Pricing Callable Bonds

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References
Abstract

In this paper, we value callable bonds. The interest rate process considered follows the Vasicek or the Cox-Ingersoll-Ross (CIR) models. Our analytical results are reached using the implicit Euler finite difference method. This paper is organized as follows: Section I introduces different interest rate models for fixed income securities. We study the European options in Section II. The finite difference method for solving the partial differentiation equations (PDE) is shown in Section III. The finite difference method applied for pricing callable bonds is investigated in Section IV. Finally, Section V concludes the paper. All the programming is done in MATLAB, and the corresponding code can be found in the Appendix.

Index Terms

Callable bonds, finite difference, Vasicek, CIR, European option.
I. Introduction

Financial innovation has developed rapidly during recent years. The traditional financial products offered by the corporations and governments to raise funds, such as equity, debt, preferred stock and convertibles, gave way to the new innovations which include options, bonds with embedded options, securitized assets, etc.

A callable bond is a type of bond which allows the issuing entity to retire the bond with a strike price at some date before the bond reaches the date of maturity [1]. We can view the callable bond as a combination of a non-option bond and a call option which is based on that bond. The writer of the call option is the holder of the bond, and the holder of the call option is the bond issuing corporation. Thus the price of a callable bond is the value of the straight bond less the value of the call option [2]. The value of the call option must converge to zero if the bond price is lower than the strike price or the bond close to maturity.

We should note that the call option of the callable bond is not a separable option in the sense that it could be traded in the open market. In other words, the bond and call option are always traded together. The issuer pays a higher coupon rate for the callable bond because of the call option. The bond will be retired at the call date if the interest rate in the market has gone down, which means the price of the bond has gone up. In this situation, the issuer will be able to refinance its debt (bond) at a cheaper level and it will be incentivized to call the bonds it originally issued. So, the value of the callable bond relates tightly to the interest rate. The callable bond is a choice for the issuers who want to avoid the risk of interest rate decreasing (bond price increasing). Each time an issuer use his right to call such a bond, the issuer is able to issue another callable bond with lower coupon (or higher price of zero-coupon bond). However, by comparing two zero-coupon bonds identical in all respects except that one of them is a callable bond, we may infer that the price of the callable bond must be lower than the price of the non-option zero-coupon bond to induce the investors to buy the callable
bond. In effect, the strategy of repeatedly calling and reissuing new callable bonds is like “marking to market” changes in interest rate [2].

In our paper, we will analyze the problem of pricing the zero-coupon bond based on the Vasicek and CIR interest rate models by the finite difference method.
II. INTEREST RATE MODELS

Zero-coupon bonds are the kinds of discount bonds which are sold at a price below the face value at time \( t \) and pay the face value at the time of maturity \( T \). We define the value of the zero-coupon bond as \( B(t, T) \). Then, we know that \( B(t, T) \) is the value of the bond at time \( t \) and \( B(T, T) = 1 \). By this definition, the bond value will keep increasing to 1 from time \( t \) to \( t = T \). When interest rate is constant, we can express the simplest model as

\[
B(t, T) e^{(T-t)R(t,T)} = 1
\]

(1)

where \( R(t,T) \) is the yield to maturity of the zero-coupon bond \( B(t, T) \).

First, we introduce the following basic definitions to the readers:

- **Instantaneous risk-free interest rate** (short-term interest rate) \( r(t) \):
  \[
  r(t) = \lim_{T \to t} R(t, T).
  \]
  (2)

- **Forward rate** \( f(t, T_1, T_2) \):
  \[
  f(t, T_1, T_2) = \frac{\ln B(t, T_1) - \ln B(t, T_2)}{T_1 - T_2},
  \]
  (3)

  it means the interest rate which is agreed at time \( t \) for the risk-free loan beginning at time \( T_1 \) and ending at time \( T_2 \).

- **Instantaneous forward rate** \( f(t, T) \):
  \[
  f(t, T) = f(t, T, T),
  \]
  (4)

  it is the limitation of forward rate \( f(t, T_1, T_2) \).

### A. Single-factor models

In single-factor model of interest rate, we assume that all security prices and rates depend on only one factor – the short-term interest rate. In our model, the short-term
interest rate can be given by the stochastic differential equation (SDE) as,

\[ dr(t) = \mu_r(t)dt + \sigma_r(t)dW(t) \]  

(5)

where \( \mu_r \equiv \mu_r(t, r(t)) \), \( \sigma_r \equiv \sigma_r(t, r(t)) \) are defined as real-value functions and \( W(t), t \geq 0 \) is the random Wiener process.

There are a lot of single-factor models for the short-term interest rate. Generally speaking, the different short-term interest rate SDE depends on the different kinds of functions \( \mu_r \) and \( \sigma_r \). Most of the SDE models are named after the people who invented them.

