$ fulfills a Leibniz rule,

$$\boxed{a * (b * c) = (a * b) * c + b * (a * c)} . \tag{4.10}$$

An algebra with this property, i.e., left multiplication acts as a derivation, is called a *Leibniz algebra*.

From skew symmetry, we get $a * b + \lambda \langle a_\lambda b \rangle = -b * a + (\lambda + \partial) \langle b_{-\lambda-\partial} a \rangle$. Set $\lambda = 0$ in this equation, and we get

$$a * b + b * a = \partial \langle b_{-\partial} a \rangle . \tag{4.11}$$

Let us define a symmetric binary operation $\langle\,,\,\rangle$ by

$$\langle a, b \rangle \equiv \frac{1}{2} \left(\langle a_{-\partial} b \rangle + \langle b_{-\partial} a \rangle \right) . \tag{4.12}$$

From (4.11) we see that

$$\boxed{a*b+b*a=\partial\langle a,b\rangle}\,. \tag{4.13}$$

The boxed formulas are the axioms of a weak Courant–Dorfman algebra. In addition, we can prove one more identity.

Using sesquilinearity, we see that $\{\partial a_\lambda \partial b\} = -\lambda\{a_\lambda \partial b\}$. From the definition, using (4.3), we also have $\{\partial a_\lambda \partial b\} = \lambda\langle \partial a_\lambda \partial b\rangle$, and thus

$$\langle \partial a_\lambda \partial b\rangle = -\{a_\lambda \partial b\}\,. \tag{4.14}$$

Using skew symmetry and sesquilinearity, we have

$$\{a_\lambda \partial b\} = -\{\partial b_{-\lambda-\partial}\, a\} = (\lambda+\partial)\{b_{-\lambda-\partial}\, a\}\,. \tag{4.15}$$

Setting $\lambda = -\partial$, we get $\{a_{-\partial}\,\partial b\} = 0$, which gives

$$\boxed{\langle \partial a,\partial b\rangle = 0}\,. \tag{4.16}$$

In the definition of a weak Courant–Dorfman algebra (def. 4.1), we have two different vector spaces, $\mathcal{R}$ and $\mathcal{E}$. In the above discussion, all elements are part of the Lie conformal algebra $\mathcal{W}$, and, in general, we have to set $\mathcal{R} = \mathcal{E} = \mathcal{W}$. In some settings, we can use conformal weights to distinguish between $\mathcal{R}$ and $\mathcal{E}$. The most evident example is to consider when $\mathcal{W}$ only have elements with non-negative conformal weight, and to consider elements of conformal weight one and zero. This is investigated in the next section.

4.4 Courant–Dorfman algebra

We let $\mathcal{W}$ be a Poisson vertex algebra that can be written as $\mathcal{W} = \bigoplus_{n\geq 0} \mathcal{W}_n$, where, as discussed in section 4.2, elements from $\mathcal{W}_n$ has conformal weight Δ_n. Note that we assume that all elements has non-negative conformal weight.

If we restrict the Poisson vertex algebra, and only keep objects with conformal weight zero and one, we can get more properties out of the axioms of a Poisson vertex algebra than we did from the axioms of a Lie conformal algebra in the last section. We will show that a Poisson vertex algebra restricted in this way is equivalent to what is known as a Courant–Dorfman algebra.

Definition 4.4 (Courant–Dorfman algebra)**.** A *Courant–Dorfman algebra*, as defined in [49], is given by the data $(\mathcal{E},\mathcal{R},\partial,\langle\ ,\ \rangle,*)$. Here, $\mathcal{R}$ is a commutative algebra, and $\mathcal{E}$ is an $\mathcal{R}$-module, i.e., elements from $\mathcal{R}$ acts on elements from $\mathcal{E}$. We also have a symmetric bilinear form,

$$\langle\,,\rangle : \mathcal{E}\otimes\mathcal{E} \to \mathcal{R}\,,$$

a derivation,

$$\partial : \mathcal{R} \to \mathcal{E} \,,$$

and a *Dorfman bracket*,

$$* : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E} \,.$$

These operations satisfy the following axioms:

Axiom 1: $A * (fB) = f(A * B) + \langle A, \partial f \rangle B$

Axiom 2: $\langle A, \partial \langle B, C \rangle \rangle = \langle A * B, C \rangle + \langle B, A * C \rangle$

Axiom 3: $A * B + B * A = \partial \langle A, B \rangle$

Axiom 4: $A * (B * C) = (A * B) * C + B * (A * C)$

Axiom 5: $(\partial f) * A = 0$

Axiom 6: $\langle \partial f, \partial g \rangle = 0$

Here, $A, B, C \in \mathcal{E}$ and $f, g \in \mathcal{R}$. •

From the Dorfman bracket, we can construct an anti-symmetric bracket, the *Courant bracket*, by

$$[A, B]_{\mathrm{C}} \equiv \frac{1}{2} (A * B - B * A) \,. \tag{4.17}$$

Note that, by axiom 3, the two brackets are related by

$$[A, B]_{\mathrm{C}} = A * B - \frac{1}{2} \partial \langle A, B \rangle \,. \tag{4.18}$$

4.4.1 Examples of Courant–Dorfman algebras

As a first example of a Courant–Dorfman algebra, we can consider the case when ∂ is the zero-map: $f \mapsto 0$. Axioms 5 and 6 are then trivially satisfied. Axioms 1, 3 and 4 then says that $*$ is a Lie bracket, so $\mathcal{E}$ is a Lie algebra over $\mathcal{R}$. Axiom 2 states that this Lie algebra has a form that is invariant under the adjoint action of this Lie algebra. So, one example of a Courant–Dorfman algebra is when $\mathcal{R} = \mathbb{R}$, and $\mathcal{E}$ is a Lie algebra $\mathfrak{g}$. The Dorfman bracket in this example is given by the Lie bracket, $A * B = \{A, B\}_{\mathrm{Lie}}$, the derivation is zero, $\partial = 0$, and the form $\langle \, , \, \rangle$ is the Killing form of $\mathfrak{g}$.

Another example of a Courant–Dorfman algebra is when $\mathcal{E} = TM \oplus T^*M$, the sum of the tangent bundle and the cotangent bundle of a manifold M. This is the basic object of interest in generalized complex geometry, see [30]. The commutative algebra $\mathcal{R}$ is given by $C^\infty(M)$ and the derivation is $\partial = 0 \oplus \mathrm{d}$,

where d is the de Rham–operator. The bracket in this example is the canonical Dorfman bracket, introduced by Dorfman in [17], and given by

$$(v_1 \oplus \beta_1) * (v_2 \oplus \beta_2) = \{v_1, v_2\}_{\text{Lie}} \oplus \mathcal{L}_{v_1}\beta_2 - \iota_{v_2} \, d\beta_1 \, , \qquad (4.19)$$

where $\mathcal{L}$ is the Lie derivative, and ι means contraction. The anti-symmetric version, the Courant bracket, is

$$[v_1 \oplus \beta_1, v_2 \oplus \beta_2]_C = \{v_1, v_2\}_{\text{Lie}} \oplus \mathcal{L}_{v_1}\beta_2 - \mathcal{L}_{v_2}\beta_1 - \frac{1}{2} \, d(\iota_{v_1}\beta_2 - \iota_{v_2}\beta_1) \, , \quad (4.20)$$

and this corresponds to the original Courant bracket introduced by Courant in [13]. The bilinear form is given by *the natural pairing*,

$$\langle v_1 \oplus \beta_1 \, , \, v_2 \oplus \beta_2 \rangle = \iota_{v_1}\beta_2 + \iota_{v_2}\beta_1 \, . \qquad (4.21)$$

4.4.2 Courant–Dorfman algebras and Poisson vertex algebras

We now want to investigate relations between Courant–Dorfman algebras and Poisson vertex algebras. We state the following theorem, that seems to be known, but not explicitly stated in this form anywhere in the literature. For different versions of the statement, see [3, 34, 12].

Theorem 4.1. *The Poisson vertex algebras, $(\mathcal{W}, \{ _\lambda \}, \cdot)$, that are graded by conformal weight, and generated by elements of conformal weight zero and one, are in a one-to-one correspondence with the Courant–Dorfman algebras, defined in def. 4.4.*

Proof. The proof is simple and straight forward. We are going to be explicit, and spell out the different steps. We are interested in the subspace of $\mathcal{W}$ that has objects with conformal weight one and zero: $\mathcal{W}_0 \oplus \mathcal{W}_1$. We let $\mathcal{R} = \mathcal{W}_0$, the subspace of $\mathcal{W}$ with vectors with conformal weight zero. We denote these $f, g, \ldots \in \mathcal{R}$. These form a commutative algebra, we can multiply them and still get an object in $\mathcal{R}$: $f \cdot g \in \mathcal{R}$. We denote the conformal weight one objects by capital letters, $A, B, \ldots$, and denote this space by $\mathcal{E} = \mathcal{W}_1$. Since we can multiply objects from $\mathcal{R}$ with objects from $\mathcal{E}$, and get objects in $\mathcal{E}$, $\mathcal{E}$ is an $\mathcal{R}$-module: $f \cdot A \in \mathcal{E}$.

We now want to construct λ-brackets for our objects and see what the axioms of a Poisson vertex algebra imply for the possible brackets. We already know that we have a weak Courant–Dorfman algebra, but since these properties easily unfold, we include them here as well.

Since we only have objects with non-negative conformal weight, the possible form of a bracket between objects in $\mathcal{E}$ is

$$\{A_\lambda B\} = A * B + \lambda \langle A, B \rangle \, , \qquad (4.22)$$

where

$$\ast : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E} , \qquad\qquad \langle \, , \, \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{R} . \qquad (4.23)$$

The ansatz (4.22) is just saying that $1 + 1 = 1 + 1 = 2 + 0$, counting the conformal weights. Since $1 + 0 = 1 + 0$, we let

$$\{A_\lambda f\} = A \star f, \qquad\qquad \star : \mathcal{E} \otimes \mathcal{R} \to \mathcal{R} \qquad (4.24)$$

The bracket between objects of conformal weight zero must vanish, since there is "not enough scaling" to allow the δ-function:

$$\{f_\lambda g\} = 0 . \qquad (4.25)$$

We also have the map $\partial : \mathcal{R} \to \mathcal{E}$. We now are going to use the different properties of a Poisson vertex algebra to show that these imply the properties of a Courant–Dorfman algebra.

SKEW-SYMMETRY   The skew-symmetry of the λ-bracket says that $\{A_\lambda B\} = -\{B_{-\lambda-\partial} A\}$, which gives that

$$A \ast B + \lambda \langle A, B \rangle = -B \ast A + \lambda \langle A, B \rangle + \partial \langle B, A \rangle . \qquad (4.26)$$

The terms with different coefficients of λ do not talk to each other, and we have

$$\boxed{\langle A, B \rangle = \langle B, A \rangle} , \qquad (4.27)$$

$$\boxed{A \ast B + B \ast A = \partial \langle A, B \rangle} . \qquad (4.28)$$

Also,

$$\{f_\lambda A\} = -\{A_{-\lambda-\partial} f\} = -A \star f . \qquad (4.29)$$

SESQUILINIARITY   We have $\{\partial f_\lambda A\} = \partial \ast A + \lambda \langle \partial f, A \rangle$ from the ansatz (4.22). At the same time, we have

$$\{\partial f_\lambda A\} = -\lambda \{f_\lambda A\} = \lambda A \star f , \qquad (4.30)$$

where we used sesquiliniarity and (4.29). So, the product $\star$ can be expressed in terms of $\langle , \rangle$, by the relation

$$A \star f = \langle A, \partial f \rangle . \qquad (4.31)$$

Also,

$$\boxed{\partial f \ast A = 0} . \qquad (4.32)$$

Using sesquiliniarity and (4.25), we have $\{\partial f_\lambda \partial g\} = -\lambda(\partial + \lambda)\{f_\lambda g\} = 0$. From the ansatz, we have $\{\partial f_\lambda \partial g\} = \partial f \ast \partial g + \lambda \langle \partial f, \partial g \rangle$, and comparing the two expressions, we get $\partial f \ast \partial g = 0$, which already follows from (4.32), but also

$$\boxed{\langle \partial f, \partial g \rangle = 0} . \qquad (4.33)$$

JACOBI IDENTITY   The Jacobi identity for objects of conformal weight one is

$$\{A_\lambda\{B_\mu C\}\} = \{\{A_\lambda B\}_{\mu+\lambda} C\} + \{B_\mu\{A_\lambda C\}\}. \qquad (4.34)$$

Using the ansatz and expanding it, we get, using (4.29) and (4.31),

$$A*(B*C) + \lambda\langle A, B*C\rangle + \mu\langle A, \partial\langle B, C\rangle\rangle =$$
$$(A*B)*C + (\lambda+\mu)\langle A*B, C\rangle - \lambda\langle C, \partial\langle A, B\rangle\rangle$$
$$+ B*(A*C) + \mu\langle B, A*C\rangle + \lambda\langle B, \partial\langle A, C\rangle\rangle \quad (4.35)$$

Reading off the coefficients of μ, we see that

$$\boxed{\langle A, \partial\langle B, C\rangle\rangle = \langle A*B, C\rangle + \langle B, A*C\rangle}, \qquad (4.36)$$

and setting $\lambda = \mu = 0$, we get

$$\boxed{A*(B*C) = (A*B)*C + B*(A*C)} \qquad (4.37)$$

LEIBNIZ   The Leibniz rule gives that

$$\{A_\lambda fB\} = \{A_\lambda f\}B + f\{A_\lambda B\} = \langle A, \partial f\rangle B + fA*B + \lambda f\langle A, B\rangle. \quad (4.38)$$

Also, from the ansatz, we have $\{A_\lambda fB\} = A*(fB) + \lambda\langle A, fB\rangle$. Equating these two expressions and setting $\lambda = 0$, we see that

$$\boxed{A*(fB) = f(A*B) + \langle A, \partial f\rangle B} \qquad (4.39)$$

The framed equations are the axioms of a Courant–Dorfman algebra, and a Poisson vertex algebra restricted to conformal weight one and zero thus imply the existence of a Courant–Dorfman algebra.

Now we are going to prove that the opposite also is true, that the axioms of a Courant–Dorfman algebra implies the Poisson vertex algebra axioms.

4.4.3   The Courant–Dorfman-algebra gives a Poisson vertex algebra

Now, assume we have a Courant–Dorfman algebra $(\mathcal{E}, \mathcal{R}, \partial, \langle\,,\,\rangle, *)$ as given in definition 4.4. Let us define a bracket by

$$\{A_\lambda B\} \equiv A*B + \lambda\langle A, B\rangle, \qquad (4.40)$$
$$\{A_\lambda f\} \equiv \langle A, \partial f\rangle, \qquad (4.41)$$
$$\{f_\lambda A\} \equiv -\langle A, \partial f\rangle, \qquad (4.42)$$
$$\{f_\lambda g\} \equiv 0, \qquad (4.43)$$

where $A, B, \ldots \in \mathcal{E}$ and $f, g, \ldots \in \mathcal{R}$. We first want to show that this bracket is a λ-bracket that gives a Lie conformal algebra, as defined in def. 4.2.

The sesquilinearity for $\mathcal{R} \otimes \mathcal{R}$, $\mathcal{E} \otimes \mathcal{R}$, and $\mathcal{R} \otimes \mathcal{E}$ are fulfilled by the definition of the bracket. For $\mathcal{E} \otimes \mathcal{E}$ we have, first,

$$\{\partial f \,_\lambda A\} = \underset{=0 \text{ by ax. 5}}{\partial f * A} + \lambda \langle \partial f, A \rangle \underset{(4.42)}{=} -\lambda \{f \,_\lambda A\}\,. \tag{4.44}$$

Secondly,

$$\begin{aligned} \{A \,_\lambda \partial f\} &= A * \partial f + \lambda \langle A, \partial f \rangle \\ &= (\partial + \lambda)\{A \,_\lambda f\} \end{aligned} \tag{4.45}$$

where axiom 3, and (4.41) where used. The sesquilinearity property is thus fulfilled.

As with sesquilinearity, the property skew-symmetry is fulfilled trivially, from the definition of the bracket, for $\mathcal{R} \otimes \mathcal{R}$, $\mathcal{E} \otimes \mathcal{R}$, and $\mathcal{R} \otimes \mathcal{E}$. In addition, we have

$$\begin{aligned} \{A \,_\lambda B\} &= A * B + \lambda \langle A, B \rangle = -B * A + \partial \langle A, B \rangle + \lambda \langle A, B \rangle \\ &= -(B * A + (-\lambda - \partial)\langle A, B \rangle) = -\{B \,_{-\lambda-\partial} A\}\,, \end{aligned} \tag{4.46}$$

where axiom 3 was used.

For $A, B, C \in \mathcal{E}$, we have

$$\begin{aligned} \{\{A \,_\lambda B\}_{\mu+\lambda} C\} + \{B_\mu \{A \,_\lambda C\}\} - \{A \,_\lambda \{B_\mu C\}\} = \\ (A * B) * C + B * (A * C) - A * (B * C) \\ + \mu \big(\langle A * B, C \rangle + \langle B, A * C \rangle - \langle A, \partial \langle B, C \rangle \rangle \big) \\ + \lambda \big(\langle A * B, C \rangle - \langle C, \partial \langle A, B \rangle \rangle + \langle B, \partial \langle A, C \rangle \rangle - \langle A, B * C \rangle \big) \end{aligned} \tag{4.47}$$

The constant term, without λ and μ, is zero due to axiom 4. The coefficient of μ is zero due to axiom 2. The two first terms in the λ-term are equal to $-\langle C, B * A \rangle$, using axiom 2. Combine this with the last term, and we get $-\langle B, \partial \langle A, C \rangle \rangle$, using axiom 2 once more. The Jacobi identity is thus satisfied for objects from $\mathcal{E}$.

