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Twisting and Gluing

On Topological Field Theories, Sigma Models and Vertex Algebras

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ACTA
UNIVERSITATIS
UPSALIENSIS
UPPSALA
2012

ISSN 1651-6214
ISBN 978-91-554-8379-1
urn:nbn:se:uu:diva-173225

Dissertation presented at Uppsala University to be publicly examined in Häggsalen, Ångströmlaboratoriet, Lägerhyddsvägen 1, Uppsala, Friday, June 8, 2012 at 13:15 for the degree of Doctor of Philosophy. The examination will be conducted in English.

Abstract
Källén, J. 2012. Twisting and Gluing: On Topological Field Theories, Sigma Models and Vertex Algebras. Acta Universitatis Upsaliensis. *Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology* 938. 95 pp. Uppsala. ISBN 978-91-554-8379-1.

This thesis consists of two parts, which can be read separately. In the first part we study aspects of topological field theories. We show how to topologically twist three-dimensional N=2 supersymmetric Chern-Simons theory using a contact structure on the underlying manifold. This gives us a formulation of Chern-Simons theory together with a set of auxiliary fields and an odd symmetry. For Seifert manifolds, we show how to use this odd symmetry to localize the path integral of Chern-Simons theory. The formulation of three-dimensional Chern-Simons theory using a contact structure admits natural generalizations to higher dimensions. We introduce and study these theories. The focus is on the five-dimensional theory, which can be understood as a topologically twisted version of N=1 supersymmetric Yang-Mills theory. When formulated on contact manifolds that are circle fibrations over a symplectic manifold, it localizes to contact instantons. For the theory on the five-sphere, we show that the perturbative part of the partition function is given by a matrix model.

In the second part of the thesis, we study supersymmetric sigma models in the Hamiltonian formalism, both in a classical and in a quantum mechanical setup. We argue that the so called Chiral de Rham complex, which is a sheaf of vertex algebras, is a natural framework to understand quantum aspects of supersymmetric sigma models in the Hamiltonian formalism. We show how a class of currents which generate symmetry algebras for the classical sigma model can be defined within the Chiral de Rham complex framework, and for a six-dimensional Calabi-Yau manifold we calculate the equal-time commutators between the currents and show that they generate the Odake algebra.

Keywords: Topological field theory, Chern-Simons theory, Contact geometry, Sigma models, Poisson vertex algebras, Vertex algebras, String theory

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ISSN 1651-6214
ISBN 978-91-554-8379-1
urn:nbn:se:uu:diva-173225 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-173225)

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I J. Källén , *Cohomological localization of Chern-Simons theory*, Journal of High Energy Physics 1108, 008 (2011).

II J. Källén , M. Zabzine, *Twisted supersymmetric 5D Yang-Mills theory and contact geometry*, `arXiv:1202.1956 [hep-th]`. Accepted for publication in Journal of High Energy Physics.

III J. Ekstrand, R. Heluani, J. Källén and M. Zabzine, *Non-linear sigma models via the chiral de Rham complex*, Advances in Theoretical and Mathematical Physics 13, 1221 (2009).

IV J. Ekstrand, R. Heluani, J. Källén and M. Zabzine, *Chiral de Rham complex on special holonomy manifolds*, `arXiv:1003.4388 [hep-th]`. Under review in Communications in Mathematical Physics.

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Part I:

Chern-Simons type theories and localization

1. Introduction

Quantum field theories were developed in order to describe and understand different aspects of Nature, for example elementary particles and their interactions. In the late 1980's, an interesting and rich interplay between a special type of quantum field theories and various branches of pure mathematics, for instance low dimensional topology, began with the work of Witten [84, 85]. These special types of quantum field theories are called *topological quantum field theories*. In order to define a quantum field theory, we must in general specify a space where the theory "lives". Topological quantum field theories are characterized by the fact that the correlation functions of the theory only depend on global features of the space where the theory is defined. In particular, the result of any calculation will be independent of the choice of metric on this space. Therefore, these theories calculate topological invariants of the spaces where the theories are defined and such objects are interesting from a purely mathematical point of view. Apart from producing interesting mathematics, topological quantum field theories can be seen as a simple class of quantum field theories and many times are exactly solvable. In addition, they are closely related to more physically interesting supersymmetric quantum field theories. Therefore, the study of these simpler types of quantum field theories can also be motivated by the fact that they can help us understand more complicated, and physically more interesting, quantum field theories. An example of a powerful quantum field theory method which was first understood in the setting of topological field theories, and later applied to supersymmetric quantum field theories, is the method of localization. This method of performing calculations is the main theme of this part of the thesis. For an excellent review of topological field theories, see [17].