1) The Merton model (1973): It is one of the first works to propose a stochastic model for the short-term interest rate,

\[ dr(t) = \mu_r dt + \sigma_r dW(t) \]  

(6)

where \( \mu_r \) and \( \sigma_r \) are constant and the risk market premium \( \lambda \) is assumed to be constant.

2) The Vasicek model (1977): Vasicek use a mean-reverting Ornstein-Uhlenbeck process to model the short-term interest rate,

\[ dr(t) = K(\theta - r(t))dt + \sigma dW(t) \]  

(7)

where \( K \), \( \theta \) and \( \sigma \) are positive constants and he assume the risk market premium \( \lambda \) is constant.

This model fit well for the zero-coupon bonds and several European-style interest rate derivatives. However, with very low probability, this model has the undesirable property of allowing negative interest rate.

3) The Brennan and Schwartz model (1980): Brennan and Schwartz were forerunner on the area of pricing the options embedded bonds. The model of the short-term interest
rate which they are given as,

\[ dr(t) = K(\theta - r(t))dt + \sigma r(t)dW(t) \]  

(8)

where \( K, \theta \) and \( \sigma \) are positive constants and the risk market premium \( \lambda \) is assumed to be constant.

In this model, the interest rate move as a geometric Brownian motion which is a mean-reverting proportional process to price the conversion options.

4) The Marsh Rosenfeld model (1983): The Marsh-Rosenfeld model corresponds to the constant-elasticity-of-variance process and is expressed as,

\[ dr(t) = -Kr(t)dt + \sigma r(t)^\alpha dW(t) \]

(9)

where \( K, \alpha \) and \( \sigma \) are positive constants and the model assumes the risk market premium \( \lambda \) is constant.

This model is considered as an alternative process for the short-term interest rate among others.

5) The Cox, Ingersoll and Ross (CIR) model (1985): The CIR model used the mean-reverting square-root process to describe the movements of short-term interest rate. It is given by,

\[ dr(t) = K(\theta - r(t))dt + \sigma \sqrt{r(t)}dW(t) \]

(10)

where \( K, \theta \) and \( \sigma \) are positive constants. The market risk premium at equilibrium is expressed as,

\[ \lambda(r, t) = \lambda \sqrt{r(t)}. \]

(11)

The CIR model is extensible to several factors and make sure the interest rates to be strictly positive. Based on this model, we can derive the closed-form solutions for zero-coupon bonds and some European-style interest rate derivatives.
All of the above models cannot be calibrated with yield curves. Therefore, some new models were introduced to overcome these problems and are consistent with the above models.

6) The Hull-White model (1993): The Hull-White model leads to the generalized Vasicek and CIR models and is given by,

\[ dr(t) = ((\theta(t) - K(t))r(t))dt + \sigma(t)r^3(t)dW(t) \]  

where all the coefficients in the model are functions of time and can be used to calibrate exactly the model to current market prices. The market risk premium is expressed as,

\[ \lambda(r, t) = \lambda r^\nu, \text{ with } \lambda \geq 0 \text{ and } \nu \geq 0. \]  

The disadvantage of this model is that we cannot derive the analytical result of the bond option price. However, we can derive the numerical result by the other methods, such as finite difference method.

7) The Lognormal model (1987): The Lognormal model is different from all the other models which we mentioned before. The previous models above model the interest rate as Gaussian process. Using Gaussian process, the interest rates can be negative with a positive probability and this implies arbitrage opportunities. However, the Lognormal model does not have this problem and can be expressed as,

\[ d \log(r(t)) = (\theta(t) - K \log(r(t)))dt + \sigma_r dW(t) \]  

This incorporates the mean reversion feature of the interest rate.

In our paper, we are pricing the zero-coupon callable bond based on the Vasicek and CIR short-term interest rate models.
III. OPTIONS

Options are the kinds of derivative financial instrument. They are sold by the option writer to the option holder by contract. In the contracts (Options), the buyer (holder) has the right, but not the obligation, to buy (call) or sell (put) a security or other financial asset at an agreed price (the strike price) during a certain period of time (American options) or on a specific (exercise) date (European options). Options can be used in many different ways as extremely versatile securities. They can be used to avoid the currency exchange risk and trading risk. Traders use options to speculate, which is a relatively risky practice, while hedgers use options to reduce the risk of holding an asset.

A. Value the option

We denote the value of the option by \( V(S, t) \). It means the value \( V \) is a function of the current value of the underlying asset, \( S \), and time parameter, \( t \). At time \( t_1 \), we know that \( V_{t_1} = V(S_{t_1}, t_1) \). \( \sigma \) is the volatility of the underlying asset. The exercise price of the option is noted by \( X \). \( T \) is the expiry time and also the interest rate is \( r \) as we mentioned before.

\[ 1) \text{Black-Scholes equation:} \] We will introduce the Black-Scholes equation (model) before we price the options. Black and Scholes started the serious study of the theory of option pricing. All further advances in this field have been extensions and refinements of the original idea expressed in [3]. In our work, we follow the assumptions which involved in the derivation of the Black-Scholes equation.