For two objects from $\mathcal{E}$ and one from $\mathcal{R}$, we have, e.g.,

$$\begin{aligned} \{\{A \,_\lambda B\}_{\mu+\lambda} f\} + \{B_\mu \{A \,_\lambda f\}\} - \{A \,_\lambda \{B_\mu f\}\} = \\ \langle A * B, \partial f \rangle + \langle B, \partial \langle A, \partial f \rangle \rangle - \langle A, \partial \langle B, \partial f \rangle \rangle\,. \end{aligned} \tag{4.48}$$

Using axiom 2 to rewrite the first term of the right-hand side, we get

$$\langle B, \partial \langle A, \partial f \rangle - A * \partial f \rangle \tag{4.49}$$

which, by axiom 3 and 5, is zero. The other ways of inserting A, B, and f in (4.49) also gives zero in a similar way. It is actually enough to prove it for one triple of elements, the other permutations of the elements are then automatically satisfied [14, Remark 2.4]. For two objects from $\mathcal{R}$, the Jacobi is also satisfied. All terms will contain one bracket between two objects in $\mathcal{R}$, and such brackets are by construction zero. For three objects from $\mathcal{R}$, the Jacobi is trivially satisfied.

We thus have a Lie conformal algebra. What remains is to show the Leibniz rule of a Poisson vertex algebra.

LEIBNIZ RULE For three objects from $\mathcal{R}$, the Leibniz identity is trivially satisfied.

For two objects from $\mathcal{R}$, we have either

$$\{f_\lambda gA\} - \{f_\lambda g\}A - g\{f_\lambda A\} = -\langle gA, \partial f \rangle + g\langle A, \partial f \rangle = 0 , \quad (4.50)$$

where the last identity is due to the bilinearity of $\langle \ , \ \rangle$, or

$$\{A_\lambda fg\} - \{A_\lambda f\}b - f\{A_\lambda g\} = \\ \langle A, \partial(fg) \rangle - \langle A, \partial f \rangle g - f\langle A, \partial g \rangle = 0 , \quad (4.51)$$

where we used that ∂ is a derivative, and $\mathcal{R}$ is commutative.

Finally, for two objects from $\mathcal{E}$, we have

$$\{A_\lambda fB\} - \{A_\lambda f\}B - f\{A_\lambda B\} = \\ A * (fB) + \lambda\langle A, fB \rangle - \langle A, \partial f \rangle B - fA * B - \lambda f\langle A, B \rangle = 0 , \quad (4.52)$$

where we used the bilinearity of $\langle \ , \ \rangle$, and axiom 1.

Thus, the Leibniz rule is satisfied.

$\square$

5. Vertex algebras

The mathematical notion of vertex algebras was developed in 1986 by Richard Bocherds in the article [10]. The original motivation was to prove the Moonshine conjecture, which relates the coefficients in the expansion of a certain modular function to the dimensions of representations of the Monster group. The Monster group is the largest of the so-called sporadic groups, see, e.g., [23]. The vertex algebra structure, however, found applications way beyond this original application of the ideas. It provides a rigorous mathematical formulation of the chiral part of two-dimensional conformal field theory. The axioms of a vertex algebra are obtained from an abstract treatment of the properties of quantum field theories, and of operator product expansions in two dimensions.

In this chapter, we will give an introduction to formal distributions, and then go through the definition of a vertex algebra. We will define two operations, the λ-bracket and the normal ordered product, and derive certain identities they fulfill. We will then give some examples of vertex algebras, and conclude with a manifest supersymmetric version of vertex algebras.

5.1 Formal distributions

We will here give a brief introduction to formal distributions. For a more complete exposé, see, e.g. [39]. Formal distributions is an algebraic way of giving meaning to "functions" and distributions, like the Dirac δ-function. In the algebraic approach, many difficulties are disregarded, compared to the analytic approach. These two approaches are related but not the same. In a way, the algebraic treatment is a simplified setting, e.g., convergence issues are disregarded, but it is a walkable path that still capture enough to describe interesting structures.

Let us start by considering infinite sums of the form

$$u(z) = \sum_{n=-\infty}^{\infty} u_n z^n \, .$$

(5.1)

Here, the elements u_n are vectors of a given vector space W, and we interpret z as a "formal parameter". We do not require that the sum necessarily should converge. We can think of $u(z)$ as a generating function for some (perhaps infinite) sequence $\{u_n\}_{n=-\infty}^{\infty}$. Sums of the type (5.1) can be added together, and the scalar multiplication of W gives a scalar multiplication of the sum. Sums of this type thus forms a vector space, and this space is denoted $W[[z^{\pm}]]$.

We can also consider more than one formal parameter, say z and w. The corresponding vector space $W[[z^{\pm}, w^{\pm}]]$ has elements of the form $\sum_{n,m} u_{n,m} z^n w^m$.

We are also interested in the subspace of $W[[z^{\pm}]]$ where the elements consist of a finite number of vectors u_n. These sums are of the form $\sum_{-N}^{M} u_n z^n$, for some finite N and M, and the subspace will be denoted by $W[z^{\pm}]$. The elements of this vector space are called *Laurant polynomials*.

To give an example, consider the simplest possible case, when $W = \mathbb{R}$. The vectors u_n will then be real numbers. As an example of elements from $\mathbb{R}[[z^{\pm}]]$, consider

$$a(z) = \sum_{n=-\infty}^{\infty} z^n, \quad b(z) = z + z^{-1}, \quad \text{and} \quad c(z) = z^2 + 2 + z^{-2}. \quad (5.2)$$

The two elements $b(z)$ and $c(z)$ are Laurant polynomials. These can obviously be multiplied together, as

$$b(z)b(z) = (z + z^{-1})(z + z^{-1}) = z^2 + 2 + z^{-2} = c(z). \quad (5.3)$$

The formal sum $a(z)$ can also be multiplied with $b(z)$ and $c(z)$. We have

$$a(z)b(z) = \sum_{n=-\infty}^{\infty} z^n(z + z^{-1}) = \sum_{n=-\infty}^{\infty} z^{n+1} + \sum_{n=-\infty}^{\infty} z^{n-1} = 2\,a(z), \quad (5.4)$$

and, in the same way, $a(z)c(z) = 4\,a(z)$. However, if we try to multiply $a(z)$ with itself, it is hard to make sense of the resulting double sum $\sum \sum_{n,m=-\infty}^{\infty} z^{n+m}$, as the coefficients of the z^n's will be infinity. In particular, it is not an element in $\mathbb{R}[[z^{\pm}]]$ and hence multiplication of elements from $\mathbb{R}[[z^{\pm}]]$ may not be closed. Looking closer at the example (5.4), we realize that $a(z)$ multiplied with any Laurant polynomial $d(z) \in \mathbb{R}[z^{\pm}]$, gives $a(z)d(z) = d(1)a(z)$. So, it is natural to identify $a(z)$ as the *formal δ-function* centered at 1, $a(z) = \delta(1 - z)$.

The elements of $W[[z^{\pm}]]$ are called *formal distributions*. We define a derivative $\partial_z u(z)$, by differentiating each term in the sum, i.e.,

$$\partial_z u(z) = \partial_z \left(\sum_{n=-\infty}^{\infty} u_n z^n \right) = \sum_{n=-\infty}^{\infty} u_{n+1}(n+1)z^n \quad (5.5)$$

We define the *formal residue* Res_z by

$$\text{Res}_z u(z) = \text{Res}_z \left(\sum_{n=-\infty}^{\infty} u_n z^n \right) \equiv u_{-1}. \quad (5.6)$$

The formal residue resembles the analytic counterpart. Let us interpret z as a coordinate on a Riemann surface, e.g., $\mathbb{C}$, and integrate a meromorphic function $u(z)$ around a closed loop around the origin, in other words, take the residue of the function. We would then, up to a factor of $2\pi i$, pick out the coefficient of the term $1/z$, and this coefficient is u_{-1}.

With the definitions (5.5) and (5.6), we have $\operatorname{Res}_z\left(\partial_z u(z)\right) = 0$. If the product of $f(z)$ and $g(z)$ is defined, then "partial integration" holds under Res:

$$\operatorname{Res}_z\left(\,\partial_z\left(f(z)\right)\,g(z)\,\right) = -\operatorname{Res}_z\left(\,f(z)\,\partial_z\left(g(z)\right)\,\right)\,. \tag{5.7}$$

The formal distributions are evaluated on the Laurant polynomials, which plays the role of test functions, using the formal residue. Given a distribution $f(z) \in W[[z^{\pm}]]$, and a Laurant polynomial $p(z) \in W[z^{\pm}]$, the value of $f(z)$, evaluated on $p(z)$, is given by $\operatorname{Res}_z\left(f(z)p(z)\right)$.

The distribution $a(z) \in \mathbb{R}[[z^{\pm}]]$ given in (5.2) were interpreted as a δ-function centered at 1. We define the *formal δ-function* centered at w by

$$\delta(z-w) = \frac{1}{z}\sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n\,. \tag{5.8}$$

To see that this definition gives the desired properties, consider the Laurant polynomial $f(z) = az^m$. The distribution $\delta(z-w)$ evaluated on $f(z)$ gives

$$\operatorname{Res}_z\delta(z-w)f(z) = a\operatorname{Res}_z\sum_n w^n z^{-n+m-1} = aw^m = f(w)\,. \tag{5.9}$$

By linearity, we have $\operatorname{Res}_z\delta(z-w)f(z) = f(w)$ for general $f(z) \in W[z^{\pm}]$.

We now want to show an alternative way of writing the δ-function, that often is more transparent. The series $1 + w/z + (w/z)^2 + \ldots$ converges for $|w| < |z|$. Multiplying the series with $1 - w/z$, and we get 1. Thus,

$$\frac{1}{z-w} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n,\ \text{for } |w| < |z|\,. \tag{5.10}$$

We let $i_w f$ mean that we expand a function f in positive powers of w. From (5.10) we have

$$i_w\frac{1}{z-w} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n\,, \tag{5.11}$$

and correspondingly

$$i_z\frac{1}{z-w} = -\frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n = -\frac{1}{z}\sum_{n=-\infty}^{-1}\left(\frac{w}{z}\right)^n\,, \tag{5.12}$$

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so we can write the δ-function (5.8) as

$$\delta(z - w) = (i_w - i_z) \frac{1}{z - w} \ . \tag{5.13}$$

From this way of writing the δ-function, we can derive some important properties. We can differentiate both sides of (5.13), and get the formula

$$\partial_w^j \delta(z - w) = (i_w - i_z) \frac{j!}{(z - w)^{j+1}} \ . \tag{5.14}$$

Observe that, for a non-negative N, we have $(i_w - i_z)(z - w)^N = 0$. Multiplying (5.14) with $(z - w)^N$, and we get the, for us, important formula

$$(z - w)^N \partial_w^j \delta(z - w) = 0 \ , \text{ for } N \geq j + 1 \ . \tag{5.15}$$

We are now ready to formulate the following proposition.

Proposition 5.1. Let $f \in V[[z^\pm, w^\pm]]$. Then, for some integer $N \geq 1$, we have that

$$(z - w)^N f(z, w) = 0 \tag{5.16}$$

if, and only if, f can be written as

$$f(z, w) = \sum_{j=0}^{N-1} c_j(w) \partial_w^j \delta(z - w) \tag{5.17}$$

where $c_j(w) \in V[[w^\pm]]$.

Proof. From (5.15) we see that a distribution of the form (5.17) fulfills (5.16). Now assume that (5.16) holds. Let $g(z, w) = (z - w)^{N-1} f(z, w)$, and expand g as $g(z, w) = \sum_{n,m} g_{n,m} z^n w^m$. That $(z - w)g(z, w) = 0$ implies that $g_{n-1,m} = g_{n,m-1}$, for all n and m. Using this relation, we can set the first index to zero, and we have $g_{n,m} = g_{0,n+m}, \forall n, m$, and consequently

$$g(z, w) = \sum_{n,m} g_{0,n+m} z^n w^m = \sum_{n,m} g_{0,n+m} w^{n+m+1} \frac{1}{w} \left(\frac{z}{w} \right)^n$$

$$= c(w)\delta(z - w) \ . \tag{5.18}$$

Replacing g with its definition, we thus have

$$(z - w)^{N-1} f(z, w) = c(w)\delta(z - w) \ . \tag{5.19}$$

Dividing with $(z - w)^{N-1}$, and using (5.14), we see that

$$f(z, w) = \frac{c(w)}{(N-1)!} \partial^{N-1} \delta(z - w) + f_1(z, w) \ , \tag{5.20}$$

where $f_1(z, w)$ is a function fulfilling $(z - w)^{N-1} f_1(z, w) = 0$. Continuing the recursion will eventually lead to the relation $f_N(z, w) = 0$. The function $f(z, w)$ can thus be written in the form (5.17). $\square$

If elements of W can act on each other, then a pair of distributions, $a(z)$ and $b(w)$, are said to be *mutually local* if their commutator $[a(z), b(w)] \in W[[z^\pm, w^\pm]]$ fulfills (5.16). Due to (5.17), we can then express the commutator as a sum of distributions in one of the formal parameters, and δ-functions with a finite number of derivatives acting on them. If we would interpret w and z as points on a Riemann surface, we would see that the commutator only have local support, and hence the name.

5.2 Definitions of a vertex algebra

We start by defining what we will mean by a field.

Definition 5.1 (Field). Let $\text{End}(V)$ be the space of *endomorphisms* of the vector space V, i.e., the space of maps from V to itself. A *field* $A(z)$ is defined as an $\text{End}(V)$-valued formal distribution in a parameter z: $A(z) \in \text{End}(V)[[z^\pm]]$. We expand $A(z)$ as:

$$A(z) = \sum_{j \in \mathbb{Z}} \frac{1}{z^{j+1}} A_{(j)}, \quad \text{where } A_{(j)} \in \text{End}(V) , \quad (5.21)$$

and for all $B \in V$, $A(z)B$ contains only finitely many negative powers of z. The field $Y(A, z)$ will also be denoted by $A(z)$. The endomorphisms $A_{(j)}$ of $Y(A, z)$ are called the *Fourier modes* of the field. •

We are now ready to give the definition of a vertex algebra.

Definition 5.2 (Vertex algebra). A *vertex algebra* is the data $(V, |0\rangle, Y, \partial)$, where V is a vector space, called the *space of states*, with a vector $|0\rangle \in V$, called the *vacuum*. The map Y is called the *state-field correspondence*, and it is a map from a given state $A \in V$ to a field $Y(A, z) \in \text{End}(V)[[z^\pm]]$. The map $\partial : V \to V$ is an endomorphism of V, and it is called the *translation operator*.

These structures are subject to the following axioms:

Axiom 1: The vacuum is invariant under translations:

$$\partial |0\rangle = 0$$

Axiom 2: The state-field correspondence creates a given state A from the vacuum in the limit $z \to 0$:

$$Y(A, z)|0\rangle|_{z=0} = A_{(-1)}|0\rangle = A .$$

Also, the field corresponding to the vacuum state is the identity field, i.e.,

$$Y(|0\rangle, z) = I_V ,$$

where I_V is the identity endomorphism for V.

Axiom 3: Acting with ∂ on the endomorphisms of a field, should be the same as differentiation of the field with respect to the formal parameter z:

$$[\partial, Y(A,z)] = Y(\partial A, z) = \partial_z Y(A,z) \ .$$

Axiom 4: All fields in the vertex algebra are mutually local, i.e., the commutator between the two fields has a finite highest pole:

$$(z-w)^N [Y(A,z), Y(B,w)] = 0 \ ,$$

for some $N \gg 0$.

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As we will see, there are many different, but equivalent, definitions of a vertex algebra. The definition 5.2 is a formalization of an axiomatic description of two-dimensional quantum field theory (the *Wightman axioms*, see [39]), and many of the physical properties of an vertex algebra is apparent (the vacuum is invariant, locality, etc.). We will later give an equivalent definition, def. 5.4, that is more suited for computations, but where the physical content might be more hidden.

It is straight-forward to generalize the definition of a vertex algebra to the case when V is a super vector space. The vectors of V then have a $\mathbb{Z}_2$-grading, and V can be written as a sum of the vector space with even vectors, V_{even}, and the odd vectors, V_{odd}. The grading is called *parity*. We let $(-1)^A = +1$ if $A \in V_{\text{even}}$, and $(-1)^A = -1$ if $A \in V_{\text{odd}}$. The definition of a vertex algebra then has the following extra requirements:
- The endomorphism ∂ is even, i.e., $\partial : V_{\text{even/odd}} \to V_{\text{even/odd}}$.
- The state-field correspondence is parity preserving.
- The commutator in axiom 4 should be interpreted as a super commutator: $[Y(A,z), Y(B,w)]_S = Y(A,z)Y(B,w) - (-1)^{AB} Y(B,w)Y(A,z)$.

In the following calculations, we treat V as a purely even vector space. The generalization to a super vector space introduces appropriate factors of (-1) in the formulas. To keep the calculations a bit cleaner, we avoid these factors. They will be reintroduced in section (5.2.4).

In chapter 6, we are going to construct automorphisms between vertex algebras. For later convenience, we here give the definition of a homomorphism, i.e., a map that preserves the algebraic structures, between vertex algebras.