A prime example of a topological quantum field theory is Chern-Simons theory, which is defined on a three-dimensional space. It produces topological invariants of three-manifolds, knots and, more generally, links. Quantum field theories are typically defined using the so called path integral, an object which in general is not rigorously defined mathematically. The main advantage of having a path integral formulation of three-manifold and knot invariants is that one can apply different quantum field theory techniques to the path integral and in this way understand different aspects of the invariants which are not obvious from a rigorous mathematical definition. For example, studying Chern-Simons theory in a perturbative expansion one can extract so called perturbative invariants of the three-manifold. Choosing different gauge fixings for the path integral leads to different formulations of the perturbative invariants. For a review of this aspect of Chern-Simons theory, see for example [51]. Another quantum field theory technique which can be used for gauge groups $U(N)$, $SO(N)$ and $Sp(N)$ is the so called $1/N$ expansion. When described in this way, Chern-Simons theory is, for some three-manifolds, equivalent to a topological string theory and this provides an interesting connection between invariants related to three-manifolds and Gromov-Witten invariants. For a review of this aspect of Chern-Simons theory see for example [61].

Chern-Simons theory can be understood from many different points of view. In this thesis, we will work in the path integral formulation of the theory. In this formulation, exact results were first obtained in [19] using the method of *abelianisation*. In [19], Chern-Simons theory on the three-manifold $S^1 \times M_2$, where M_2 is a Riemann surface, was considered. The method of abelianisation was later generalized in [21] and applied to Chern-Simons theory on so called Seifert manifolds, which are three-manifolds that are S^1 fibrations over a Riemann surface. At about the same time as [21], another method of doing exact path integral calculations in Chern-Simons theory on Seifert manifolds was introduced by Beasley and Witten in [11]; the method of *non-abelian localization*. Recently, in [47], a new method which could be applied to Chern-Simons theory on S^3 was introduced. This method uses a slightly different approach to localization of the path integral, as compared to [21, 11]. However, the method is, as it is formulated in [47], only applicable† to Chern-Simons theory on S^3. In this thesis, we will show how to generalize this method to a broader class of three-manifolds. In short, the method introduces a set of auxiliary fields in Chern-Simons theory together with an odd symmetry. For a general three-manifold, the notion of contact geometry plays an important role when adding the auxiliary fields together with the odd symmetry. Using this new formulation of the theory, we will derive known expressions for the partition function and knot invariants on Seifert manifolds using a slightly different localization method as compared to [21, 11].

Interestingly, this new formulation of Chern-Simons theory with a set of auxiliary fields can be straightforwardly generalized to higher dimensions. Hence, formally, we can define a set of topological field theories for odd dimensional manifolds and their formulation will in general depend on the contact structure. This, in turn, is a generalization of the construction by Baulieu, Losev and Nekrasov in [10]. In that work, similar topological field theories

† The main motivation of [47] was not to do calculations for pure Chern-Simons theory, but for Chern-Simons theories coupled to matter, theories which are not topological but physically very interesting.

are constructed on $2n+1$ dimensional manifolds of the form $S^1 \times M_{2n}$, where M_{2n} is a $2n$ dimensional manifold. Roughly speaking, in order to define these higher dimensional analogs of Chern-Simons theory we have to introduce some extra structure on the manifold where the theory lives. In [10], this extra structure is one where the manifold is a product between a circle and another manifold, whereas in this thesis the extra structure will be a contact structure. In the case of a five-dimensional manifold of the form $S^1 \times M_4$, where M_4 is a symplectic four-manifold, a similar theory has also been considered in [64, 65, 55].