- We know the risk-free interest rate \( r(t) \) and the volatility of asset \( \sigma \) are functions of time \( t \) over the life of the options and zero-coupon bonds.
- The market falls into the efficient market hypothesis. It means that the market is liquid, has price-continuity, is fair, no arbitrage opportunities and provides all players with equal access to available information. It implies that the Black-Scholes
model assume no transaction cost, underlying security is perfectly divisible and the short selling with full use of proceeds is possible.

- The asset price follows the lognormal random process.
- No dividend is paid during the life of the options.

Based on [4], we assume that there is a general derivative $V$ whose value is a function of the value of the underlying security $S(t)$. $S(t)$ follows the stochastic process,

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t),$$

(15)

where the constant coefficient $\alpha$ is the average growth rate of the underlying security and $\sigma$ is the volatility.

We suppose that the derivative can be traded on an ideal market and its process has the form $V(t, S(t))$. We obtain the following theorem.

**Theorem 1:** From the no-arbitrage condition, the pricing function of $V(t, S(t))$ is the function when $V(t, S(t))$ is the solution of the following boundary PDE problem in the domain $[0, T] \times r_{\text{max}},$

$$\frac{\partial V(t, S(t))}{\partial t} + rS(t)\frac{\partial V(t, S(t))}{\partial S(t)} + \frac{1}{2}S(t)^2\sigma^2(t, S(t))\frac{\partial^2 V(t, S(t))}{\partial S(t)^2} - rV(t, S(t)) = 0$$

(16)

with the boundary at $t = T$,

$$V(T, S(T)) = \Phi(S(T))$$

(17)

where $\Phi(S(t))$ is a simple claim of $S(t)$ at time $t$.

The function $V(t, S(t))$ will be the price of the European option if we consider $\Phi(S(T))$ to be the payoff function at time $T$. Meanwhile, we have to confirm the boundary and final conditions in order to have a unique solution when we solve the above PDE problem.
2) **European options:** The European options are the options that can only be exercised at expiration. The value of the options at expiration date are,

\[
V(S_T, T) = \begin{cases} 
(S_T - X)^+ & \text{call options} \\
(X - S_T)^+ & \text{put options} 
\end{cases}
\]  

(18)

where \((a - b)^+ = \max(0, a - b)\).

Specially, we just think about the pricing problem of European call option in our paper. The value of the call options are denoted by \(C(t, S)\) with the expiry date \(T\) and strike price \(X\). We derive that the value of the call option at time \(t = T\) can be expressed as the payoff function,

\[
C(t, S) = \max(S - X, 0)
\]  

(19)

This is the final condition of option pricing problem at time \(T\). Meanwhile, we also need to find out the boundary conditions for our problem. For general call options, the boundary conditions can be find out at \(S = 0\) and \(S \rightarrow \infty\). However, the maximum price of the zero-coupon bonds is their face value. It means the boundary conditions of the European call options based on the zero-coupon bonds are applied at \(S = 0\) and \(S = B_T\). In [5], we know the price of call options will remain zero, if \(S\) is ever zero. So, we get the boundary conditions as,

\[
C(t, 0) = 0, \quad \text{when} \quad S = 0,
\]  

(20)

\[
C(t, S) = B_T - X, \quad \text{when} \quad S = B_T.
\]  

(21)

With the conditions above, we can derive the Black-Scholes’ value of the European call options.
IV. Finite Difference Methods

The goal of our paper is to develop an accurate and efficient numerical method to price the zero-coupon callable bonds in quantitative finance based on our knowledge of finance and PDE. However, it is very difficult or even impossible to find the exact closed-form solutions for the PDE problems. Also, it may be very difficult to calculate even if we can find the closed-form solutions. For these reasons, we have to involve the approximate methods. There are several commonly used approximate methods as followings which mentioned in [6],

- Lattice method. It includes the binomial and trinomial models, assuming that the underlying stochastic process is discrete i.e. the underlying asset ”jumps” to a finite number of values (each associated with a certain probability) with a small advancement in time.

- Monte Carlo method. It is a class of computational algorithms that rely on repeated random sampling to compute their results. It is based on the law of great number. Monte Carlo methods are often used in simulating physical and mathematical systems. These methods are most suited to calculate by a computer and tend to be used when it is infeasible to compute an exact result with a deterministic algorithm. This method is also used to complement the theoretical derivations.

- Finite Difference method. In mathematics, finite-difference methods are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives. They consist of discretizing the PDEs and the given boundary conditions to form a set of difference equations and can be solved either directly or iteratively. These methods have a long history and have been applied for more than 200 years to approximate the solutions of PDEs in physical sciences and engineering.