Definition 5.3 (Vertex algebra homomorphism). A homomorphism from a vertex algebra V to a vertex algebra W, is a linear map $\varphi : V \to W$, that preserves parity, and is such that

$$\varphi(Y_V(A,z)B) = Y_W(\varphi(A),z)\varphi(B) \ , \quad \forall A, B \in V,$$

and $\varphi(|0\rangle_V) = |0\rangle_W$.

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We now want to define two operations, the λ-bracket and the normal ordered product, that makes computations in a vertex algebra easier and more transparent. We also want to derive some properties and relations between these operations. These properties will all be listed in definition 5.4 on page 46.

5.2.1 The λ-bracket

From the Fourier modes $A_{(j)}$ of the field $Y(A,z)$ corresponding to the state A, we can define an operation $V \otimes V \to V[\lambda]$, which we call the λ-*bracket* [14]:

$$[A_\lambda B] = \sum_{j \geq 0} \frac{\lambda^j}{j!} (A_{(j)}B) , \quad A, B \in V . \tag{5.22}$$

Since $Y(A,z)$ is a field, the series (5.22) terminates, i.e., it only contains finite powers of λ for all A and B. The λ-bracket can be viewed as a formal Fourier transformation of $Y(A,z)B$:

$$[A_\lambda B] = \mathrm{Res}_z \, e^{\lambda z} \, Y(A,z)B . \tag{5.23}$$

We now want to show that this bracket gives the vertex algebra the structure of a Lie conformal algebra, as defined in definition 4.2. We mostly follow the treatment in [5], but also [39] and [15]. From the way (5.23) of writing the bracket, we see, using axiom 3 and partial integration (5.7), that

$$[\partial A_\lambda B] = \mathrm{Res}_z \, e^{\lambda z} \, Y(\partial A,z)B = \mathrm{Res}_z \, e^{\lambda z} \, \partial_z Y(A,z)B$$
$$= -\mathrm{Res}_z \, \partial_z(e^{\lambda z}) \, Y(A,z)B = -\lambda \, [A_\lambda B] . \tag{5.24}$$

We also have

$$\partial[A_\lambda B] = \mathrm{Res}_z \, e^{\lambda z} \, \partial \, (Y(A,z)B)$$
$$= \mathrm{Res}_z \, e^{\lambda z} \, ([\partial, Y(A,z)]B + Y(A,z)\partial B)$$
$$= \mathrm{Res}_z \, e^{\lambda z} \, (Y(\partial A,z)B + Y(A,z)\partial B)$$
$$= [\partial A_\lambda B] + [A_\lambda \partial B] . \tag{5.25}$$

This shows that the bracket fulfills the sesquilinearity axiom of a Lie conformal algebra.

Let us borrow the following identity from [39, Prop. 4.2], that follows from the locality and translational covariance axioms:

$$Y(B,z)A = e^{z\partial}Y(A,-z)B . \tag{5.26}$$

From this relation, we see that

$$[B_\lambda A] = \mathrm{Res}_z \, e^{\lambda z} \, Y(B,z)A$$
$$= \mathrm{Res}_z \, e^{\lambda z}e^{z\partial} \, Y(A,-z)B \tag{5.27}$$
$$= -\mathrm{Res}_{-z} \, e^{-z(-\lambda-\partial)} \, Y(A,-z)B = -[A_{-\lambda-\partial}B]$$

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The skewsymmetry axiom is thus fulfilled.

We now want to show the remaining axiom, the Jacobi identity. We define the *n-th product* by

$$Y(A,w)_{(n)}Y(B,w) \equiv \text{Res}_z \big(i_w(z-w)^n \, Y(A,z)Y(B,w) \\ - i_z(z-w)^n \, Y(B,w)Y(A,z) \big) \quad (5.28)$$

Since the fields in the vertex algebra are mutual local, which is the meaning of axiom 4, we know from proposition 5.1, that

$$[Y(A,z),Y(B,w)] = \sum_j C_j(w)\partial_w^j \delta(z-w) . \quad (5.29)$$

The field $C_j(w)$ is given by

$$C_j(w) = \frac{1}{j!}\text{Res}_z(z-w)^j [Y(A,z),Y(B,w)] = \frac{1}{j!}Y(A,w)_{(n)}Y(B,w) \quad (5.30)$$

We need the following identity that is true for a vertex algebra, called the *n-th product identity* [39, Prop. 4.4]:

$$Y(A,z)_{(n)}Y(B,z) = Y(A_{(n)}B,z) . \quad (5.31)$$

Expanded out in modes, this identity is the so called *Borcherds identity*, which is part of the axioms of a vertex algebra in Borcherds original formulation [10]. In [39], it is shown that the Borcherds definition of a vertex algebra and the definition given in def. 5.2 are equivalent.

So, we have, for two fields $A(z)$ and $B(w)$,

$$[A(z),B(w)] = \sum_j \frac{1}{j!}Y(A_{(n)}B,w)\partial_w^j \delta(z-w) \quad (5.32)$$

This equation is the *operator product expansion* (OPE) of the commutator between two fields. Often one assumes $|z| > |w|$, and only writes the singular part of the equation as

$$A(z)B(w) \sim \sum_j \big(A_{(n)}B \big)(w)\frac{1}{(z-w)^{j+1}} , \quad (5.33)$$

where (5.14) is used. Note that the λ-bracket (5.22) gives the same information as this more traditional way of writing an OPE.[†]

Now, apply (5.32) to a state C, multiply with $e^{\lambda z}$ and take the residue Res_z, and we get

$$[A_\lambda B(w)C] - B(w)[A_\lambda C] = Y([A_\lambda B],w)e^{\lambda w}C , \quad (5.34)$$

[†] In the λ-bracket notation, there is no need to specify the parameters z and w though.

where we used $\partial_w^j \delta(z-w) = (-1)^j \partial_z^j \delta(z-w)$, and partial integration. Multiplying this with $e^{\mu w}$, where μ is a formal parameter on the same footing as λ, and then taking the residue Res_w finally gives

$$[A_\lambda[B_\mu C]] - [B_\mu[A_\lambda C]] = \mathrm{Res}_w\, e^{(\lambda+\mu)w} Y([A_\lambda B], w)C$$
$$= [[A_\lambda B]_{\lambda+\mu} C]\,, \tag{5.35}$$

and the Jacobi identity is fulfilled.

A vertex algebra, as given in definition 5.2 thus imply the existence of a Lie conformal algebra, as defined in definition 4.2, with the bracket defined by (5.22).

5.2.2 The normal ordered product

Next, we introduce the *normal ordered product*. One way to define a normal ordered product of operators in quantum field theory is by point splitting. The singular terms captured by the OPE between the operators, are subtracted from the product:

$$(AB)(z) = \lim_{w \to z} (A(w)B(z) - \text{singular terms})\,, \tag{5.36}$$

and the result is a well-defined operator at z. In vertex algebras, we can define a product, $\cdot : V \otimes V \to V$, in a similar way; by acting with the non-singular part of a field $A(z)$, in the limit $z \to 0$:

$$A \cdot B \equiv \mathrm{Res}_z z^{-1} Y(A, z) B = A_{(-1)} B\,. \tag{5.37}$$

From the n-th product identity (5.31), with $n = -1$, we see that the field corresponding to the vector $A \cdot B$ is given by

$$Y(A_{(-1)}B, w) = \mathrm{Res}_z \left(\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n A(z)B(w) \right.$$
$$\left. - \frac{1}{z} \sum_{n=-\infty}^{-1} \left(\frac{w}{z}\right)^n B(w)A(z) \right) \tag{5.38}$$
$$= A(w)_+ B(w) + B(w)A(w)_-\,,$$

where the field $A(w)$ is split in two parts, $A(w) = A(w)_+ + A(w)_-$. The parts are given by $A(w)_+ = \sum_{j \leq -1} \frac{1}{z^{j+1}} A_{(j)}$, and $A(w)_- = \sum_{j \geq 0} \frac{1}{z^{j+1}} A_{(j)}$. From the vacuum axioms, $A(w)_-$ are required to annihilate the vacuum. At the level of the corresponding fields, the definition (5.37) thus resembles the familiar procedure of normal ordering by moving the annihilation operators to the left, and the creation operators to the right.

We want to investigate this product and its relation to the λ-bracket. We start with proving the following proposition.

Proposition 5.2. The state $|0\rangle$ is an identity element of the algebra $(V, \cdot)$.

Proof. First, by axiom 2, we have $|0\rangle \cdot A = \mathrm{Res}_z z^{-1} A = A$. To show that $A \cdot |0\rangle = A$, we first need to investigate $Y(A,z)|0\rangle$. Axiom 2 shows that the non-negative modes of $Y(A,z)$ annihilates the vacuum, so $Y(A,z)|0\rangle = (A_{-1} + zA_{-2} + z^2 A_{-3} + \ldots)|0\rangle$. Acting with ∂ on both sides, using axiom 3, and letting $z \to 0$, we see that

$$A_{-2}|0\rangle = \partial A . \tag{5.39}$$

Repeating this argument, we see that

$$Y(A,z)|0\rangle = (A + z\partial A + \frac{z^2}{2}\partial^2 A + \ldots) = e^{z\partial} A . \tag{5.40}$$

Using (5.40), we finally see that $A \cdot |0\rangle = \mathrm{Res}_z z^{-1} e^{z\partial} A = A$. $\qquad\square$

Equation (5.25) showed that ∂ is a derivation with respect to the λ-bracket. We can use the same argument to show that it is a derivation with respect to the normal ordered product:

$$\partial(A \cdot B) = \partial A \cdot B + A \cdot \partial B . \tag{5.41}$$

5.2.3   Relations between the λ-bracket and the normal ordered product

We now want to derive the important *non-commutative Wick formula*. It allows us to calculate λ-brackets of composite fields, i.e., of fields that are constructed out of other fields, using the normal ordered product. We start with equation (5.34). Multiply both sides with w^{-1} and take the residue Res_w, and we get

$$\mathrm{Res}_w w^{-1} e^{\lambda w} Y([A_\lambda B], w)C = [A_\lambda B \cdot C] - B \cdot [A_\lambda C] \tag{5.42}$$

To rewrite the left hand side in a more transparent way, we first note that $w^{-1} e^{\lambda w} = \int_0^\lambda e^{\mu w} d\mu + w^{-1}$. Using this, we have

$$[A_\lambda B \cdot C] = [A_\lambda B] \cdot C + B \cdot [A_\lambda C] + \int_0^\lambda [[A_\lambda B]_\mu C] d\mu , \tag{5.43}$$

which is the sought non-commutative Wick formula.

Taking equation (5.26), multiplying with z^{-1} and taking the residue with respect to z, we get

$$B \cdot A = \mathrm{Res}_z z^{-1} e^{z\partial} Y(A, -z)B = \mathrm{Res}_z z^{-1} e^{-z\partial} Y(A, z)B . \tag{5.44}$$

Using the equation $z^{-1} e^{-z\partial} = z^{-1} - \int_{-\partial}^0 e^{\lambda z} d\lambda$, we can rewrite the right hand side, and we have the following equation, called *quasi-commutativity*,

$$A \cdot B = B \cdot A + \int_{-\partial}^0 [A_\lambda B] d\lambda . \tag{5.45}$$

The normal ordered product is thus not commutative.

We now want to show one last property, the *quasi-associativity*. Taking $n = -1$ in the n-th product identity (5.31) and taking the normal ordered product with a state C, we get

$$(A \cdot B) \cdot C = \text{Res}_{w,z} \big(w^{-1} i_w (z-w)^n \, Y(A,z) Y(B,w) C$$
$$- w^{-1} i_z (z-w)^n \, Y(B,w) Y(A,z) C \big) . \quad (5.46)$$

Using the expansion defined in (5.11), the first term of the right hand side is

$$\text{Res}_{z,w} \sum_{j=0} \sum_{m,n \in \mathbb{Z}} w^{j-m-2} z^{-j-n-2} A_{(n)} B_{(m)} C =$$
$$\sum_{j=0} A_{(-j-1)} B_{(j-1)} C = A \cdot (B \cdot C) + \sum_{k=0} A_{(-k-2)} B_{(k)} C \quad (5.47)$$

To rewrite the last term in (5.47), we note that, using the translational covariance axiom, we have

$$(\partial^n A) \cdot B = \text{Res}_z \sum_{j=0} z^{-1} \partial_z^n \frac{1}{z^{j+1}} A_{(j)} B = n! \, A_{(-1-n)} B . \quad (5.48)$$

Using this, (5.47) can be written

$$A \cdot (B \cdot C) + \left(\int_0^{\partial} d\lambda \, A \right) \cdot [B_\lambda C] . \quad (5.49)$$

The second term of the right hand side of (5.46) is, using the expansion (5.12),

$$\text{Res}_{z,w} \sum_{j=0} \sum_{m,n \in \mathbb{Z}} w^{-3-j-m} z^{-n-1+j} B_{(m)} A_{(n)} C =$$
$$\sum_{j=0} A_{(-j-2)} B_{(j)} C = \left(\int_0^{\partial} d\lambda \, B \right) \cdot [A_\lambda C] \quad (5.50)$$

In total, we have

$$(A \cdot B) \cdot C = A \cdot (B \cdot C) + \left(\int_0^{\partial} d\lambda \, A \right) \cdot [B_\lambda C] + \left(\int_0^{\partial} d\lambda \, B \right) \cdot [A_\lambda C] , \quad (5.51)$$

which are the quasi-associativity formula.

We see that in a vertex algebra, the normal ordered product is neither commutative, nor associative. We will, when necessary, use parenthesis to indicate in which order the normal ordered product acts.

5.2.4 Alternative definition in terms of bracket and a product

We have shown that the axioms of a vertex algebra imply the existence of a λ-bracket, that endowes V the structure of a Lie conformal algebra. In addition, we can define a normal ordered product. The endomorphism ∂ is a derivation with respect to both products. The products are related by the non-commuting Wick formula (5.43), the quasi-commutativity (5.45), and the quasi-associativity (5.51). It is important to stress that the normal ordered product $\cdot$ is neither commutative, nor associative.

In [5] it is shown that the opposite is also true, that these formulas imply the axioms of a vertex algebra. We therefore can formulate an alternative, but equivalent, definition of a vertex algebra.

Definition 5.4 (Vertex (super) algebra, alternative definition)**.** A vertex algebra is a super vector space V, together with an even vector $|0\rangle \in V$, an even endomorphism ∂, an even product $\cdot$, $V \otimes V \to V$, and an even binary operation $V \otimes V \to V[\lambda]$ denoted $[\,._{\lambda}.\,]$. We have that ∂ is a derivation and $|0\rangle$ is a unit, with respect to $\cdot$. In addition, these operations satisfy

Sesquilinearity:

$$[\partial A_{\lambda}B] = -\lambda[A_{\lambda}B]\,, \qquad [A_{\lambda}\partial B] = (\partial + \lambda)[A_{\lambda}B]\,. \qquad (5.52)$$

Skew-symmetry:

$$[A_{\lambda}B] = -(-1)^{AB}[B_{-\lambda-\partial}A]\,. \qquad (5.53)$$

Jacobi identity:

$$[A_{\lambda}[B_{\mu}C]] = [[A_{\lambda}B]_{\mu+\lambda}C] + (-1)^{AB}[B_{\mu}[A_{\lambda}C]]\,. \qquad (5.54)$$

Quasi-commutativity:

$$A\cdot B - (-1)^{AB}B\cdot A = \int_{-\partial}^{0}[A_{\lambda}B]\,\mathrm{d}\lambda\,. \qquad (5.55)$$

Quasi-associativity:

$$(A\cdot B)\cdot C - A\cdot(B\cdot C) =$$
$$\left(\int_{0}^{\partial}\mathrm{d}\lambda\,A\right)\cdot[B_{\lambda}C] + (-1)^{AB}\left(\int_{0}^{\partial}\mathrm{d}\lambda\,B\right)\cdot[A_{\lambda}C]\,. \qquad (5.56)$$

Non-commutative Wick formula: (also called quasi-Leibniz)

$$[A_{\lambda}B\cdot C] = [A_{\lambda}B]\cdot C + (-1)^{AB}B\cdot[A_{\lambda}C] + \int_{0}^{\lambda}[[A_{\lambda}B]_{\mu}C]\,\mathrm{d}\mu\,. \qquad (5.57)$$

●

Note that a vertex algebra is a Lie conformal algebra (def. 4.2).

In Paper IV, a Mathematica package is presented, where the rules in definition 5.4 are implemented, together with the rules for the $N = 1$ supersymmetric vertex algebra, to be described in section 5.6. This allows one to do vertex algebra calculations using a computer. Usually, the parts of these calculations are easy, but as a whole, the calculations are error prone and time consuming. It is therefore often of great value to be able to use a computer. In Paper III, the use of this package was an integral part in many of the calculations.

There are more, equivalent, definitions of vertex algebras. In [15], five of them are listened, including the two we have presented, and they are all shown to be equivalent.

The definition given here, (definition 5.4), is the most useful for doing concrete calculations. Also, it is more transparent when constructing vertex algebras. Start by postulating a "times table" for the products between the elements of V. Then impose the relations above by modding out the relations they generate. If something is left, we have a vertex algebra. However, the relations are not that easy to fulfill, and there are not that many examples of vertex algebras. We will soon consider some of the simplest ones, but first we discuss relations between vertex algebras and Poisson vertex algebras.