In this thesis, we will concentrate on the five-dimensional theory. When this theory is formulated on a five-manifold which is a circle fibration over a four dimensional symplectic manifold of integral class, we will show that it localizes to a set of equations which we call contact instantons. These equations are a natural generalization of the four-dimensional instanton equations to a five-dimensional contact manifold. We will also derive the full perturbative partition function for a certain class of the above described five-manifolds. The field content of the five-dimensional theory can be identified with the physical $\mathcal{N}=1$ supersymmetric Yang-Mills theory and it can be considered as a topologically twisted version of this theory.

This part of the thesis is organized as follows. We begin with a short background on Chern-Simons theory in chapter 2. The next three chapters give a review of different mathematical notions which we will need in order to localize the Chern-Simons path integral. In chapter 3, we will describe the concept of localization. First we do it in a finite-dimensional setting and we will then describe how these techniques can be implemented for the infinite dimensional path integrals. In chapter 4, we will introduce the basics of contact geometry. In chapter 5, we will review an important tool when it comes to computing one-loop determinants, namely the Atiyah-Singer index theorem. After these preliminaries we will, in chapter 6, give a short review of paper I which shows how to formulate three-dimensional Chern-Simons theory in a way suitable for path integral localization. In chapter 7, we will review the content of paper II, which shows how to generalize the three-dimensional construction to higher dimensions and the paper also studies the five-dimensional theory in some detail. We end with a discussion of possible applications of the five-dimensional theory in chapter 8.

2. Chern-Simons theory

2.1 Basic aspects of Chern-Simons theory

Chern-Simons theory was introduced as a topological field theory by Witten in [85]. The theory is formulated as follows. Let M_3 be a compact three-manifold, and let G be a compact, simple and simply connected Lie group. Consider a trivial principal G-bundle P over M_3, and let A represent a connection on this principal bundle. A is a one-form taking value in the Lie algebra $\mathfrak{g}$ of G: $A \in \Omega^1(M, \mathfrak{g})$. Under a gauge transformation, A transforms as

$$A \to g^{-1}Ag + g^{-1}dg \ , \qquad g : M \to G \ . \tag{2.1}$$

With this data, the Chern-Simons action is defined by

$$S_{CS} = \frac{k}{4\pi}\mathrm{Tr}\int_{M_3}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \ , \tag{2.2}$$

where Tr denotes an invariant inner product on $\mathfrak{g}$. As first discussed in [27], the action (2.2) is not invariant under the gauge transformations (2.1), but it transforms as

$$S_{CS} \to S_{CS} + 2\pi k\cdot\frac{1}{24\pi^2}\mathrm{Tr}\int_{M_3}\left(g^{-1}dg\right)^3 \ . \tag{2.3}$$

The extra term on the right hand side is the winding number of the map g. If we choose $k \in \mathbb{Z}$, and normalize the Tr appropriately, the quantity of interest in quantum field theories, namely

$$e^{iS_{SC}} \tag{2.4}$$

is gauge invariant, since S_{CS} is gauge invariant modulo 2π. The quantity k is known as the level. Since the action (2.2) does not involve the metric on M_3, the partition function of the theory,

$$Z(M_3) = \int \mathcal{D}A\ e^{iS_{CS}} \ , \tag{2.5}$$

is formally a topological invariant of the manifold M_3. The integral in (2.5) is a path integral, and the integration is over the space of all connections on P

modulo gauge transformations. The invariant $Z(M_3)$ can be defined rigorously, [74], and is sometimes called the Witten-Reshetikhin-Turaev invariant.