In our paper, we use the finite difference methods to solve the PDEs of pricing the zero-coupon callable bonds.
### A. Basic knowledge of numerical differentiation

Supposing, we have a real-valued function of a real variable, such as [7],

\[ y = f(x). \] \hspace{1cm} (22)

The things which we are most interested are how to find the approximations to the first and second derivatives of the function \( f(x) \). In general, we do not know the form of the function \( f(x) \). So, we can not calculate the derivatives of \( f(x) \) analytically. In this situation, we involve the methods to numerical approximations. For example, we want to approximate the first derivative of \( f(x) \) at point \( a \) and \( h \) is a small (usually) positive constant number. The first derivative of function \( f(x) \) at point \( a \) can be given by,

- Centred difference formula: \( f'(a) = \frac{f(a+h) - f(a-h)}{2h} \);
- Forward difference formula: \( f'(a) = \frac{f(a+h) - f(a)}{h} \);
- Backward difference formula: \( f'(a) = \frac{f(a) - f(a-h)}{h} \);

In our paper, we use the following notations for short,

\[ D_0 f(a) = \frac{f(a + h) - f(a - h)}{2h}, \] \hspace{1cm} (23)

\[ D_+ f(a) = \frac{f(a + h) - f(a)}{h}, \] \hspace{1cm} (24)

\[ D_- f(a) = \frac{f(a) - f(a - h)}{h}, \] \hspace{1cm} (25)

**Lemma 1:** The centred difference formula gives a second order approximation to the first derivative if \( h \) is small enough and if \( f(x) \) has continuous derivatives up to 3.
Proof: We use the Taylor’s expansion to expand \( f(x) \) at point \( a \) as,

\[
f(a \pm h) = f(a) \pm hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(\eta_{\pm})
\]  

(26)

where

\[
\eta_- \in (a - h, a), \text{ and } \eta_+ \in (a, a + h).
\]  

(27)

Substituting equation (26) into equation (23), we derive,

\[
D_0 f(a) = f'(a) + \frac{h^2}{3!} \left( \frac{f'''(\eta_+) + f'''(\eta_-)}{2} \right).
\]  

(28)

From the equation above, we proofed lemma 1.

In the same way, we can say that the forward and backward difference formulas give first order approximation to the first derivative of \( f(x) \) at point \( a \), as,

\[
D_+ f(a) = f'(a) + \frac{h}{2} f''(\eta_+), \quad \eta_+ \in (a, a + h)
\]  

(29)

and

\[
D_- f(a) = f'(a) - \frac{h}{2} f''(\eta_-), \quad \eta_- \in (a - h, a).
\]  

(30)

The one side schemes place low continuity constraints on the function \( f(x) \) because they are first order accurate. We just need to assume that its second order derivative is continuous.

For the second derivative of \( f(x) \) at point \( a \), we use the following three points formula [8],

\[
D_+ D_- f(a) \equiv \frac{f(a - h) - 2f(a) + f(a + h)}{h^2}.
\]  

(31)

It is a second order approximation to the second order derivative of \( f(x) \) at point \( a \) and we assume that the function \( f(x) \) has continuous derivatives up to and including
order 4. The error of discretisation is given by,

$$D_+ D_- f(a) = f''(a) + \frac{h^4}{4!} (f^{(iv)}(\eta_+) + f^{(iv)}(\eta_-)).$$

(32)

B. Types of finite difference methods

We need to consider the properties of consistency, stability and convergence of the scheme when we are using the finite difference methods. The details of the proof of these properties can be found in [7] and [9]. The main finite difference methods are as following,

- Explicit finite difference method. In this method, we use the forward difference formula.
- Implicit finite difference method. In this method, we use the backward difference formula.
- Crank-Nicholson finite difference method. In this method, we use the centred difference formula.

In our paper, we use the implicit Euler scheme to solve our PDEs problem of pricing the callable bond.
V. PRICING CALLABLE BONDS

The callable bonds are the zero-coupon bonds embedded call option (European).

Generally, we can say that,

\[ \text{Price of callable bond} = \text{Price of option free zero - coupon bond} - \text{Price of embedded option}. \] (33)

Next, we will price the option-free zero-coupon bonds and callable bonds separately.

A. Pricing zero-coupon bonds with the CIR model

We use the implicit method to solve this PDEs problem. We discretize the interest rate \( r \) into \( N \) equally spaced units of \( \delta r \), and the time variable \( t \) into \( M \) equally spaced units of \( \delta t \),

\[ r_j = j\delta r, \quad j = 0, \ldots, N; \] (34)
\[ t_i = i\delta t, \quad i = 0, \ldots, M. \] (35)

1) Pricing zero-coupon bonds: In this part, we price the zero-coupon bonds based on the CIR model.

When \( j = N \), \( r_j = r_{\text{max}} \). Please note that we are using the engineering’s time. It means that \( t_i = T \) when \( i = 0 \).