5.3   Quantum corrections and the semi-classical limit

Recall the definition of a Poisson vertex algebra (see page 26). If we drop the integral terms in the definition of a vertex algebra (in the relations (5.55), (5.56) and (5.57)), we would have a Poisson vertex algebra. The integral terms can therefore be thought of as "quantum corrections" to a Poisson vertex algebra, they measure the failure of the normal ordered product to be commutative and associative, and of the λ-bracket to fulfill Leibniz rule.

Let us introduce an additional parameter into the game: $\hbar$. In some cases, e.g., in the following example, this will allow us to keep track of the quantum corrections.

Consider the case when a vertex algebra V, as a module for $\mathbb{C}[\partial]$, is generated by a set of fields $\{A^i\}$. The fields have the λ-brackets $[A^i {}_\lambda A^j] = F^{ij}$, where F^{ij} is a polynomial in λ, with coefficients in the vertex algebra V. Each coefficient is built up, using the normal ordered product, by the fields A^i and their derivatives. We can then consider the vertex algebra $V_\hbar$, which is generated by

$$[A^i {}_\lambda A^j] = \hbar F^{ij} . \tag{5.58}$$

We can think of $V_\hbar$ as a family of vertex algebras, parametrized by $\hbar$. Elements in the family of vertex algebras will be polynomials in $\hbar$, and we write $a \in V[\hbar]$. The power of $\hbar$ in an expression is a measure for how many times the basic

λ-bracket (5.58) has been applied. The quantum corrections, i.e., the integral terms, all contains an extra application of the λ-bracket, and will therefore be of higher order in $\hbar$. If we define a rescaled bracket by

$$\{A^i\,_\lambda A^j\} \equiv \lim_{\hbar \to 0} \frac{1}{\hbar}[A^i\,_\lambda A^j]\,, \tag{5.59}$$

then this bracket, on the elements in $V_0 = V[\hbar]_{\hbar=0}$ with the normal ordered product $\cdot$ projected on V_0, is a Poisson vertex algebra. The product $\cdot$ is associative and commutative, and the bracket $\{\,_\lambda\,\}$ fulfills Leibniz rule. This limit is called *the semi-classical limit*. We will refer to the parts of vectors in $V[\hbar]$ that contains factors of $\hbar$ as *quantum corrections*, as these terms are not present in V_0. Also, the result of a λ-bracket calculation, using (5.58), will necessarily be at least linear in $\hbar$. All terms in the result that are of higher order in $\hbar$ will also be denoted as quantum corrections, as they will disappear if the calculation is performed with the bracket $\{\,_\lambda\,\}$.

5.4   Examples of vertex algebras

We will here consider some basic examples of vertex algebras. These examples will be the building blocks of our later considerations.

5.4.1   *The $\beta\gamma$-system and the bc-system*

Let H be the even vector space with a basis $\{a_i, b_i\}, i \in \mathbb{Z}$ and c. The space is equipped with a Lie bracket, given by $[a_i, b_j] = \delta_{i,-j}\, c$. The other Lie brackets are zero. This Lie algebra is called the *Heisenberg algebra*. Let this algebra act on the vector space V_H, which has one even vector $|0\rangle$ in it. Let $b_n |0\rangle = 0$ if $n \geq 1$, and $a_n |0\rangle = 0$ if $n \geq 0$, and $c |0\rangle = |0\rangle$. We can then define two fields, $\beta(z)$ and $\gamma(z)$, by

$$\beta(z) = \sum_{n\in\mathbb{Z}} \frac{1}{z^{n+1}} a_n\,, \qquad \gamma(z) = \sum_{n\in\mathbb{Z}} \frac{1}{z^{n+1}} b_{n+1}\,. \tag{5.60}$$

These corresponds to the states

$$\beta = a_{-1} |0\rangle\,, \qquad\qquad \gamma = b_0 |0\rangle\,. \tag{5.61}$$

The λ-bracket between these states are given by

$$[\beta\,_\lambda\gamma] = \mathrm{Res}_z\, e^{\lambda z} \sum_{n\in\mathbb{Z}} \frac{1}{z^{n+1}} a_n b_0 |0\rangle \tag{5.62}$$

In order to have a z^{-1}-term, the sum over $n \in \mathbb{Z}$ is reduced to $n \geq 0$. We have $a_n b_0 |0\rangle = ([a_n, b_0] + b_0 a_n) |0\rangle = \delta_{n,0} |0\rangle$, and

$$[\beta_\lambda \gamma] = |0\rangle \ . \tag{5.63}$$

In the same way, $[\beta_\lambda \beta] = [\gamma_\lambda \gamma] = 0$. The corresponding OPE's, as written in (5.33) and discussed in section 3.3, are

$$\beta(z)\gamma(w) \sim \frac{1}{(z-w)} \ , \qquad \beta(z)\beta(w) \sim 0 \ , \qquad \gamma(z)\gamma(w) \sim 0 \ . \tag{5.64}$$

Instead of the even vector space V_H, we can consider the space with odd elements $\{\hat{a}_i, \hat{b}_i\}$ and even c as its basis, fulfilling the same algebra as before, but now with the super commutator. This algebra is a *Clifford algebra*. We can construct the corresponding fields $b(z)$ and $c(z)$, and they will have the λ-brackets

$$[b_\lambda c] = |0\rangle \ , \qquad [b_\lambda b] = 0 \ , \qquad [c_\lambda c] = 0 \ , \tag{5.65}$$

and, in the same way as before, the OPE's are

$$b(z)c(w) \sim \frac{1}{(z-w)} \ , \qquad b(z)b(w) \sim 0 \ , \qquad c(z)c(w) \sim 0 \ . \tag{5.66}$$

5.4.2 The Virasoro algebra

The Virasoro algebra is a Lie algebra spanned by the elements $L_n, n \in \mathbb{Z}$ and C. The Lie algebra is given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12} c \delta_{n+m,0} \, C \ . \tag{5.67}$$

The element C is central, and c is a number, called *the central charge*. Let the vector space V, which has one even vector $|0\rangle$ in it, be a module for this algebra. Define the field $L(z)$ by

$$L(z) = \sum_{n \in \mathbb{Z}} \frac{1}{z^{n+2}} L_n \ , \tag{5.68}$$

i.e., with $L_{(n)} = L_{n-1}$. For the vacuum axioms to be fulfilled, we need that $L_n |0\rangle = 0$, for $n \geq -1$. Also, let $C |0\rangle = |0\rangle$, as in the previous examples. We have $L = L_{-2} |0\rangle$. The λ-bracket is given by

$$[L_\lambda L] = \text{Res}_z \sum_{m \geq 0} \sum_{n \in \mathbb{Z}} \frac{1}{m!} \lambda^m z^{m-n-2} L_n L_{-2} |0\rangle$$

$$= \sum_{m \geq 0} \frac{1}{m!} \lambda^m L_{m-1} L_{-2} |0\rangle = \sum_{m=0,1,2,3} \frac{1}{m!} \lambda^m [L_{m-1}, L_{-2}] |0\rangle \tag{5.69}$$

$$= (L_{-3} + \lambda 2 L_{-2} + \lambda^3 \frac{c}{12} C) |0\rangle$$

Using that $L_{-3}|0\rangle = \partial L$, which follow from translational covariance, we have

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3 \,, \tag{5.70}$$

where the vacuum $|0\rangle$ is implicit in the last term. We will often omit it. This bracket is consistent with all the axioms of the vertex algebra.

Note that the skew-symmetry axiom puts constraints on the possible λ-brackets between the same elements, as $[L_\lambda L] = -[L_{-\lambda-\partial}L]$.

The operator product expansion corresponding to (5.70) is

$$L(z)L(w) \sim \frac{1}{(z-w)}\partial L(w) + \frac{2}{(z-w)^2}L(w) + \frac{c}{2(z-w)^4} \,, \tag{5.71}$$

where c is the central charge.

The vertex algebras that have a field that fulfills the Virasoro algebra are of central interest to us and they deserve a definition.

Definition 5.5 (Conformal vertex algebra). A vertex algebra is called *conformal* [39], with central charge c, if it has a vector L, that has the λ-bracket (5.70), and the field $Y(L,z)$ is such that when expanded as in (5.68), the Fourier mode L_{-1} is the translational operator ∂, and L_0 is diagonalizable on V, with a smallest eigenvalue. The eigenvalue for a state ϕ is called the *conformal weight* of ϕ, and it is denoted as Δ_ϕ. •

In a conformal vertex algebra, due to the above requirements on the way the modes L_{-1} and L_0 acts, there is a basis $\{\phi_k\}$ of V, so that the algebra of L and ϕ_k is

$$[L_\lambda \phi_k] = (\partial + \Delta_{\phi_k})\phi_k + \mathcal{O}(\lambda^2) \,. \tag{5.72}$$

The fields where the terms $\mathcal{O}(\lambda^2)$ are absent are called *primary fields*.

5.5 Example calculations

In this section, we do a simple calculation using the λ-bracket and the normal ordered product. Hopefully, this elucidate the rules in definition 5.4. We are going to use a tensor product of d vertex algebras of the type defined in section 5.4.1.

5.5.1 Virasoro vector in $\beta\gamma$-system

We have

$$[\beta_i {}_\lambda \gamma^j] = \delta_i^j \,, \quad i,j = 1,\dots,d \,. \tag{5.73}$$

Let us show that the vector $L_b \equiv \partial\gamma^i \cdot \beta_i$ is a conformal vector, i.e., that it gives the λ-bracket (5.70). First, we note that skew-symmetry gives that $[\gamma^j {}_\lambda \beta_i] =$

$-\delta_i^j$. Then, we use the non-commutative Wick formula (5.57) to calculate $[\gamma^i{}_\lambda L_{\mathrm{b}}]$.

$$[\gamma^i{}_\lambda L_{\mathrm{b}}] = [\gamma^i{}_\lambda \partial\gamma^j \cdot \beta_j] = \partial\gamma^j \cdot (-\delta^i_j) = -\partial\gamma^i . \qquad (5.74)$$

We then use skew-symmetry (5.53) to calculate $[L_{\mathrm{b}}{}_\lambda \gamma^i]$:

$$[L_{\mathrm{b}}{}_\lambda \gamma^i] = -[\gamma^i{}_{-\lambda-\partial} L_{\mathrm{b}}] = \partial\gamma^i . \qquad (5.75)$$

This shows that, given that L_{b} fulfills the algebra (5.70), and indeed is a conformal vector, γ^i is a primary conformal field of conformal weight 0 with respect to L_{b}. From (5.75) we see, using sesquilinearity, that $[L_{\mathrm{b}}{}_\lambda \partial\gamma^i] = (\partial+\lambda)\partial\gamma^i$. The field $\partial\gamma^i$ is possibly also a primary field, with weight 1.

Now, in a similar way, we have $[\beta_i{}_\lambda \partial\gamma^j] = (\partial+\lambda)\delta_i^j = \lambda\,\delta_i^j$, and

$$[\beta_i{}_\lambda L_{\mathrm{b}}] = [\beta_i{}_\lambda \partial\gamma^j \cdot \beta_j] = \lambda\beta_i , \qquad (5.76)$$

and consequently $[L_{\mathrm{b}}{}_\lambda \beta_i] = (\lambda+\partial)\beta_i$, and also β_i is potentially a primary field of conformal weight 1 with respect to L_{b}. Putting it together, we have

$$[L_{\mathrm{b}}{}_\lambda L_{\mathrm{b}}] = [L_{\mathrm{b}}{}_\lambda \partial\gamma^i \cdot \beta_i] = \big((\lambda+\partial)\partial\gamma^i\big)\cdot\beta_i + \partial\gamma^i\cdot\big((\lambda+\partial)\beta_i\big)$$
$$+ \int_0^\lambda [(\lambda+\partial)\partial\gamma^i{}_\mu\beta_i]\,\mathrm{d}\mu \qquad (5.77)$$

Using that ∂ is a derivation of the normal ordered product, the two first terms can be written $(\partial+2\lambda)\partial\gamma^i\cdot\beta_i = (\partial+2\lambda)L_{\mathrm{b}}$. Using sesquilinearity, and recalling that d is the number of $\beta\gamma$-field we have, so $\operatorname{tr}\delta_j^i = d$, the integral term is calculated as

$$\int_0^\lambda [(\lambda+\partial)\partial\gamma^i{}_\mu\beta_i]\,\mathrm{d}\mu = \int_0^\lambda (-\lambda\mu+\mu^2)[\gamma^i{}_\mu\beta_i]\,\mathrm{d}\mu$$
$$= (-\delta_i^i)\int_0^\lambda (-\lambda\mu+\mu^2)\,\mathrm{d}\mu = \frac{2d}{12}\lambda^3 . \qquad (5.78)$$

So, L_{b} generates the algebra (5.70), with central charge $2d$, and the vertex algebra generated by the $\beta\gamma$-system thus have the structure of a conformal vertex algebra.

5.5.2 Virasoro vector in the bc-system and supersymmetry

For the fermionic bc-system, we have

$$[b_i{}_\lambda c^j] = \delta_i^j , \quad i,j = 1,\ldots,d . \qquad (5.79)$$

Let $L_{\mathrm{f}} \equiv \frac{1}{2}\big((\partial b_i)\cdot c^i + (\partial c^i)\cdot b_i\big)$. In a calculation similar to the one in the last subsection, we can show that L_{f} is a conformal vector, with central charge

$c = d$. The main difference compared to the bosonic calculation is that special care must be taken to signs, since we are dealing with fermions.

We can take the tensor product of the $\beta\gamma$-system, and the bc-system. The vectors L_b and L_f commutes, and we then have a conformal vector $L_{\text{tot}} \equiv L_b + L_f$, with a central charge of $c = 3d$. This system has an odd symmetry between the bosons and the fermions: a supersymmetry. Let us define the odd vector G, given by $G \equiv c^i \cdot \beta_i + (\partial\gamma^i) \cdot b_i$. This vector, and L_{tot}, which we now call just L, has the λ-brackets

$$[G_\lambda G] = 2L + \frac{d}{3}\lambda^2 \,, \qquad\qquad [L_\lambda G] = (\partial + \frac{3}{2}\lambda)G \,. \qquad (5.80)$$

We see that G "squares" to L. We have a supersymmetry, together with a conformal symmetry, and the algebra (5.80), together with (5.70), is the $N = 1$ superconformal algebra.

The supersymmetry can be written and interpreted in a (super-) geometric way. Introducing odd formal variables, and considering fields that are covariant with respect to an odd translational operator, we are led to the definition of supersymmetric (SUSY) vertex algebras, which is the subject of the next section.

5.6 SUSY vertex algebras

We want to give a short introduction to supersymmetric vertex algebras, as developed in [31]. The idea is to consider odd formal parameters $\theta^1, \theta^2, \ldots, \theta^N$, in addition to the even formal parameter z. We can then consider superfields, which are required to be translational covariant with respect to odd translational operators. The supersymmetry is then said to be manifest.

We can think of the formal parameters as coordinates on $\mathbb{C}^{1|N}$. There are different ways of constructing the supermanifolds $\mathbb{C}^{1|N}$, compare with the construction of the supercircle $S^{1|1}$ in section 3.4. The meaning of covariance under odd translations depends on which supermanifold, and which translational operator, one considers. Different choices give rise to different SUSY vertex algebras. The possible ways to construct a Lie conformal (super) algebra (def. 4.2 generalized to graded elements) are classified in [18]. Of these, there are two series, K_N and W_N, that are of special interest to us. In both these cases, in addition to the even formal parameter z, one introduce a set of N odd (Grassman) formal parameters $\theta^1, \theta,^2, \ldots, \theta^N$. In W_N, the translational operators acts as ∂_{θ^i}. This operator squares to zero. In K_N, on the other hand, we have $D_i = \partial_{\theta^i} + \theta^i\partial_z$, an operator that squares to the even translational operator ∂_z. From a K_2-algebra, one can construct a W_1-algebra, by taking linear combinations of the translational operators.

Our main interest is the SUSY vertex algebra corresponding to K_1, which we denote as $N_K = 1$ SUSY vertex algebra. This is the framework suitable for our treatment of the chiral de Rham complex, as will be explained in section 6.3. Therefore, in this section, we will consider when we have one supersymmetry manifest, in the K_1 scenario. We will comment on more general settings later on, in section 6.4, when we consider K_2 supersymmetry.

5.6.1 $N_K = 1$ SUSY vertex algebra

Let us consider a vertex algebra V with one odd endomorphism D. We require that D applied twice to a state is the same as applying the translational operator ∂. Let θ be an odd formal parameter, that squares to zero: $\theta^2 = 0$, and commutes with z. Using the state-field correspondence $Y(A,z)$ of V, we can then construct a superfield for each state $A \in V$, by combining $Y(A,z)$ with $Y(DA,z)$. The latter field has opposite parity compared to the first. Using θ, we get a field that is of homogenous parity:

$$Y(A,z,\theta) = Y(A,z) + \theta\, Y(DA,z) \,. \tag{5.81}$$

We require that $[D, Y(A,z)] = Y(DA,z)$. We then have

$$
\begin{aligned}
[D, Y(A,z,\theta)] &= [D, Y(A,z)] - \theta[D, Y(DA,z)] \\
&= Y(DA,z) - \theta Y(\partial A,z) = (\partial_\theta - \theta\partial_z)Y(A,z,\theta) \,.
\end{aligned} \tag{5.82}
$$

This will be the requirement that $Y(A,z,\theta)$ is (super) translational covariant. Also note that from (5.81) we have

$$Y(DA,z,\theta) = Y(DA,z) + \theta\, Y(\partial A,z) = (\partial_\theta + \theta\partial_z)Y(A,z,\theta) \,. \tag{5.83}$$

We will denote the derivative by $D_\theta \equiv \partial_\theta + \theta\partial_z$. The subscript may be omitted, and D is used instead. Hopefully, it is clear from context which operator is meant.