A natural set of observables in Chern-Simons theory are Wilson loops, which are constructed as follows. Let γ be an oriented closed curve in M_3, and let R be a representation of G. Then the Wilson loop $W_R(\gamma)$ is defined by

$$W_R(\gamma) = \mathrm{Tr}_R\, \mathcal{P}\, \exp\Big[\oint_\gamma A\Big]\,, \tag{2.6}$$

where $\mathcal{P}$ denotes path ordering. Mathematically, we are computing the holonomy of a connection A around γ; this gives us an element in G defined up to conjugation. Taking the trace gives a gauge invariant object, which is also independent of any choice of metric on M_3. Hence, the expectation value of $W_R(\gamma)$, that is, the path integral

$$Z(M_3,\gamma) = \int \mathcal{D}A\; W_R(\gamma)\; e^{iS_{CS}} \tag{2.7}$$

is also formally a topological invariant associated to the curve γ. A closed curve in M_3 is also known as a knot, and it was shown in [85] that (2.7) indeed gives knot invariants. In the simplest case, when $M_3 = S^3$, $G = SU(2)$ and R is the fundamental representation, $Z(M_3,\gamma)$ will be a polynomial in the variable $q = \exp[\frac{2\pi i}{k+2}]$, and this polynomial is in fact the Jones polynomial [44]. For other groups and representations, other knot invariants will be obtained. For more details on knot invariants in Chern-Simons theory, and also different aspects of the theory and applications, see for example the book [60] and references therein.

An unusual and fascinating aspect of Chern-Simons theory is that a method to solve the theory was introduced right from the beginning by Witten in [85]. The method exploits a relation between Chern-Simons theory on a three-manifold and Wess-Zumino-Witten models, and using two-dimensional conformal field theory techniques it is possible to obtain explicit expressions for the various invariants. However, in many cases these expressions are quite complicated. For a special sort of three-manifolds and knots, and for gauge group $SU(2)$, these expressions were manipulated into a more manageable form by Lawrence and Rozansky in [53]. The three-manifolds in question are called Seifert manifolds, and they are circle fibrations over a Riemann surface, where the Riemann surface is allowed to have orbifold points, whereas the knots are required to wrap the fibers of the Seifert manifold. These expressions were later generalized by Mariño to any simply laced gauge group in [59]. The expressions are given by sums and integrals over flat connections. A connection A is called flat if its curvature $F = dA + A\wedge A$ vanishes. The part of the expressions which comes from an isolated flat connection is given by a matrix model. By analyzing this matrix model explicit expressions for the perturbative invariants of a certain class of Seifert manifolds were obtained in [59], comparing to the perturbative expansion of the path integral. Also, these matrix model descriptions of Chern-Simons theory are important when analyzing the $1/N$ expansion of Chern-Simons theory and relate it to topological strings. Again, for more details of this story we refer to the book [60].

That the partition function and expectation values of Wilson loops only receive contributions from flat connections calls for an understanding directly from the path integral. A powerful tool on the market to calculate path integrals is the method of localization, and such an understanding was obtained in [11] and [21], where the path integral was calculated using two different localization techniques, non-abelian localization and abelianisation, respectively. Later, in a different context, Kapustin, Willett and Yaakov [47] introduced yet another method of calculating the partition function of Chern-Simons theory on S^3. This time the method was based on a supersymmetric version of the theory, and it again used localization. One of the goals of this thesis is to generalize the localization method introduced in [47] to a broader class of three-manifolds. As we will see, the formulation of Chern-Simons theory that we will arrive at admits quite natural generalizations to higher dimensions.

As a first preparation for these considerations, we will in the next chapter review the method of localization of path integrals, as it is usually formulated for supersymmetric quantum field theories.

3. Equivariant cohomology and localization of integrals

In this chapter, we will study localization of integrals. On manifolds which admit a group action, we will introduce the concept of equivariant differential forms. We will then consider integration of such forms, and review *the equivariant localization formula*, which states that the integration of an equivariantly closed differential form can be localized to the fixed points of the group action. We will then generalize this concept to quantum field theories and path integrals, reviewing under which circumstances the path integral can be localized to a finite dimensional integral. We will basically follow [79, 20]. For a detailed discussion of the finite dimensional case, we refer to [16].

3.1 Equivariant differential forms

Let $\mathcal{M}$ be a compact, smooth n-dimensional manifold, and let H be a compact Lie group acting on $\mathcal{M}$:

$$\begin{aligned} H \times \mathcal{M} &\to \mathcal{M} \\ (h,x) &\to h \cdot x \ . \end{aligned} \tag{3.1}$$

This is called a group action if $e \cdot x = x$ for all $x \in \mathcal{M}$, where e is the identity element, and if the action respects the multiplication law of the group, that is $h_1 \cdot (h_2 \cdot x) = (h_1 h_2) \cdot x$ for all $h_1, h_2 \in H$.