Letting \( a = K\theta \) and \( K = b \), the pricing problem of zero-coupon bond can be expressed as,

\[ -\frac{\partial B(r(t), t)}{\partial t} + \frac{1}{2}r(t)^2\sigma^2\frac{\partial^2 B(r(t), t)}{\partial r(t)^2} + (a - br(t))\frac{\partial B(r(t), t)}{\partial r(t)} - r(t)B(r(t), t) = 0, \] (36)

The boundary conditions were investigated thoroughly in [11] and can be expressed
as,

\[
B(\max, t) = 0, \; t > 0; \tag{37}
\]

\[
- \frac{\partial B(0, t)}{\partial t} + a \frac{\partial B(0, t)}{\partial r} = 0, \; t > 0 \text{ and } a > 0; \tag{38}
\]

\[
B(r, 0) = 1 \text{ (facevalue)}, \; 0 < r < \max. \tag{39}
\]

Notice, we do not know the bond value at \( r = 0 \) and use the implicit Euler scheme for the Black-Scholes equation as,

\[
- \frac{B_{j+1}^{i+1} - B_j^i}{\delta t} + \frac{1}{2} \sigma^2 r_j^{i+1} D_+ D_- B_j^{i+1} + (a - br_j^{i+1})D_0 B_j^{i+1} - r_j B_j^{i+1} = 0, \; 1 \leq j \leq N - 1; \tag{40}
\]

where \( B_j^i = B(r_j, t_i) \).

This equation can be rewritten as,

\[
- \frac{B_{j+1}^{i+1} - B_j^i}{\delta t} + \frac{1}{2} \sigma^2 r_j^{i+1} \frac{B_j^{i+1} - 2B_j^{i+1} + B_{j+1}^{i+1}}{(\delta r)^2} + (a - br_j^{i+1}) \frac{B_j^{i+1} - B_{j-1}^{i+1}}{2\delta r} - r_j B_j^{i+1} = 0, \; 1 \leq j \leq N - 1. \tag{41}
\]

Easily, we have,

\[
B_j^i = a_j B_{j-1}^{i+1} + b_j B_j^{i+1} + c_j B_{j+1}^{i+1}, \; \text{with } j = 1, \cdots, N - 1 \text{ and } i = 1, \cdots, M, \tag{42}
\]

where

\[
a_j = \frac{1}{2} a \frac{\delta t}{\delta r} - b \delta t \frac{1}{2} (j - 1) - \sigma^2 \frac{\delta t}{\delta r} (j - 1); \tag{43}
\]

\[
b_j = 1 + \delta r (j - 1) \delta t + \sigma^2 \frac{\delta t}{\delta r} (j - 1); \tag{44}
\]

\[
c_j = b \delta t \frac{1}{2} (j - 1) - a \delta t \frac{1}{2\delta r} - \sigma^2 \delta t \frac{1}{2\delta r} (j - 1). \tag{45}
\]

For boundary conditions, we have,
Fig. 1: Price of 5, 7, 10 and 20 years zero-coupon bonds with expiration for different interest rate, based on the CIR model.

\[
-\frac{B_{j}^{i+1} - B_{j}^{i}}{\delta t} + a \frac{B_{j+1}^{i+1} - B_{j}^{i+1}}{\delta r} = 0, \quad \text{when} \ j = 0 \ (r = 0). \tag{46}
\]

\[
B_{N}^{i} = 0, \quad \text{when} \ j = N \ (r = r_{\text{max}}). \tag{47}
\]

\[
B_{j}^{0} = \text{face value}, \quad \text{when} \ t = 0 \ (\text{at expiration date}). \tag{48}
\]

Fig. 1 depicts the value of 5 years zero-coupon bond with face value 1. We suppose that the parameters in CIR model as \( a = b \delta, \ b = 0.54958 \) and \( \delta = 0.38757 \). It shows the interest rate changes from 0 to 350% and the bond price decrease from face value.
to 0. It fit our knowledge well that the bond price decrease when the time of maturity increase.

2) Pricing bonds with European call options: The bonds with European call options just have a single possible call date. We denote the call date by $\tau_c$ and define the engineering time $t_c = T - \tau_c$. Meanwhile, we denote the notice time as $\tau_n$ and the engineering time as $t_n = T - \tau_n$. Also, we denote the call price is $X$. In general, we assume that it is optimal for the issuer to minimize the value of the contract. It means the issuer will exercise the options if the price of the callable bonds exceed the exercise price at the notice date. Otherwise, the issuer will give up the right of the call options and the callable bonds price are equal to the price of non-option bonds.

We call the interest rate $r_b$ is the “break-even” interest rate if the issuer is indifferent between exercising the options or not doing so with the interest rate $r_b$ at the notice date $\tau_n$. In [10], we can find the “break-even” interest rate by,

$$XB(r_b, t_n - t_c) - P(r_b, t_n^-) = 0,$$

(49)

where $B(r_b, t_n - t_c)$ is the value at time $t_n$ of the zero-coupon bond maturing at $t_c$ with face value and satisfies the equation (36), $t_n^+$ ($t_n^-$) means the time immediately after (before) the notice date and $P(r_b, t_n^-)$ denote the price of the callable bonds an instant before the notice date.