Let capital letter Z, W denote pairs of even and odd formal parameters. E.g., $Z = (z, \theta)$. We then write $Y(A,Z)$, which means $Y(A,z,\theta)$.

The definition 5.1 of a field is generalized to the superfield case.

Definition 5.6 (Superfield). A *superfield* $A(Z)$ is defined as an End(V)-valued formal distribution in an even parameter z and an odd parameter θ, $A(Z) \in$ End(V)$[[z^\pm]][\theta]$. We expand $A(Z)$ as:

$$A(Z) = \sum_{j\in\mathbb{Z}} \frac{1}{z^{j+1}} \left(A_{(j|1)} + \theta\, A_{(j|0)} \right), \quad \text{where } A_{(j|*)} \in \text{End}(V) \,, \tag{5.84}$$

and for all $B \in V$, $A(Z)B$ contains only finitely many negative powers of z. •

We can now formulate the definition of an $N_K = 1$ SUSY vertex algebra, in the same spirit as definition 5.2.

Definition 5.7 ($N_K = 1$ SUSY vertex algebra)**.** An $N_K = 1$ *SUSY vertex algebra* is the data $(V, |0\rangle, Y, D)$, where V is a super vector space, called the *space of states*, with an even vector $|0\rangle \in V$, called the *vacuum*. The map Y is called the *state-superfield correspondence*, and it is a parity preserving map from a given state $A \in V$ to a superfield $Y(A, Z) \in \text{End}(V)[[z^{\pm}]][\theta]$. The map $D : V \to V$ is an odd endomorphism of V, and it is called the *odd translation operator*. This data obeys the following axioms:

Axiom 1:   The vacuum is invariant under odd translations: $D|0\rangle = 0$.

Axiom 2:   The state-superfield correspondence and the vacuum is related by:

$$Y(A, Z)|0\rangle|_{z=0, \theta=0} = A_{(-1|1)}|0\rangle = A , \qquad Y(|0\rangle, Z) = I_V .$$

where I_V is the identity endomorphism for V.

Axiom 3:   Translational covariance: $[D, Y(A, Z)] = (\partial_\theta - \theta \partial_z) Y(A, Z)$.

Axiom 4:   All superfields in the SUSY vertex algebra are mutual local, i.e., for some $N \gg 0$, we have $(z - w)^N [Y(A, Z), Y(B, W)] = 0$.

$$\bullet$$

We denote the square of D as ∂, $[D, D] = 2\partial$. Axiom 3 implies $[\partial, Y(A, Z)] = \partial_z Y(A, Z)$, as for an ordinary vertex algebra. We will often drop the prefix *super-*, and, e.g., call superfields just fields. Hopefully, the meaning will be clear from the context.

5.6.2 Λ-*bracket and the normal ordered product*

Analogue to the λ-bracket (5.22), we can describe how the Fourier modes of a superfield $A(Z)$ act on another superfield by introducing two formal parameters, one even, λ, and one odd, χ. We denote these collectively by $\Lambda = (\lambda, \chi)$. We then define the so-called Λ-bracket:

$$[A_{\Lambda} B] = \sum_{j \geq 0} \frac{\lambda^j}{j!} \left(A_{(j|0)} B + \chi A_{(j|1)} B \right) , \qquad (5.85)$$

where we use a capital Λ to distinguish this bracket from the λ-bracket in vertex algebras. Note that this bracket has odd parity, so, e.g., the bracket of two even states will yield an odd expression. The formal parameters obey the algebra $\chi^2 = -\lambda$. We also have $[D, \chi] = 2\lambda$, while $[\partial, \chi] = 0$, and λ commutes with D and ∂.

The Λ-bracket (5.85) can also be written in the more traditional way of writing OPEs, analogously to (5.33). We consider two different points, Z_1 and Z_2, and define the displacements $Z_{12} = z_1 - z_2 - \theta_1\theta_2$ and $\theta_{12} = \theta_1 - \theta_2$. The OPE between the superfields $A(Z)$ and $B(Z)$ can now be written as

$$A(Z_1)B(Z_2) \sim \sum_{j=0} \frac{\theta_{12}(A_{(j|0)}B)(Z_2) + (A_{(j|1)}B)(Z_2)}{Z_{12}^{j+1}} \ . \qquad (5.86)$$

Note that the poles are expanded as

$$\frac{1}{Z_{12}^n} = \frac{1}{(z_1 - z_2)^n} + \frac{n\theta_1\theta_2}{(z_1 - z_2)^{n+1}} \ , \qquad \frac{\theta_{12}}{Z_{12}^n} = \frac{\theta_{12}}{(z_1 - z_2)^n} \ , \qquad (5.87)$$

so, e.g., Z_{12}^{-1} contains a double pole in $z_1 - z_2$.

The normal ordered product is given by

$$A \cdot B = A_{(-1|1)}B \ . \qquad (5.88)$$

As with an ordinary vertex algebra, the axioms in definition 5.7 is equivalent to a set of rules for the Λ-bracket and the normal ordered product [31]. These rules can be summarized as follows:

Sesquilinearity:

$$[DA_\Lambda B] = \chi[A_\Lambda B], \qquad [A_\Lambda DB] = (-1)^{A+1}(D+\chi)[A_\Lambda B], \quad (5.89)$$
$$[\partial A_\Lambda B] = -\lambda[A_\Lambda B], \qquad [A_\Lambda \partial B] = (\partial + \lambda)[A_\Lambda B]. \quad (5.90)$$

Skew-symmetry:

$$[A_\Lambda B] = (-1)^{AB}[B_{-\Lambda-\nabla}A] \ . \qquad (5.91)$$

The right-hand side is computed by first calculating $[B_\Gamma A]$, where $\Gamma = (\rho, \eta)$, ρ even and η odd, and then replacing Γ with $(-\lambda - \partial, -\chi - D)$.

Jacobi identity:

$$[A_\Lambda[B_\Gamma C]] = -(-1)^A[[A_\Lambda B]_{\Gamma+\Lambda}C]$$
$$- (-1)^{AB+A+B}[B_\Gamma[A_\Lambda C]] \quad (5.92)$$

where the first bracket on the right hand side is computed as in (5.91).

Quasi-commutativity:

$$A \cdot B - (-1)^{AB}B \cdot A = \int_{-\nabla}^0 [A_\Lambda B] \, d\Lambda \qquad (5.93)$$

The integral $\int d\Lambda$ is defined as $\partial_\chi \int d\lambda$. After the integration, there will thus be no χ, and the limits of the integral mean that the λ's should be replaced by $-\partial$'s, together with an overall minus sign.

Quasi-associativity:

$$(A \cdot B) \cdot C - A \cdot (B \cdot C) =$$

$$\left(\int_0^\nabla \mathrm{d}\Lambda \, A \right) \cdot [B_\Lambda C] + (-1)^{AB} \left(\int_0^\nabla \mathrm{d}\Lambda \, B \right) \cdot [A_\Lambda C] \quad (5.94)$$

The right hand side is to be understood as follows. First, the Λ-brackets $[B_\Lambda C]$ and $[A_\Lambda C]$, are calculated. The λ's and χ's are integrated as in (5.93). This will give an operator of the form ∂ to some power times some numerical factor. This operator acts on A respectively B, and the results are normal ordered with the resulting operators from the brackets.

Quasi-Leibniz: (non-commutative Wick formula)

$$[A_\Lambda B \cdot C] = [A_\Lambda B] \cdot C + (-1)^{AB+B} B \cdot [A_\Lambda C]$$
$$+ \int_0^\Lambda [[A_\Lambda B]_\Gamma C] \, \mathrm{d}\Gamma \quad (5.95)$$

The Γ is the same as in (5.91), and the integration is to be understood as in (5.93). The limits of the integral amounts to replacing ρ by λ.

This set of rules constitutes a very efficient way of calculating OPEs of superfields. As mentioned, the rules are implemented in the Mathematica package presented in Paper IV. Note that, as in the non-SUSY case, the integral terms determine the non-commutativity and non-associativity of the normal ordered product, and the failure for the Λ-bracket to fulfill the Leibniz rule. In the SUSY case, these integrals are over a "super space" with coordinates λ and χ. The integrals vanish if there are no χ-terms. Therefore, in some settings, the SUSY vertex algebra becomes more "classical", cf. with section 5.3. This is utilised, e.g., in the calculations presented in section 7.6.

In section 6.4, we will briefly discuss the case of an $N_K = 2$ SUSY vertex algebra. This is a natural generalization of the objects described above, but now with two odd translation operators, two odd formal parameters, etc. For a more detailed description, see Paper V. The rules above is, modulo signs, essentially unmodified. The integrals now contain two Berezin integrals and in some settings the integral terms will typically vanish, see (2.3).

6. Sheaves of vertex algebras

We want to use vertex algebras to describe, and get hold of, geometrical data of manifolds. We are going to assign a vertex algebra to each local patch of the manifold we investigate and then try to "glue" these on the overlaps, using automorphisms of the vertex algebra. The object we get doing this is a *sheaf of vertex algebras*. We start with describing the concept of a sheaf. Thereafter, we investigate sheaves of vertex algebras of the type of a $\beta\gamma$-system, then sheaves of $N_K = 1$ SUSY vertex algebras, and finally sheaves of $N_K = 2$ SUSY vertex algebras. Our main interest is the case of $N_K = 1$ SUSY vertex algebras, which will lead to an appropriate description of the so called chiral de Rham complex, which will be further described in chapter 7.

6.1 Sheaves

A sheaf over a manifold is a collection of local data, together with a set of functions. In a way, it resembles of the concept of a fiber bundle. For a fiber bundle, $E \xrightarrow{\pi} M$, however, the fiber F is itself assumed to be a manifold. Together with the consistence requirements on the transition functions, this makes the total space E to also be a manifold. This is often a too restrictive construction, and one needs an assignment of local data to patches $\{U_\alpha\}$ that can be glued consistently, but without the need for the data to be of the form of a manifold. A sheaf provides such a notion. We here basically follow [28].

Definition 6.1. A *sheaf* $\mathcal{F}$ on a topological space X is an assignment of some data $\mathcal{F}(U)$ to each open set $U \subset X$. The elements of $\mathcal{F}(U)$ is called the sections of $\mathcal{F}$ over U. We will only consider the case when X is a manifold. In many definitions, the data $\mathcal{F}(U)$ is of the form of an abelian group, but one can allow the data to be more general.

In order to have a sheaf, we must also have a set of *restriction maps*. For each subset V of U, we should have a map $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$. The restriction maps must fulfill three properties:
  1. If we have $W \subset V \subset U$, then the composition $r_{V,W} \circ r_{U,V}$ must equal the restriction map $r_{U,W}$. For a section $\sigma \in \mathcal{F}(U)$, we write $\sigma|_V$ for $r_{U,V}(\sigma)$.

2. If we have $U, V \subset X$, and two sections: $u \in \mathcal{F}(U)$ and $v \in \mathcal{F}(V)$, and u equals v on the overlaps of U and V, i.e. $u|_{U \cap V} = v|_{U \cap V}$, then there must exist a section $w \in \mathcal{F}(U \cup V)$, such that $w|_U = u$ and $w|_V = v$.
3. For a section $w \in \mathcal{F}(U \cup V)$, if $w|_U = 0$ and $w|_V = 0$, then $w = 0$.

●

If, for a manifold M with an open covering $\{U_\alpha\}$, we have a section $u_\alpha \in \mathcal{F}(U_\alpha)$ for each U_α, such that for each pair $U_\alpha \cap U_\beta \neq 0$ the sections equal each other:

$$u_\alpha|_{U_\alpha \cap U_\beta} = u_\beta|_{U_\alpha \cap U_\beta} \,,$$

then, by property 2, we have a *global section* $u \in \mathcal{F}(M)$. This is how we will construct global sections in the coming sections. We will also use the name *well defined* sections as a synonym for global sections.

Examples of sheaves includes bundles, but, as said, the definition allows more general constructions.

The perhaps simplest example of a sheaf is the so called *structure sheaf* $\mathcal{O}_M$ of a manifold M. To each open subset U of M, $\mathcal{O}_M(U)$ is the algebra of smooth functions from U to $\mathbb{C}$, and the product of the algebra is pointwise multiplication.

Another simple example which illustrates the use of sheaves is the sheaf $\mathcal{O}_\partial$ of holomorphic functions over a manifold M. If M is compact, there are no global sections except for constant functions, but we can still work locally, with $\mathcal{O}_\partial(U)$.

6.2   Sheaf of $\beta\gamma$ vertex algebras

Our first task in our investigation of sheaves of vertex algebras is to construct a sheaf of the vertex algebra that a $\beta\gamma$-system generates (as described in section 5.4.1).

To a local patch $U_\alpha \cong V_\alpha \subset \mathbb{R}^d$, of a d-dimensional manifold M, we assign a tensor product of d vertex algebras corresponding to the $\beta\gamma$-system. We have

$$[\beta_{i\,\lambda}\beta_j] = 0 \,, \quad [\gamma^i{}_\lambda \gamma^j] = 0 \,, \quad [\beta_{i\,\lambda}\gamma^j] = \delta_i^j \,, \quad i, j = 1, \ldots, d \,. \quad (6.1)$$

Call this vertex algebra $V_{\beta\gamma}$, so $\mathcal{F}(U_\alpha) = V_{\beta\gamma}$. Consider another patch U_β, that overlaps with U_α, and let $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$. We now need to construct an automorphism, i.e., an invertible morphisms, see def. 5.3, of this algebra. The automorphism should be defined on $U_{\alpha\beta}$, and it should map the algebra generated by (6.1) defined on U_α, to the same algebra, but defined on U_β. From this automorphism, we can then create the restriction maps. Note that we can use geometrical data defined on U_α to construct objects in $\mathcal{F}(U_\alpha)$. These

objects may not be invariant under the automorphism and may not be global sections.

We give γ^i the interpretation of a coordinate on U_α. We also want it to be a scalar field, i.e., of conformal weight zero, under scalings of the formal parameters z. If $\{X^i\}$ are the coordinates on U_α, the coordinates on U_β are given by a set of invertible functions: $\tilde{X}^a = f^a(X)$, where the functions f^a corresponds to the map $\psi_{\beta\alpha}$ in the definition 2.1 of a manifold. Let g be the inverse of f: $g^i \equiv (f^{-1})^i(\tilde{X})$. Also, let $g^i_{,a} \equiv \frac{\partial g^i}{\partial \tilde{X}^a}$.

Given a polynomial $h(X) \in \mathbb{C}[X^1,\ldots,X^d]$, we can map this expression to a field $h(\gamma)$ by replacing each occurrence of X^i by γ^i, and using the normal ordered product between factors of γ. The field γ has a vanishing λ-bracket with itself, so normal ordered products between factors of γ are commutative and associative, and $h(\gamma)$ is a well defined expression. In [44] it is shown that this mapping consistently extends to arbitrary functions of X and not just polynomials.

We now want to use the functions f to define the field $\tilde{\gamma}^a$:

$$\tilde{\gamma}^a = f^a(\gamma) . \tag{6.2}$$

From (6.2) we get the wanted relation $[\tilde{\gamma}^a{}_\lambda \tilde{\gamma}^b] = 0$.

Classically, β transform as a one-form. As a first attempt, we define the field $\tilde{\beta}$ by $\tilde{\beta}_a = g^i_{,a} \cdot \beta_i$. With this definition, we have $[\tilde{\beta}_a{}_\lambda \tilde{\gamma}^b] = \delta^b_a$ as wanted, but $[\tilde{\beta}_a{}_\lambda \tilde{\beta}_b] \neq 0$. To try to cure this, we make the most general ansatz that are compatible with the required scaling of our fields. Since γ is a scalar, β must scale as a field of conformal weight one, from the considerations of the scaling of the λ-bracket, see section 4.2. The only other such field we have available is $\partial\gamma$ and the ansatz is thus given by

$$\tilde{\beta}_a = g^i_{,a} \cdot \beta_i + B_{aj} \cdot \partial\gamma^j , \tag{6.3}$$

where B_{aj} is some unknown functions of the γ's. The requirement $[\tilde{\beta}_a{}_\lambda \tilde{\beta}_b] = 0$ gives a set of equations for B_{aj}. As explained in great detail in [45], the obstruction to solve these equations is given by the so called *first Pontryagin class* of M. It is thus not always possible to consistently define the β-fields over M, we may get *anomalies*, and we can therefore not assign a sheaf of $\beta\gamma$ vertex algebras for a general manifold.

As we will see in the next section, introducing fermions allows us to alter the transformations of the β's and with fermions included, we can always create a set of consistent gluing rules of the vertex algebras.

6.3 Sheaves of $N_K = 1$ SUSY vertex algebras

We start by describing the chiral de Rham complex. The chiral de Rham complex, denoted $\Omega^{\mathrm{ch}}(M)$, is a sheaf of vertex algebras constructed by the mathematicians Malikov, Schechtman and Vaintrob, in [44].

For a manifold M of dimension n and with a covering $\{U_\alpha\}$, we assign to each local patch U_α the vertex algebra generated by the tensor product of n copies of a $\beta\gamma$-system, and n copies of a bc-system. We then have, in addition to (6.1), also

$$[b_{i\,\lambda} b_j] = 0\,, \qquad [c^i{}_\lambda c^j] = 0\,, \qquad [b_{i\,\lambda} c^j] = \delta_i^j\,, \tag{6.4}$$

and

$$[b_{i\,\lambda}\beta_j] = [b_{i\,\lambda}\gamma^j] = [c^i{}_\lambda\beta_j] = [c^i{}_\lambda\gamma^j] = 0\,, \quad i,j = 1,\ldots,d\,. \tag{6.5}$$

We have the local data $\mathcal{F}(U_\alpha) = V_{\beta\gamma} \times V_{bc}$.

The remarkable thing is, that the introduction of the fermions allows us to create automorphisms of this vertex algebra and thus consistently glue these relations on the overlaps of the patches, creating a sheaf of vertex algebras. The fermions "cancels" the anomaly observed for the purely bosonic vertex algebra investigated in the last section.

We consider the same coordinate transformations as in the last section, where the coordinates on U_β is given by $\tilde{X}^a = f^a(X)$, where X are the coordinates on U_α, and f is defined on the overlap $U_\alpha \cap U_\beta$. As before, g is the inverse of f. The automorphism is now given by:

$$\tilde{\gamma}^a = f(\gamma)^a\,, \tag{6.6a}$$

$$\tilde{\beta}_a = \beta_i \cdot g^i_{,a}(f(\gamma)) + \left(\left(g^i_{,ab}(f(\gamma)) \cdot f^b_{,j}\right) \cdot c^j\right) \cdot b_i\,, \tag{6.6b}$$

$$\tilde{c}^a = f^a_{,i} \cdot c^i\,, \tag{6.6c}$$

$$\tilde{b}_a = g^i_{,a}(f(\gamma)) \cdot b_i\,. \tag{6.6d}$$

The new fields $\tilde{\gamma}$, $\tilde{\beta}$, $\tilde{b}$ and $\tilde{c}$ will fulfill the same relations (6.1), (6.4) and (6.5), as the old fields. Note that there is no real need for the parenthesis in (6.6b), we can move them freely.

6.3.1 $N = 1$ SUSY formulation

The vertex algebra described in the last section can conveniently be described as an $N = 1$ SUSY vertex algebra. This was noted, and described, in [8]. The $N = 1$ SUSY vertex algebra is described and defined in section 5.6.1. From the fields of the vertex algebra, we can define two $N = 1$ fields, one even,

$$\phi^i = \gamma^i + \theta\, c^i\,, \tag{6.7}$$

and one odd,

$$S_i = b_i + \theta \beta_i \ . \tag{6.8}$$

These field will have the brackets

$$[\phi^i {}_\wedge S_j] = \delta^i_j \ , \qquad [\phi^i {}_\wedge \phi^j] = 0 \ , \qquad [S_i {}_\wedge S_j] = 0 \ , \tag{6.9}$$

which in terms of the components (6.7) and (6.8) is equivalent to the brackets (6.1), (6.4) and (6.5).

The automorphism (6.6) can now be described very compactly, by

$$\tilde{\phi}^a = f^a(\phi) \ , \qquad \tilde{S}_a = g^i_{,a}(f(\phi)) \cdot S_i \ . \tag{6.10}$$

The field ϕ transform as a coordinate, and the odd field S_i as a one-form, under a change of coordinates of the target manifold.

6.3.2 Courant algebroids and sheaves of vertex algebras

We now want to describe a relationship between *Courant algebroids* and $N = 1$ SUSY vertex algebras. Courant algebroids was first defined in [43]. We follow [33].[†]

Definition 6.2 (Courant algebroid). A *Courant algebroid* is a vector bundle E over a smooth manifold M, with a non-degenerate symmetric bilinear form $\langle \, , \, \rangle$, and a bilinear bracket $*$ on $\Gamma(E)$. The form and the bracket must be compatible, in the meaning defined below, with the vector fields on M. We must have a smooth bundle map, *the anchor*, from E to the tangent bundle of M,

$$\pi : E \to TM \ .$$

These structures should satisfy the following five axioms, for all $A, B, C \in \Gamma(E)$ and $f \in C^\infty(M)$.

Axiom 1: $\pi(A * B) = [\pi(A), \pi(B)]_{\text{Lie}}$.

Axiom 2: $A * (B * C) = (A * B) * C + B * (A * C)$.

Axiom 3: $A * (fB) = (\pi(A)f)B + f(A * B)$.

Axiom 4: $\langle A , B * C + C * B \rangle = \pi(A)\langle B , C \rangle$.

Axiom 5: $\pi(A)\langle B , C \rangle = \langle A * B , C \rangle + \langle B , A * C \rangle$.

<div align="right">●</div>

[†] There exists slightly different definitions of Courant algebroids. For example, the requirement that the symmetric form should be non-degenerate is sometimes dropped, see [49, Remark 2.9].

From the above data, we can define a map, $\partial : C^\infty(M) \to \Gamma(E)$ by

$$\langle \partial f, A \rangle = \pi(A)f, \quad \forall f \in C^\infty(M) \, , \, \forall M \in \Gamma(E) \, . \qquad (6.11)$$

Since the vector fields fulfill Leibniz rule when acting on products of functions, the map ∂ will be a differential. Let $\mathcal{R} = C^\infty(M)$ and $\mathcal{E} = \Gamma(E)$, and we see that a Courant algebroid gives a Courant–Dorfman algebra, see definition 4.4 on page 28.

The standard example of a Courant algebroid is given by $E = TM \oplus T^*M$, with the anchor map given by $\pi(v \oplus \lambda) \mapsto v$. This example is considered in section 4.4.1, and the bracket and form are given in (4.19) and (4.21), respectively.

In [34], it is shown that a Courant algebroid gives a sheaf of $N = 1$ SUSY vertex algebras, as we now will describe.

We can use the parity change functor Π, discussed on page 4, to construct an odd vector bundle ΠE where the fibers now are considered to be odd. The operations defined above carries over to ΠE. We get a bilinear form $\langle \, , \, \rangle$, that will be skew-symmetric, and a bilinear bracket $*$. The bracket will have parity one, since it maps two odd elements to an odd element. We get an odd differential $\mathrm{D} : C^\infty(M) \to \Gamma(\Pi E)$, analogue to ∂ above.

Theorem 6.1 (Heluani [34])**.** *For each Courant algebroid E over a differentiable manifold M, there exists a sheaf $U^{\mathrm{ch}}(E)$ of $N = 1$ SUSY vertex algebras on M. This sheaf is generated by functions $C^\infty(M)$ and sections $\Gamma(\Pi E)$. We let $i : C^\infty(M) \to U^{\mathrm{ch}}(E)$ and $j : \Gamma(\Pi E) \to U^{\mathrm{ch}}(E)$. The Λ-brackets are given by*

$$[j(A) \,_\Lambda\, j(B)] = j(A * B) + \chi i \langle A, B \rangle \, , \qquad (6.12)$$
$$[j(A) \,_\Lambda\, i(f)] = i(\pi(A)f) \, . \qquad (6.13)$$

The maps i and j fulfill the following relations:
  1. *$i(1) = |0\rangle$, and $i(fg) = i(f) \cdot i(g)$, where we use the normal ordered product in $U^{\mathrm{ch}}(E)$ on the right hand side.*
  2. *$j(fA) = i(f) \cdot j(A)$.*
  3. *$j(\mathrm{D}f) = \mathrm{D}(i(f))$.*

$\bullet$

Here, we used the same symbol for the odd differential that maps functions on M to sections of ΠE, and the odd translational operator in the vertex algebra.

This theorem should be compared with theorem 4.1 on page 30, where a Poisson vertex algebra is constructed from a Courant–Dorfman algebra.

The theorem allows us to work with Courant algebroids instead of vertex algebras directly, when constructing sheaves of $N = 1$ vertex algebras. Also, results from generalized geometry can be used in a vertex algebra–context.

In the particular case when $E = TM \oplus T^*M$, i.e., the standard Courant algebroid, the sheaf $U^{ch}(E)$ is the chiral de Rham complex, $\Omega^{ch}(M)$.

The global sections of $\Omega^{ch}(M)$, and the symmetries associated to these, will be the subject of the next chapter. But first, we end our investigation of sheaves of vertex algebras by considering a sheaf of $N = 2$ SUSY vertex algebras.

6.4   Sheaf of $N_K = 2$ SUSY vertex algebras

In Paper V, a sheaf of $N_K = 2$ SUSY vertex algebras is constructed. For a short review of the $N_K = 2$ SUSY vertex algebra formalism, see Paper V, for a more detailed exposition, see [31], where the formalism is developed. For our purposes here, we note that an $N_K = 2$ SUSY vertex algebra is similar to the $N_K = 1$ case. Modulo signs, it fulfills the rules presented in section 5.6.2. We will have two odd indeterminates, χ_1 and χ_2, and the superfields are distributions over two odd formal parameters, θ_1 and θ_2, in addition to the even z.

One peculiarity in the case of two supersymmetries, is that it allows a non-trivial Λ-bracket between fields that are invariant under scale transformations of the formal parameter z. Recall that the Λ-bracket can be understood as a Fourier transformation, see, e.g., (5.23). In the $N = 2$ formalism, the residue of a super distribution is defined as the term with $\theta_1 \theta_2 z^{-1}$ in front of it. Under the transformation $z \to \alpha z$, the odd formal parameters transforms as $\theta_i \to \sqrt{\alpha} \theta_i$. The combination $\theta_1 \theta_2 z^{-1}$ is therefore invariant, and we can have a Λ-bracket involving only fields of conformal weight zero. This is what is utilized in Paper V.

We here review the construction in Paper V, but in local coordinates, making the presentation a bit more explicit.

We want to construct a sheaf of $N = 2$ vertex algebras associated to a manifold M, and we only want to include fields that can transforms as coordinates of M under target space diffeomorphisms. We also want these fields to be of conformal weight zero. We don't explicitly write out the normal ordered product $\cdot$ between fields.

Let us consider a set of even $N = 2$ superfields Φ^μ, $\mu = 1, \ldots, \dim M$, that are associated to the local coordinates of a given patch. We make the ansatz

$$[\Phi^\mu {}_\Lambda \Phi^\nu] = \Pi^{\mu\nu}(\Phi) \,, \tag{6.14}$$

where $\Pi^{\mu\nu}$ is some yet unknown matrix, built out of some geometrical data of M. Skew-symmetry of the Λ-bracket implies $\Pi^{\mu\nu}$ is anti-symmetric.

Let us change coordinates on the target manifold. Let $\tilde{\Phi}^i = f^i(\Phi)$, where, as before, f^i is the invertible function associated to the change of coordinates

from X to $\tilde{X}$. Assume that f^i is a polynomial in Φ, and write

$$f^i(\Phi) = A^{(0)i} + A^{(1)i}_\mu \Phi^\mu + \frac{1}{2} A^{(2)i}_{\mu_1\mu_2} \Phi^{\mu_1} \Phi^{\mu_2} + \dots , \qquad (6.15)$$

where $A^{(n)}_{\dots}$ are constants. Since no odd derivatives occur and since there are no χ_i-terms in the ansatz (6.14), the normal ordering will be associative and commutative in expressions like (6.15) and the expression is well defined. Then

$$\begin{aligned}
[\Phi^\nu {}_\Lambda f^i(\Phi)] &= [\Phi^\nu {}_\Lambda A^{(1)i}_\mu \Phi^\mu + \frac{1}{2} A^{(2)i}_{\mu_1\mu_2} \Phi^{\mu_1} \Phi^{\mu_2} + \dots] \\
&= A^{(1)i}_\mu [\Phi^\nu {}_\Lambda \Phi^\mu] + \frac{1}{2} A^{(2)i}_{\mu_1\mu_2} [\Phi^\nu {}_\Lambda \Phi^{\mu_1} \Phi^{\mu_2}] + \dots \qquad (6.16) \\
&= A^{(1)i}_\mu \Pi^{\nu\mu} + A^{(2)i}_{\mu_1\mu_2} \Phi^{\mu_1} \Pi^{\nu\mu_2} + \dots ,
\end{aligned}$$

so

$$[\tilde{\Phi}^i {}_\Lambda \tilde{\Phi}^j] = f^i_{,\mu} f^j_{,\nu} \Pi^{\mu\nu} . \qquad (6.17)$$

So, if we let $\Pi^{\mu\nu}$ transform as a bivector, we have $[\tilde{\Phi}^i {}_\Lambda \tilde{\Phi}^j] = \tilde{\Pi}^{ij}$, and (6.14) is possible to glue across patches.

The Jacobi identity of the Λ-bracket gives $[\Phi^\mu {}_\Lambda \Pi^{\nu\rho}] + \text{cyclic} = 0$, so

$$\Pi^{\mu\tau} \Pi^{\nu\rho}_{,\tau} + \Pi^{\rho\tau} \Pi^{\mu\nu}_{,\tau} + \Pi^{\nu\tau} \Pi^{\rho\mu}_{,\tau} = 0 , \qquad (6.18)$$

i.e., Π must be a Poisson structure, see (2.10).

So, in order for a manifold to admit a sheaf of $N_K = 2$ SUSY vertex algebras generated by the coordinates of the manifold, it must be a Poisson manifold, cf. definition 2.3. Conversely, to any Poisson manifold with a Poisson structure Π, we can associate a sheaf of $N = 2$ vertex algebras, generated by the relation (6.14).

If the Poisson structure Π is invertible, its inverse is a symplectic structure, and we denote it by ω. The target M is then a symplectic manifold. The sheaf of $N = 2$ SUSY vertex algebras is then isomorphic to the chiral de Rham complex, described in the previous section. The relation between the $N = 2$ field Φ, and the $N = 1$ fields ϕ and S, is

$$\Phi^\mu(z, \theta^1, \theta^2) = \phi^\mu(z, \theta^1) - \theta^2 \Pi^{\mu\nu}(\phi(z, \theta^1)) S_\nu(z, \theta^1) . \qquad (6.19)$$

In the next chapter, we will discuss global sections of the chiral de Rham complex. Let us first conclude our discussion of the sheaf of $N = 2$ vertex algebras by stating some global sections of this sheaf, and the algebra they generate. In Paper V, we reproduce the results from [34]: on a Calabi–Yau manifold, we have two commuting $N = 2$ superconformal algebras, each with a central charge $c = \frac{3}{2} \dim M$. The novelty here is that the calculation is performed in a manifest $N = 2$ formalism. The $N = 2$ superconformal algebra

with central charge c can be written very compact in terms of one single operator $\mathcal{G}$:

$$[\mathcal{G} \wedge \mathcal{G}]_{N=2} = (2\lambda + 2\partial + \chi_1 D_1 + \chi_2 D_2)\mathcal{G} + \lambda \chi_1 \chi_2 \frac{c}{3} \,. \qquad (6.20)$$

In the next chapter, in section 7.2, the corresponding expressions and brackets are written in $N = 1$ formalism.

We define two operators, $\mathcal{G}_\omega$ and $\mathcal{H}$, by

$$\mathcal{G}_\omega = \tfrac{1}{2}\omega_{\mu\nu}(D_1\Phi^\mu D_1\Phi^\nu + D_2\Phi^\mu D_2\Phi^\nu) \,, \qquad (6.21)$$

$$\mathcal{H} = (g_{\alpha\bar{\beta}}D_2\Phi^\alpha)D_1\Phi^{\bar{\beta}} - (g_{\alpha\bar{\beta}}D_1\Phi^\alpha)D_2\Phi^{\bar{\beta}} \,. \qquad (6.22)$$

The operator $\mathcal{G}_\omega$ is well defined on any symplectic manifold, and generates an $N = 2$ superconformal algebras with a central charge $c = 3 \dim M$. The operator $\mathcal{H}$ can be written as it stand in coordinates where the holomorphic volume form is constant, but in general complex coordinates, it needs an additional term to be well defined. On a Calabi–Yau manifold, the linear combinations $\mathcal{G}_\pm = \mathcal{G}_\omega \mp \tfrac{1}{2}\mathcal{H}$ generates the mentioned two commuting copies of the $N = 2$ superconformal algebra. Note that (6.22) coincides in form with the classical Hamiltonian density (3.51) derived for the $N = 2$ supersymmetric sigma model in section 3.5.

7. The chiral de Rham complex

As mentioned, the chiral de Rham complex was introduced by the mathematicians Malikov, Schechtman, and Vaintrob, in [44]. As described in section 6.3, it is a sheaf of vertex algebras constructed by gluing n copies of the $\beta\gamma - bc$ vertex algebra on the overlaps of local patches, isomorphic to subsets of $\mathbb{R}^n$, of a manifold M. We denote this sheaf by $\Omega^{\mathrm{ch}}_\hbar(M)$, where the subscript $\hbar$ will be motivated below.

In the original article [44], the authors considered an embedding of the *de Rham complex* into this sheaf. Also, in the traditional approach to vertex algebras, the interpretation is that it captures the algebra of *chiral* operators of a two-dimensional field theory. In this perspective, the term *chiral de Rham complex* was a well-motivated name to describe this sheaf of vertex algebras. As we shall see, we allow for non-chiral interpretations of the vertex algebras, and we are not particularly interested in the embedding of the de Rham complex. For our purposes, the name is not representative but we use this name since it is established in the literature. We will mainly use the abbreviation CDR henceforth.

There are many different possible physical interpretations of the CDR. The construction appears, for example, in the theory of half-twisted $(2,0)$ supersymmetric sigma models, see, e.g., [52, 40]. It also appears in the theory of the so-called large volume limit of the sigma model, see, e.g., [22].