For finding the “break-even” interest rate, i.e. solving the equation (49), we know the value of callable bonds $P(r_b, t_n^-)$ at the time before the notice date are equal to the value of non-option bonds. The solution of equation (49) is the cross point between the curve of non-option bonds $P(r_b, t_n^-)$ and the curve of zero-coupon bonds $B(r_b, t_n - t_c)$ product the exercise price $X$.

Fig. 2 shows the “break-even” interest rate for callable bonds with strike price 0.8 and notice date at 5 years before maturity. We suppose the time between the call date and the notice date, $t_n - t_c$, is 0.1 years. The “break-even” interest rate is $r_b = 0.11$. 
For pricing the callable bonds, we should update the callable bonds price after we find the “break-even” interest rate. We price the callable bonds at the notice date because the interest rate at notice date determine the price of the zero-coupon callable bonds. At this situation, we can price the zero-coupon callable bonds at the notice date as \[ P(r_b, t_n^+) = \begin{cases} X B(r, t_n - t_c) & \text{if } r \leq r_b \\ P(r_b, t_n^-) & \text{if } r \geq r_b \end{cases} \] (50)

The process for pricing the zero-coupon callable bonds is,

- Step 1, finding the “break-even” interest rate, \( r_b \), by solving the equation (49);
- Step 2, pricing the callable bonds at notice date as equation (50) for different range of \( r \).

Fig. 3 plots the value of zero-coupon callable bonds at notice date 3 years before
the maturity. We suppose the strike price of the embedded call options is 0.8, the time between the call date and the notice date is $t_n - t_c = 0.1$ year and the face value of the callable bond is 1. We can see that the “break-even” interest rate is $r_b = 0.161$. Meanwhile, the parameters of CIR model are set as mentioned before.

Fig. 4 illustrates the value of zero-coupon callable bonds at notice date 2 years before the maturity. We derive that the “break-even” interest rate is $r_b = 0.161$. The call price is 0.7 and the time between the call date and the notice date is 2 month, i.e. $t_n - t_c = 0.167$. The bond will pay 1 at the maturity and the others parameters as mentioned before.
Fig. 4: The value of zero-coupon callable bonds with exercise price 0.7 at the notice date which is 2 years before maturity, based on the CIR model.

### B. Pricing zero-coupon bonds with Vasicek model

1) Vasicek model: Based on the Vasicek interest rate model and the assumption we mentioned before, the pricing problem of zero-coupon bond can be expressed as,

\[-\frac{\partial B(r(t), t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 B(r(t), t)}{\partial r(t)^2} + (a - br(t)) \frac{\partial B(r(t), t)}{\partial r(t)} - r(t) B(r(t), t) = 0, \tag{51}\]

where $a = K\theta$ and $b = K$. One of the disadvantages of the Vasicek model is that the interest rate may be negative in this model. Based on the practical experience, we know that the equation (51) holds the following boundary conditions when $r(t)$ is not negative.
When \( r(t) = 0 \), (51) can be transform to,
\[
- \frac{\partial B(0, t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 B(r(t), t)}{\partial r(t)^2} + a \frac{\partial B(0, t)}{\partial r} = 0, \quad t > 0; \tag{52}
\]

When \( r(t) = r_{\text{max}} \) and \( t = 0 \), we get,
\[
B(r_{\text{max}}, t) = 0, \quad t > 0; \tag{53}
\]
\[
B(r, 0) = 1 \text{ (facevalue), } 0 < r < r_{\text{max}}. \tag{54}
\]

Using the implicit Euler scheme for the Black-Scholes equation, we derive that,
\[
- \frac{B_{j}^{i+1} - B_{j}^{i}}{\delta t} + \frac{1}{2} \sigma^2 \frac{B_{j-1}^{i+1} - 2B_{j}^{i+1} + B_{j+1}^{i+1}}{(\delta r)^2} + (a - b_{j}^{i+1}) \frac{B_{j}^{i+1} - B_{j-1}^{i+1}}{2\delta r} - r_{j}B_{j}^{i+1} = 0,
\]
\[
1 \leq j \leq N - 1; \tag{55}
\]

where \( B_{j}^{i} = B(r_{j}, t_{i}) \).