In this chapter, we want to put forward a physical interpretation of the CDR as a formal quantization of the $N = 1$ supersymmetric sigma model. We discuss various operators, with accompanying algebras, that one can define in the CDR on manifolds with different properties. We will mainly work in the $N = 1$ SUSY vertex algebra formalism, as described in section 6.3.1. We start with a discussion about a semi-classical limit of the CDR.

7.1 Semi-classical limit of the CDR

We consider the vertex algebra $V = V_{\beta\gamma} \times V_{bc}$, using $N = 1$ SUSY vertex algebra formalism. We let the vertex algebra, as a module for the algebra $\mathrm{D}^2 = \partial$, be generated by the fields ϕ and S as before, but now the nontrivial bracket

reads

$$[\phi^i{}_\Lambda S_j] = \hbar\, \delta^i_j \,, \tag{7.1}$$

where we introduced a formal parameter $\hbar$.

This gives us a family $V_\hbar$ of SUSY vertex algebras. This is a SUSY vertex algebra analogue of the discussion of section 5.3, where the semi-classical limit of a vertex algebra was considered. We define a rescaled bracket on the elements in $V_0 = V_{\hbar=0}$ by

$$\{A_\Lambda B\} \equiv \lim_{\hbar\to 0} \frac{1}{\hbar}[A_\Lambda B] \,. \tag{7.2}$$

The normal ordered product $\cdot_\hbar$ of $V_\hbar$ gives a normal ordered product on V_0, by $A \cdot B = \lim_{\hbar\to 0} A \cdot_\hbar B$. In this semi-classical limit, we get an $N = 1$ SUSY Poisson vertex algebra $(V_0, \{\,_\Lambda\,\}, \cdot)$, the supersymmetric analogue of a Poisson vertex algebra.

Gluing the families $V_\hbar$ of vertex algebras, using the same automorphisms as in section 6.3.1, we get a sheaf of families of $N = 1$ SUSY vertex algebras which we denote by $\Omega^{\mathrm{ch}}_\hbar(M)$. This object is what we henceforth will refer to as the chiral de Rham complex. Taking a semi-classical limit of this sheaf, one gets a sheaf of $N = 1$ SUSY Poisson vertex algebras, $\Omega^{\mathrm{ch}}_0(M)$, cf. figure 7.1 on page 74.

We now want to discuss different superconformal structures that we can construct in the CDR for different manifolds.

7.2 Superconformal algebras

In the vertex algebra $\Omega^{\mathrm{ch}}_\hbar(U_\alpha)$ that we define locally over a patch $U_\alpha \cong V_\alpha \subset \mathbb{R}^d$, we can define the operator $\mathcal{P} = \mathrm{D}\phi^\mu \mathrm{D}S_\mu + \partial\phi^\mu S_\mu$. Recall that we use the isomorphism (6.10) to glue the vertex algebras on the overlaps of the local patches. In order to get a global section of the CDR, we need an operator that is invariant under the isomorphism (6.10). It turns out that we need to add a quantum correction, i.e., terms of order $\hbar$, to the operator $\mathcal{P}$ to get a well defined global section of the CDR. We can do this on any orientable manifold [44, 8], and the global section is given by

$$\mathcal{P} = \mathrm{D}\phi^\mu \mathrm{D}S_\mu + \partial\phi^\mu S_\mu - \hbar\partial\,\mathrm{D}\log\sqrt{g} \,, \tag{7.3}$$

where $g = \det g_{ij}$, the determinant of the metric of the target manifold[†]. This operator gives an $N = 1$ superconformal algebra with central charge $c = \dim M$,

$$[\mathcal{P}_\Lambda \mathcal{P}] = \hbar(2\partial + \chi\mathrm{D} + 3\lambda)\mathcal{P} + \hbar^2\frac{c}{3}\lambda^2\chi \,. \tag{7.4}$$

[†] We could equally well use another volume form to define a global section, see Paper II.

Note that the central charge is a quantum effect.

Expanding $\mathcal{P}$ as

$$\mathcal{P}(z,\theta) = G(z) + 2\theta L(z) , \qquad (7.5)$$

the operators G and L gives the $N = 1$ superconformal algebra (5.80).

We want to find additional symmetry generators that extends the $N = 1$ superconformal symmetry generated by $\mathcal{P}$. On a symplectic manifold, with a symplectic structure ω, we can define [34]

$$\mathcal{J}_\omega = \tfrac{1}{2}\left(\omega^{\mu\nu} S_\mu S_\nu - \omega_{\mu\nu} D\phi^\mu D\phi^\nu\right) , \qquad (7.6)$$

where $\omega^{\mu\nu}$ is the inverse of $\omega_{\mu\nu}$. This operator is a global section. The operators $\mathcal{P}$ and $\mathcal{J}_\omega$ together gives an $N = 2$ superconformal algebra. In addition to (7.4), the algebra is given by

$$[\mathcal{P} \wedge \mathcal{J}_\omega] = \hbar(2\partial + 2\lambda + \chi D)\mathcal{J}_\omega , \qquad (7.7)$$

$$[\mathcal{J}_\omega \wedge \mathcal{J}_\omega] = -\hbar\mathcal{P} - \hbar^2 \frac{c}{3}\lambda\chi . \qquad (7.8)$$

We can combine $\mathcal{P}$ and $\mathcal{J}_\omega$ into one $N = 2$ superfield $\mathcal{G}_\omega$ by

$$\mathcal{G}_\omega(z,\theta_1,\theta_2) = \mathcal{J}_\omega(z,\theta_1) - \theta_2\mathcal{P}(z,\theta_1) . \qquad (7.9)$$

This operator corresponds to the operator defined in (6.21), and the brackets (7.4), (7.7), and (7.8) can be written as the single bracket (6.20), see Paper V.

On a complex manifold, with complex structure I, and Levi-Civita connection Γ, we can define the following global section:

$$\mathcal{J}_I = I^i_j D\phi^j S_i + \hbar\Gamma^i_{jk}I^j_i\partial\phi^k . \qquad (7.10)$$

In [8] it is shown that $(\mathcal{P}, \mathcal{J}_I)$ generates an $N = 2$ superconformal algebra, if, and only if, M is a Calabi–Yau manifold.

We have found two cases of $N = 2$ structures, $(\mathcal{P}, \mathcal{J}_\omega)$ on a symplectic, and $(\mathcal{P}, \mathcal{J}_I)$ on a Calabi–Yau manifold. In general, the existence of an $N = 2$ superconformal algebra requires the manifold to be a *generalized Calabi–Yau* manifold [32]. Both symplectic and Calabi–Yau manifolds are examples of generalized Calabi–Yau manifolds. They can be understood as the natural analogue of a Calabi–Yau manifold, in the formalism of generalized complex geometry, see [30].

When M is a Calabi–Yau manifold it is also Kähler, and hence symplectic, with the symplectic form given by $\omega = gI$. In this case, we have both $N = 2$ algebras simultaneously, but they do not commute. Choosing a metric g on our Calabi–Yau manifold such that the corresponding Ricci curvature vanishes, i.e., a Ricci flat metric, we can define the operator

$$\mathcal{H} = \partial\phi^\mu D\phi^\nu g_{\mu\nu} + g^{\mu\nu} D S_\mu S_\nu + \Gamma^\rho_{\sigma\nu}g^{\nu\lambda} D\phi^\sigma(S_\lambda S_\rho) . \qquad (7.11)$$

The operator $\mathcal{H}$ is a global section when M is Calabi–Yau with a Ricci flat metric, as shown in [34]. It is an open question whether or not it is well defined for a general Riemannian manifold. We will soon return to a physical interpretation of the operator $\mathcal{H}$, but first we note that from the operators $\mathcal{P}$, $\mathcal{H}$, $\mathcal{J}_\omega$ and $\mathcal{J}_I$, we can define the following linear combinations:

$$\mathcal{H}_\pm = \tfrac{1}{2}(\mathcal{P} \pm \mathcal{H}) , \qquad\qquad \mathcal{J}_\pm = \tfrac{1}{2}(\mathcal{J}_I \pm \mathcal{J}_\omega) . \qquad (7.12)$$

These gives two commuting $N = 2$ superconformal algebras [34], $(\mathcal{H}_+, \mathcal{J}_+)$ and $(\mathcal{H}_-, \mathcal{J}_-)$, each with a central charge $c = 3/2 \dim M$.

7.3 Interpretation of the CDR as a formal quantization

In section 3.4, the phase space of the $N = 1$ supersymmetric sigma model was derived. The local coordinates were given by two maps, $\phi^\mu(\sigma)$, and $S_\mu(\sigma)$. The Poisson bracket was given by (3.43), and the Hamiltonian density by (3.40).

The classical mechanics of the sigma model can thus be described in the language of an N=1 SUSY Poisson vertex algebra, generated by the Λ-bracket

$$\{\phi^\mu {}_\Lambda S_\nu\} = \delta^\mu_\nu . \qquad (7.13)$$

This Poisson vertex algebra is the semiclassical limit of the CDR, as described in section 7.1. This suggests the natural interpretation of the CDR as a formal canonical quantization of the supersymmetric sigma model. We then interpret the Λ-brackets as describing the equal time commutators of operators. This interpretation was made in Paper II, we here review it briefly.

In the classical description of the sigma model in section 3.4, we have the worldsheet $\mathbb{R} \times S^1$, with coordinates t and σ respectively. Let us map these to a complex coordinate z, by

$$z = e^{i\sigma + t} . \qquad (7.14)$$

This is a conformal mapping of the cylinder onto the complex plane. Since $e^{i\sigma + t} \to 0$ as $t \to -\infty$, the infinite past will be mapped to the origin of the plane. A surface of fixed time on the cylinder will correspond to a circle of fixed radius around the origin in the complex plane. Let us set time to zero. The circle coordinate σ is then the coordinate on the unit circle in the complex plane. Let us interpret the coordinate z on the complex plane, as the formal parameter z that we use in our fields in the vertex algebras. With this interpretation, the λ-brackets become Fourier transformations of equal-time commutators.

Up to quantum terms, the Hamiltonian density derived from the sigma model, (3.40), takes the same form as the operator $\mathcal{H}$ in (7.11). Under the coordinate change (7.14), we have $d\sigma \, d\theta_\sigma \mathcal{H}(\sigma, \theta_\sigma) = dz \, d\theta \, iz\mathcal{H}(z, \theta)$. We therefore define as our quantum Hamiltonian, the endomorphism given by

$$H = \frac{i}{2} \oint dz \, d\theta \, z \mathcal{H}(z, \theta) . \qquad (7.15)$$

From (7.12) we see that $\mathcal{H} = \mathcal{H}_+ - \mathcal{H}_-$. Now, $\oint \mathrm{d}z\,\mathrm{d}\theta\,z\,\mathcal{H}_\pm$ are the *zero modes* of the Virasoro operators $L_\pm$, that are parts of the superfields $\mathcal{H}_\pm$, see (7.5) and the expansion in (5.68). So, we have $H = i(L_+)_0 - i(L_-)_0$. We postulate that the time behavior of an operator $\mathcal{O}$ is governed by the flow equation

$$\frac{\mathrm{d}\mathcal{O}}{\mathrm{d}t} = \frac{1}{\hbar}[H, \mathcal{O}] . \tag{7.16}$$

The operators in the CDR are interpreted as operators at a fixed time. From the flow equation, we get the time dependence of the operators.

It is important to stress that the canonical quantization of the sigma model we are advocating here is formal. We are treating the phase space, and the operators defined on it, in a formal way and not analytically. There are many analytical issues, like continuity and convergence issues, that we do not address in this treatment of operators. Also, the quantization considered here only captures "small loops", loops with no winding that can be expanded in the basis $e^{in\sigma}$. We have silently assumed that M is simply connected, so that all loops can continuously be deformed to a point. However, if M has a nontrivial first homotopy group π_1, we can take copies of the CDR, each capturing the behavior of one type of wrapping, and an appropriate formalism can be developed to describe this setting, see [2].

The interpretation of the CDR as a quantization of the sigma model raises a puzzle. In section 7.2, we saw that we have two commuting $N = 2$ superconformal algebras on a Calabi–Yau manifold with a Ricci flat metric. This is in conflict with multi-loop calculations performed in a path-integral quantization of the sigma model, where the model still has the desired conformal symmetry but for a non Ricci-flat metric, see [46]. If we allow us to speculate, the discrepancy might be due to differences in the Hamiltonian and the Lagrangean approach to path integral quantization. The two formalisms are not necessarily equal on curved manifolds, see, e.g., [16].

7.4   Algebra extensions

In the last section, we saw that there is a substantial relation between the CDR and the $N = 1$ supersymmetric sigma model: the CDR can be viewed as a formal quantization of the sigma model. From this interpretation of the CDR, we can use insights from sigma models to construct interesting operators in the CDR. Especially, we are interested in operators that corresponds to symmetries of the model.

In Paper III, we take as our starting point the observations made by Howe and Papadopoulos in [36, 37] about 20 years ago. They observed that a covari-

Holonomy	dim M	Name of manifold
$SO(n)$	n	Orientable
$U(n)$	$2n$	Kähler
$SU(n)$	$2n$	Calabi–Yau
$Sp(n)$	$4n$	Hyperkähler
$Sp(n) \cdot Sp(1)$	$4n$	Quaternionic Kähler
G_2	7	G_2-manifold
$Spin(7)$	8	$Spin(7)$-manifold

Table 7.1: Berger's list of possible holonomy groups.

antly constant form gives rise to a symmetry of the classical $N = 1$ supersymmetric sigma model.[†]

The existence of covariantly constant forms is ultimately related to the holonomy group of the manifold. Therefore, there is a direct correspondence between additional symmetries of the sigma model and manifolds with special geometries, i.e., nontrivial holonomy groups.

Given some assumptions, the possible holonomy groups have been classified by Berger in 1955 [9], see [38] for a review. The classification states that, given that M is simply-connected, the metric g is irreducible, and M is not locally a Riemannian symmetric space, then the holonomy group is one of the following seven different groups: $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, $Sp(n) \cdot Sp(1)$, G_2, or $Spin(7)$, see table 7.1. Each of these cases has covariantly constant forms associated to them. In Paper III, we first investigate the classical algebras the currents associated to these forms generate, using the Poisson vertex algebra formalism. We then raise the question: can we "lift" these currents to operators in the CDR? Will they be global sections, and if not, can we modify the operators to get well defined operators? It turns out that the answer to the last question is yes, as we now briefly review.

7.5 Well-defined operators corresponding to forms

The classical symmetry current that we obtain from a covariantly constant form ω is in the Lagrangean formalism of the form

$$J_\pm = \omega_{i_1 \ldots i_n}(\Phi)\, D_\pm \Phi^{i_1} \ldots D_\pm \Phi^{i_n} , \qquad (7.17)$$

where Φ is the $N = (1,1)$ field, see section 3.5. When going to the Hamiltonian formalism, the fields $D_\pm \Phi^i$ will be reduced to linear combinations of

---

[†] This was observed earlier, but not utilized in the same way. See [36] and the references therein.

fields with conformal weight $1/2$. This motivates us to define the following combinations:

$$e^i_\pm \equiv \tfrac{1}{\sqrt{2}} \left(g^{ij} S_j \pm \mathrm{D}\phi^i \right) \ . \tag{7.18}$$

We have $e^i_+ = \mathrm{D}_+ \Phi^i|_{\theta_0=0}$ and $e^i_- = -i\mathrm{D}_- \Phi^i|_{\theta_0=0}$.

The symmetry generators in the Poisson vertex algebra formalism are now of the form

$$J_\pm \sim \omega_{i_1 \dots i_n} e^{i_1}_\pm \dots e^{i_n}_\pm \ . \tag{7.19}$$

We want to "lift" this expression to the CDR. We concentrate on the plus-sector, the minus-sector are treated analogously. We first note that the e^i_+'s do not commute, and in particular, the bracket $[e^i_{+\,\Lambda}\, e^j_+]$ contains a χ-term. This makes the order of the parenthesis in an expression like $e^i_+ \cdot (e^j_+ \cdot e^k_+)$ important. Furthermore, such expressions do not transform like tensors under a change of coordinates.

In Paper III we show that, using the Levi-Civita connection, it is possible to create an object with an arbitrary number of anti-symmetric contravariant indices that do transform as a tensor. This object can be contracted with a form to create a target space diffeomorphism invariant operator. By construction, this will be a well defined global section of the CDR. For the explicit formulas, see Paper III. We here exemplify by writing the objects with two and three indices:

$$e^i_+ \cdot e^j_+ + \hbar \Gamma^i_{kl} g^{jk} \partial \phi^l \ , \tag{7.20}$$

and

$$e^i_+ \cdot (e^j_+ \cdot e^k_+) + 3\hbar \Gamma^i_{lm} g^{lj} \partial \phi^m e^k \ , \tag{7.21}$$

where i, j, k should be anti-symmetrized in the last expression. We can construct similar objects for arbitrary number of anti-symmetric indices.

It is an open question if this is possible for tensors in general, see figure 7.1. For example, the operator $\mathcal{H}$ given in (7.11) corresponds to a symmetric two-tensor: the metric. We are only able to show that this operator is well defined when M is a Calabi–Yau manifold. As we already pointed out, it is an open question if it is a global section of a general Riemannian manifold.

Anyway, we have shown that we can construct well-defined operators corresponding to forms and in particular, we can construct well-defined sections out of the forms associated to the different holonomy groups in table 7.1, and try to calculate the quantum algebra they generate. In fact, we have already covered the Kähler case in section 7.2.

In Paper III we cover the other, from the perspective of string theory compactification, physically interesting cases; when the manifold is a Calabi–Yau three-fold, when it is G_2, and when it is $\mathrm{Spin}(7)$. In the two latter cases, we calculate the algebra when the manifold is flat, i.e., with a constant metric. We are unfortunately unable to perform the general calculations due to the
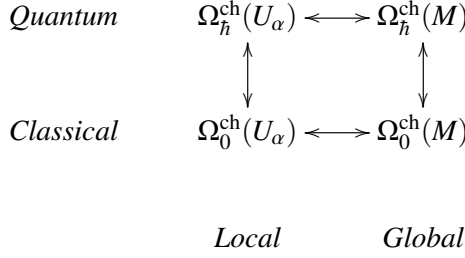
$$\begin{array}{ccc}
\textit{Quantum} & \Omega^{\mathrm{ch}}_{\hbar}(U_\alpha) \longleftrightarrow \Omega^{\mathrm{ch}}_{\hbar}(M) \\
& \updownarrow \qquad\qquad \updownarrow \\
\textit{Classical} & \Omega^{\mathrm{ch}}_{0}(U_\alpha) \longleftrightarrow \Omega^{\mathrm{ch}}_{0}(M)
\end{array}$$

$$\textit{Local} \qquad\qquad \textit{Global}$$

Figure 7.1: Locally defined sections in a Poisson vertex algebra (classical setting) can be extended to global classical sections. The local sections can be lifted and defined in a vertex algebra. Some, or all, of these sections can be extended to global sections of the CDR.

complexity of the calculation. This is due to the lack of coordinates where the covariantly constant forms on these manifolds are constant. Even the flat case is quite involved, and we need to rely on the software presented in Paper III.

7.6 The Odake algebra

In the Calabi–Yau three-fold case, we calculate the full quantum algebra generated by the operators associated to the holonomy group for a general curved Calabi–Yau manifold. This algebra is called the *Odake algebra*, since it was first investigated by Odake, in [47]. A Calabi–Yau three-fold means a six-dimensional manifold that has $\mathrm{SU}(3)$ as holonomy group. The covariantly constant forms are the holomorphic and antiholomorphic volume forms Ω and $\bar{\Omega}$. From these, we construct the operators

$$\mathcal{X}_+ \equiv \frac{1}{3!}\Omega_{\alpha\beta\gamma}e^\alpha_+ e^\beta_+ e^\gamma_+ \, , \qquad \mathcal{X}_- \equiv \frac{i^3}{3!}\Omega_{\alpha\beta\gamma}e^\alpha_- e^\beta_- e^\gamma_- \, , \tag{7.22a}$$

$$\bar{\mathcal{X}}_+ \equiv \frac{1}{3!}\bar{\Omega}_{\bar\alpha\bar\beta\bar\gamma}e^{\bar\alpha}_+ e^{\bar\beta}_+ e^{\bar\gamma}_+ \, , \qquad \bar{\mathcal{X}}_- \equiv \frac{i^3}{3!}\bar{\Omega}_{\bar\alpha\bar\beta\bar\gamma}e^{\bar\alpha}_- e^{\bar\beta}_- e^{\bar\gamma}_- \, . \tag{7.22b}$$

Here, greek indices indicate complex coordinates. Note that the quantum corrections from the corresponding covariant object (7.21) are absent, since they involve connections with mixed indices (holomorphic and anti-holomorphic) and these vanishes on a Kähler manifold. Since a Calabi–Yau is also Kähler, we have the $(1,1)$-form ω, giving rise to the operators $\mathcal{J}_+$ and $\mathcal{J}_-$ of section 7.2, that together with $\mathcal{H}_\pm$ generates the two commuting copies of the $N=2$ superconformal algebra with central charge $c=9$. In Paper III, we show that these operators, together with the operators defined in (7.22), in addition

generate the Odake algebra:

$$
\begin{aligned}
[\mathcal{X}_{\pm} \wedge \bar{\mathcal{X}}_{\pm}] &= \frac{1}{2}\hbar\left(i\mathcal{H}_{\pm}\mathcal{J}_{\pm} + \mathrm{D}\,\mathcal{J}_{\pm}\mathcal{J}_{\pm} + \chi\mathcal{J}_{\pm}\mathcal{J}_{\pm}\right) \\
&\quad - \hbar^2\left(i\chi\partial\mathcal{J}_{\pm} - \lambda\mathcal{H}_{\pm} + i\lambda\mathrm{D}\,\mathcal{J}_{\pm} + 2i\chi\lambda\mathcal{J}_{\pm}\right) + \hbar^3\chi\lambda^2 \,, \\
[\mathcal{J}_{\pm} \wedge \mathcal{X}_{\pm}] &= +i\hbar\left(3\chi + \mathrm{D}\right)\mathcal{X}_{\pm} \,, \\
[\mathcal{J}_{\pm} \wedge \bar{\mathcal{X}}_{\pm}] &= -i\hbar\left(3\chi + \mathrm{D}\right)\bar{\mathcal{X}}_{\pm} \,, \\
[\mathcal{X}_{\pm} \wedge \mathcal{H}_{\pm}] &= \hbar(3\lambda + \chi\mathrm{D} + 2\partial)\mathcal{X}_{\pm} \,. \\
[\bar{\mathcal{X}}_{\pm} \wedge \mathcal{H}_{\pm}] &= \hbar(3\lambda + \chi\mathrm{D} + 2\partial)\bar{\mathcal{X}}_{\pm} \,, \\
[\mathcal{X}_{\pm} \wedge \mathcal{X}_{\pm}] &= 0 \,, \\
[\bar{\mathcal{X}}_{\pm} \wedge \bar{\mathcal{X}}_{\pm}] &= 0 \,.
\end{aligned}
\tag{7.23}
$$

Note that the algebra is non-linear. This algebra is related to the spectral flow [47] that is responsible for the existence of target space supersymmetry, see [27] for a review.

Acknowledgements

It is said that the acknowledgements is the most read part of a thesis. I think this is unquestionable true. Like the placement of milk in grocery stores, it is most often placed at the back.[†] The owner of a store hopes that the customer will buy something extra while walking through the store. I suspect there is a similar reason why the acknowledgements is traditionally placed at this location of the thesis, but perhaps the author is just pleased that the reader at least has opened the dissertation.

First of all, I would like to thank Maxim, my supervisor during my time as a Ph.D.-student. I have very much appreciated your ability to de-dramatize apparently complex issues and your way of patiently explaining them to me. Thanks for your guidance, and also for our collaborations: both in teaching and in research.

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I am grateful to the present and former Ph.D.-students and postdocs for creating and maintaining the nice atmosphere of the department. I especially want to thank Olof, for all discussions about physics (among other topics) and for nice traveling company, Valentina, for being so genuinely friendly, Niklas for many things, but especially for being such a good table hockey opponent, Johan K, for traveling company and many valuable discussions and finally Malin for organizing everything, from Friday cake to Strings 2011. Amjad, Andreas, Andrey, Anton, Diego, Gabriele, Itai, Jian, Johan B., Jonas, Kasper, Luís, Magdalena, Manuela, Martin, Mikael, Sergey, Shuang-Wei, Thomas V. R., Till, and Xubiao — I am grateful for nice lunch breaks and for all interesting discussions on various topics.

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[†] At least in Swedish stores milk is always at the back of the store. Swedes drink a lot of milk.

Summary in Swedish

Gå runt i cirklar
Från sigmamodeller till vertexalgebror och tillbaka

I avhandlingen undersöks *sigmamodeller* och *vertexalgebror*, samt relationer mellan dessa.

Sigmamodeller är ett samlingsnamn för en typ av modeller som beskriver och undersöker avbildningar mellan *mångfalder*. En mångfald är en matematisk beskrivning av till exempel en linje, som är en endimensionell mångfald, eller av en yta, som är en tvådimensionell mångfald. Mångfalder med högre dimensioner är generaliseringar av linjer och ytor; en mångfald kan ha vilken positiv dimension som helst. De mångfalder som används och studeras i avhandlingen kallas för differentierbara. Differentierbara mångfalder är basala byggstenar inom den teoretiska fysiken.

Vi koncentrerar oss på tvådimensionella sigmamodeller, som behandlar avbildningar från en tvådimensionell mångfald, kallad *världsytan*, till en annan mångfald som kallas för *målmångfalden*. Vi undersöker modeller som har en vanlig tvådimensionell mångfald som världsyta, men också modeller vars världsytor är *supermångfalder*. Supermångfalder gör det möjligt att på ett geometriskt sätt beskriva och hantera *supersymmetri* – en symmetri mellan två klasser av partiklar: *bosoner* och *fermioner*. På en supermångfald beskrivs vissa koordinater på mångfalden av antikommuterande tal, det vill säga av tal som uppfyller $a \cdot b = -b \cdot a$. Sådana tal kallas för *Grassman*-tal. De världsytor som vi är intresserade av ser ut som cylindrar. En cylinder har två väldigt olika riktningar: en riktning är periodisk, en är det inte. Den periodiska riktningen har formen av en cirkel och vi låter den riktningen vara en *rumslig* riktning. Den andra riktningen tolkas som en tidsriktning. Om man tänker sig att man bor på världsytan, bor man på en cirkel och kan endast gå runt denna. Tiden för en framåt i den ickeperiodiska riktningen på cylindern.

Sigmamodeller är grundläggande inom *strängteori*. Något förenklat kan strängteori sägas vara en tvådimensionell sigmamodell, kopplat till en modell för gravitation på den tvådimensionella världsytan. Strängteori har varit ett stort forskningsområde inom teoretisk och matematisk fysik de senaste 25 åren och har lett till en ökad förståelse av många fysikaliska modeller och teorier, men även gett inspiration till många nya forskningsområden inom matematiken.

Vi använder sigmamodeller till att undersöka olika typer av målmångfalder. Det finns ett intrikat samspel mellan sigmamodellens symmetrier på världsytan och egenskaper hos målmångfalden. Ett av målen med avhandlingen är att undersöka detta samspel.

Vi vill även undersöka hur man kan *kvantisera* sigmamodeller genom att använda vertexalgebror, och se hur sigmamodellens symmetrier påverkas när modellen kvantiseras. Att kvantisera en fysikalisk modell kan innebära olika saker, det finns inget entydigt tillvägagångssätt. För att kort illustrera kvantisering, ta mekanik som exempel. Man skiljer på klassisk mekanik och kvantmekanik. I klassisk mekanik beskrivs det man vill mäta, exempelvis ett fallande äpples höjd över marken, entydigt av funktioner som beror på begynnelsevärden (hur högt äpplet var när det föll från grenen o.s.v.). Inom kvantmekaniken blir kvantfenomen viktiga, till exempel kan man inte samtidigt mäta en partikels, eller för den delen ett äpples, läge och hastighet med fullständig noggrannhet. Storleken på kvantfenomenen är (oftast) proportionella mot *Plancks konstant*[†] $\hbar$, som är en naturkonstant. Uttryckt i våra vardagsenheter (kilogram, meter o.s.v.), är Plancks konstant ett väldigt litet tal: $\hbar \approx 10^{-34}$ kg m^2/s. Därför är kvantfenomen oftast försumbara i vår vardagsvärld – klassisk mekanik räcker för att beskriva hur ett äpple faller från grenen till marken. Gemensamt för de olika tillvägagångssätt som finns för att kvantisera en modell är att man har en parameter, $\hbar$, så att man i gränsen där $\hbar$ går mot noll får en modell som beter sig klassiskt, men då $\hbar$ är skilt från noll har kvantfenomen i modellen.

Det är svårt att kvantisera sigmamodeller som har krökta målmångfalder. Dessa teorier är ickelinjära, de innehåller objekt som växelverkar med varandra på ett sätt som försvårar "skolboksreceptet" för kvantisering. I den här avhandlingen används vertexalgebror för att beskriva kvantfenomen hos sigmamodeller. Vertexalgebror är matematiskt väldefinierade algebraiska teorier som fångar viktiga aspekter av konform fältteori, en viktig typ av kvantteori som kan definieras på tvådimensionella ytor.

En ökad förståelse av kvantiserade sigmamodeller är viktig, bland annat leder det till ett matematiskt stabilare fundament till strängteorin, och även till en djupare insikt i geometrin hos målmångfalderna.

Avhandlingen i korthet

Avhandlingen börjar med en genomgång av grundläggande begrepp. Vi går igenom Hamilton-formuleringen av klassisk mekanik och betonar den geometriska formuleringen därav, som ges i termer av en Poisson-klammer definierad på ett fasrum samt en speciell funktion: Hamiltonianen. Den klassiska mekani-

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[†] Plancks konstant är egentligen h, där $h = 2\pi\hbar$. Konstanten $\hbar$ kallas ibland för den reducerade Planck-konstanten eller för Diracs konstant.

ken behandlar punktpartiklars rörelser – fasrummets dimension är då ändlig. I de fall som behandlas senare i avhandlingen undersöks oändligtdimensionella fasrum.

I kapitel 3 härleder vi Poisson-klammern, fasrummet och Hamiltonianen för den *bosoniska tvådimensionella sigmamodellen*, vars världyta är en "vanlig" tvådimensionell mångfald. Genom att ersätta världsytan med en supermångfald kan en supersymmetrisk sigmamodell formuleras. Vi undersöker sådana sigmamodeller med en respektive två så kallat manifesta supersymmetrier. Vi härleder Poisson-klammern, fasrummet och Hamiltonianen även för dessa modeller.

I kapitel 4 definieras två typer av algebror: Poisson-vertexalgebror och Lie-konforma algebror. Genom dessa kan en algebraisk beskrivning av strukturer på sigmamodellernas fasrum ges. Poisson-vertexalgebror är en generalisering av egenskaperna hos Poisson-klammern, fast för oändligtdimensionella fasrum. Vi visar att en Lie-konform algebra ger en så kallad *svag Courant-Dorfman-algebra*. Vi visar också att en Poisson-vertexalgebra som är genererad av objekt med konform vikt noll och ett, står i ett ett-till-ett förhållande till en *Courant-Dorfman-algebra*.

Kapitel 5 inleds med en genomgång av *formella distributioner*. En formell distribution i en formell parameter z är ett utryck av typen $A(z) = \ldots + A_{(1)}z^{-2} + A_{(0)}z^{-1} + A_{(-1)}z^0 + A_{(-2)}z^1 + \ldots$, där punkterna representerar ett oändligt antal termer, alla med en viss potens av z i sig. Genom dessa uttryck kan *fält* definieras. Ett fält i det här sammanhanget är en formell distribution, där varje koefficient $A_{(n)}$ är en avbildning av ett givet vektorrum på sig självt.

Vi beskriver sedan vertexalgebror. En vertexalgebra består av ett oändligt-dimensionellt vektorrum, ett så kallat *Hilbertrum*, som kallas *tillståndsrummet*. Varje punkt i detta rum svarar mot ett givet tillstånd hos det fysikaliska systemet som vertexalgebran beskriver. I en vertexalgebra finns det för varje givet tillstånd ett fält som beskriver detta tillstånd. Detta kallas för *fält-tillståndskorrespondans*. Vi beskriver hur man kan definiera en vertexalgebra i termer av två operationer: λ-klammern och normalordningsprodukten. Dessa operationer gör att många beräkningar i vertexalgebrorna blir mer direkta än vad de skulle varit annars. Många vertexalgebror kan ses som kvantiseringar av Poisson-vertexalgebror. Man kan införa en parameter $\hbar$, och då $\hbar$ sätts till noll övergår vertexalgebran till en Poisson-vertexalgebra.

I kapitel 6 beskriver vi först en *kärve*[†]. Om man har en differentierbar mångfald, kan man alltid hitta ett litet område runt en given punkt på mångfalden som är i stort sett platt. Det området kallas för en koordinatavbildning. I en kärve tilldelar man ett givet algebraiskt objekt till varje koordinatavbildning. För att det ska bli en kärve ställs vissa krav på hur man gör tilldelningen, krav som inte alltid går att uppfylla.

[†] Den engelska termen är *sheaf*.

Vi är intresserade av kärven av vertexalgebror. För en godtycklig mångfald associerar vi en given vertexalgebra till varje koordinatavbildning. Vi är sedan intresserade av att se om vi kan göra detta konsistent – om vi har en kärve eller inte. Det visar sig att det går att göra detta om vi använder oss av en viss typ av vertexalgebror med en manifest supersymmetri. Vi kan då, för varje mångfald, konstruera en kärve av vertexalgebror. Den här konstruktionen kallas för *det kirala de Rham-komplexet*. Vi argumenterar för att det kirala de Rham-komplexet kan tolkas som en kvantisering av den supersymmetriska tvådimensionsionella sigmamodellen.

Vi visar även att det går att konstruera en kärve av vertexalgebror med två manifesta supersymmetrier, och att det finns ett samband mellan en sådan kärve och existensen av en Poisson-klammer på den underliggande mångfalden.

Det kirala de Rham-komplexet behandlas mer ingående i det avslutande kapitlet, där symmetrialgebror för olika typer av mångfalder diskuteras. Vi är speciellt intresserade av fallet då målmångfalden är en sexdimensionell *Calabi-Yau*-mångfald. Det är en typ av mångfalder som kan vara hur krökta eller kurviga som helst, men i ett specifikt avseende är de platta: de tillåter en *Ricci-platt* metrik. För en sådan mångfald visar vi att det kirala de Rham-komplexet innehåller två kommuterande kopior av den så kallade $N = 2$ superkonforma algebran med central laddning $c = 9$, och även att det innehåller en ickelinjär algebra som kallas för *Odake*-algebran.

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