We can rewrite the equation (55) as,
\[
- \frac{B_{j}^{i+1} - B_{j}^{i}}{\delta t} + \frac{1}{2} \sigma^2 \frac{B_{j-1}^{i+1} - 2B_{j}^{i+1} + B_{j+1}^{i+1}}{(\delta r)^2} + (a - b_{j}^{i+1}) \frac{B_{j}^{i+1} - B_{j-1}^{i+1}}{2\delta r} - r_{j}B_{j}^{i+1} = 0,
\]
\[
1 \leq j \leq N - 1.
\]

Then, we have,
\[
B_{j}^{i} = a'_{j}B_{j-1}^{i+1} + b'_{j}B_{j}^{i+1} + c'_{j}B_{j+1}^{i+1}, \quad \text{with } j = 1, \ldots, N - 1 \text{ and } i = 1, \ldots, M, \tag{57}
\]

where
\[
a'_{j} = \frac{1}{2} a \frac{\delta t}{\delta r} - \frac{1}{2} b \delta t(j - 1) - \frac{1}{2} \sigma^2 \frac{\delta t}{(\delta r)^2}; \tag{58}
\]
\[
b'_{j} = 1 + \delta r(j - 1) \delta t + \sigma^2 \frac{\delta t}{(\delta r)^2}; \tag{59}
\]
\[
c'_{j} = \frac{1}{2} b \delta t(j - 1) - a \delta t \frac{1}{2\delta r} - \sigma^2 \delta t \frac{1}{2(\delta r)^2}. \tag{60}
\]

For boundary conditions, we have,
Fig. 5 depicts the value of 5, 7, 10 and 20 years zero-coupon bonds with face value 1, based on the Vasicek model. We suppose that the parameters as $a = 0.0038$, $b = 0.025$.

\begin{equation}
- \frac{B_{j+1}^i - B_j^i}{\delta t} + \frac{1}{2} \sigma^2 \frac{B_{j+1}^i - 2B_j^i + B_{j-1}^i}{(\delta r)^2} + a \frac{B_{j+1}^i - B_j^i}{\delta r} = 0, \text{ when } j = 0 \ (r = 0).
\end{equation}

\begin{equation}
B_N^j = 0, \text{ when } j = N \ (r = r_{\text{max}}).
\end{equation}

\begin{equation}
B_0^j = \text{face value}, \text{ when } t = 0 \ (\text{at expiration date}).
\end{equation}
Fig. 6: The value of zero-coupon callable bonds with exercise price 0.8 at the notice date which is 3 years before maturity, based on Vasicek model.

and $\delta = 0.0126$. It shows the interest rate changes from 0 to 100% and the bond price decrease from face value to 0. The bond value based on the Vasicek model decrease more quickly than the value in Fig. 1. Comparing to our practical knowledge, we can say that the CIR model fit the practical situation much better.

2) Pricing bonds with European call options: We are pricing the zero-coupon bonds with European call options by the same method which we mentioned above. We suppose that the Vasicek parameters $a = 0.0038$, $b = 0.025$ and $\delta = 0.0126$ [13].

Fig. 6 shows the value of zero-coupon callable bonds at notice date 3 years before the maturity as Fig. 3. We suppose the strike price of the embedded call option is 0.8, the time between the call date and the notice date is $t_n - t_c = 0.1$ year and the face value of the callable bond is 1. The “break-even” interest rate in this situation is $r_b = 0.09$ which is below the $r_b$ in Fig. 3. We can see that the value of the zero-coupon callable
bonds based on the Vasicek model drop steeply than the value in Fig. 3. The bonds price on the notice date are higher than the price in Fig. 3.

![Graph](image)

Fig. 7: The value of zero-coupon callable bonds with exercise price 0.7 at the notice date which is 2 years before maturity, based on Vasicek model.

Fig. 7 plots the value of zero-coupon callable bonds at notice date 2 years before the maturity as Fig. 4. The “break-even” interest rate is $r_b = 0.20$. The call price is 0.7 and the time between the call date and the notice date is 2 month, i.e. $t_n - t_c = 0.167$. The bonds face value are 1. Comparing to Fig. 4, the value of the bonds decrease more quickly and the price on the notice date is a little bit higher based on the Vasicek interest rate model.

The reason of the difference between the Vasicek and CIR models is that the Vasicek interest rate may actually become negative unlike CIR. Under Vasicek model, the interest rate is normally distributed and so there is a probability for them being negative. For the CIR model this density function has the property of a Gamma distribution, not allowing
for negative interest rates. It means the random term becomes increasingly smaller as the rate approaches zero in CIR.
VI. Conclusion

In this paper, we have considered pricing callable bonds. We have investigated the problem of how to price zero-coupon European callable bonds using the implicit Euler finite difference method. Our interest rate models are based on the Vasicek and CIR interest rate processes. Moreover, we derive analytical results for pricing zero-coupon European callable bonds. From the derived expressions and numerical figures, the price of the callable bonds drop more quickly using the Vasicek model than the CIR model. The interest rate is quite low in reality. It means that the CIR model is better than the Vasicek model since the Vasicek model allows negative interest rates [14]. Our result can be obtained easily by Matlab.
APPENDIX A

MATLAB PROGRAMME FOR PRICING THE ZERO-COUPON BOND WITHOUT OPTION

(Fig. 1)

clear, clc;

Parameter Initialization

Bondprice = 1;

$dr = 0.01;$

$dt = 0.01;$

$r_{max} = 3.5;$

$T = 5;$

$a = 0.54958 \times 0.034847;$

$b = 0.54958;$

$\delta = 0.38757;$

$M = T/dt;$ Number of Time Intervals;

$N = r_{max}/dr;$ Number of Stock Price Intervals

$\text{for } j = 1 : N \text{ } f(j, 1) = Bondprice; \text{ end; }$

$r=\text{max for } i = 2 : M + 1$

$f(N, i) = 0;$

$\text{end; }$

Calculate $aa$, $bb$, $cc$ (parameters) for Model

$\text{for } j = 1 : N - 1$

$aa(j) = 0.5 \times a \times dt/dr - b \times dt \times 0.5 \times (j - 1) - \delta^2 \times 0.5 \times dt/dr \times (j - 1);$

$bb(j) = dr \times (j - 1) \times dt + 1 + \delta^2 \times dt/dr \times (j - 1);$

$cc(j) = b \times dt \times 0.5 \times (j - 1) - a \times dt \times 0.5/dr - \delta^2 \times dt \times 0.5/dr \times (j - 1);$

$\text{end; }$

$\text{for } j = 1 : N - 2$

$cc(j) = b \times dt \times 0.5 \times (j - 1) - a \times dt \times 0.5/dr - \delta^2 \times dt \times 0.5/dr \times (j - 1);$
Calculations (Solve System $Ax=d$)

Set up matrix $A$

```matlab
for j = 2 : N
    A(j, j - 1) = aa(j - 1);
    A(j, j) = bb(j - 1);
end;
for j = 2 : N - 1
    A(j, j + 1) = cc(j - 1);
end
A(1, 1) = 1 + a * dt/dr;
A(1, 2) = -a * dt/dr;
A = inv(A);
```

Solve System

```matlab
for i = 1 : 1 : M + 1
    f(:, i + 1) = A * f(:, i);
end;
```

```matlab
j = 36 : 1 : N;
plot(j * dr, f(j, M + 1));
```

APPENDIX B

MATLAB PROGRAMME FOR PRICING THE ZERO-COUPON BOND BASED ON THE VASICEK MODEL (FIG. 5)

```matlab
clear, clc;

Parameter Initialization

Bondprice = 1;

dr = 0.01;

dt = 0.01;
```
\( r_{\text{max}} = 2.5; \)
\( T = 3; \)
\( a = 0.025 \times 0.15339; \)
\( b = 0.025; \)
\( \delta = 0.0126; \)
\( M = T/dt; \) Number of Time Intervals;
\( N = r_{\text{max}}/dr; \) Number of Stock Price Intervals

\[
\text{for } j = 1 : N \ f(j, 1) = \text{Bondprice}; \quad \text{end};
\]
\[
r = \max \text{ for } i = 2 : M + 1
\]
\[
f(N, i) = 0;
\]
\[
\text{end};
\]
Calculates parameters for Model

\[
\text{for } j = 1 : N - 1
\]
\[
aa(j) = 0.5 \times a \times \frac{dt}{dr} - 0.5 \times b \times \frac{dt}{(j - 1)} - 0.5 \times \delta^2 \times \frac{dt}{((dr)^2)};
\]
\[
bb(j) = \frac{dr \times (j - 1) \times dt + 1 + \delta^2 \times dt}{((dr)^2)};
\]
\[
cc(j) = 0.5 \times b \times \frac{dt}{(j - 1)} - 0.5 \times a \times \frac{dt}{dr} - 0.5 \times \delta^2 \times \frac{dt}{((dr)^2)};
\]
\[
\text{end};
\]

\[
\text{for } j = 1 : N - 2
\]
\[
cc(j) = b \times \frac{dt}{(j - 1)} - a \times \frac{dt}{dr} - \delta^2 \times \frac{dt}{((dr)^2)};
\]
\[
\text{end};
\]
Calculations (Solve System \( Ax = d \))

Set up matrix \( A \)

\[
\text{for } j = 2 : N
\]
\[
A(j, j - 1) = aa(j - 1);
\]
\[
A(j, j) = bb(j - 1);
\]
\[
\text{end};
\]
\[
\text{for } j = 2 : N - 1
\]
\begin{verbatim}
A(j, j + 1) = cc(j − 1);
end;
A(1, 1) = 1 − 1/2 * delta^2 * dt/((dr)^2) + a * dt/dr;
A(1, 2) = delta^2 * dt/((dr)^2) − a * dt/dr;
A(1, 3) = −1/2 * delta^2 * dt/((dr)^2);
A = inv(A);
Solve System
for i = 1 : 1 : M + 1
f(:, i + 1) = A * f(:, i);
end;
j = 36 : 1 : N;
plot(j * dr, f(j, M + 1));
\end{verbatim}
REFERENCES