We will assume that the notion of differential forms and de Rham cohomology is known to the reader, and we will now describe how to generalize these concepts, taking the group action on the manifold into account. We will describe the Cartan model of equivariant cohomology. In this thesis, we will only be concerned with group actions on manifolds coming from the circle group $U(1)$, and we will restrict our description to this case. For the general case, see [79, 20]. When H is $U(1)$, there is a single vector field V generating the group action on $\mathcal{M}$. Using this vector field, we can extend the de Rham differential d to the *equivariant de Rham differential* d_V, defined by

$$d_V = d + \iota_V \ , \tag{3.2}$$

where ι_V denotes the contraction of the vector field V with a differential form. We notice that d_V is a derivation. The operator d_V acts on $\Omega^\bullet(\mathcal{M})^\dagger$. Using the Cartan identity

$$d\iota_V + \iota_V d = \mathcal{L}_V \ , \tag{3.3}$$

where $\mathcal{L}_V$ is the Lie derivative along the vector field V, we find that $d_V^2 = \mathcal{L}_V$. Therefore, in general, d_V^2 is not zero. Let us define the space $\Omega_V^\bullet(\mathcal{M})$ by

$$\Omega_V^\bullet(\mathcal{M}) = \{\alpha \in \Omega^\bullet(\mathcal{M}) \mid \mathcal{L}_V \alpha = 0\} \ . \tag{3.4}$$

Elements in $\Omega_V^\bullet(\mathcal{M})$ are called *equivariant differential forms*. When acting on $\Omega_V^\bullet(\mathcal{M})$, d_V^2 is zero, and we can define the cohomology of the operator d_V when acting on $\Omega_V^\bullet(\mathcal{M})$. An element $\alpha \in \Omega_V^\bullet(\mathcal{M})$ is called *equivariantly closed* if $d_V\alpha = 0$, and it is called *equivariantly exact* if $\alpha = d_V\beta$ for some $\beta \in \Omega_V^\bullet(\mathcal{M})$. The space of equivariantly closed forms modulo the space of equivariantly exact forms, denoted by $H_V^\bullet(\mathcal{M})$, is called the H-equivariant cohomology group of $\mathcal{M}$:

$$H_V^\bullet(\mathcal{M}) := \ker d_V|_{\Omega_V^\bullet(\mathcal{M})} / \mathrm{Im}\ d_V|_{\Omega_V^\bullet(\mathcal{M})} \ . \tag{3.5}$$

We notice that the top degree of an equivariantly exact form in $\Omega_V^\bullet(\mathcal{M})$ is exact in the de Rham sense, since the contraction of a vector field lowers the differential form degree by one.

3.2 The equivariant localization theorem

We will now consider integration of equivariant differential forms. Let $\mathcal{M}$ be an even-dimensional, orientable, compact manifold, without a boundary. Let $\dim\mathcal{M} = n$. The integration of an ordinary top degree differential form $\omega \in \Omega^n(\mathcal{M})$ can be thought of as a map $\Omega^n(\mathcal{M}) \to \mathbb{R}$:

$$\omega \to \int_{\mathcal{M}} \omega \ . \tag{3.6}$$

If ω is not a top form, its integral is defined to be zero. Moreover, by Stokes' theorem, the integration of a closed differential form ω, $d\omega = 0$, depends only on the cohomology class of ω, that is

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{M}} (\omega + d\lambda) \ , \tag{3.7}$$

† We define d_V without a parameter in front of ι_V. In the general case, one considers the operator $d_V = (d - \phi^a \iota_{V^a})$, where $a = 1, 2, \ldots, \dim H$, $\phi^a \in \mathfrak{h}^*$, $\mathfrak{h}$ the Lie algebra of H, and extents the space of differential forms $\Omega^\bullet(M)$ to $\mathbb{C}[\mathfrak{h}] \otimes \Omega^\bullet(M)$, where $\mathbb{C}[\mathfrak{h}]$ is polynomials in elements in $\mathfrak{h}^*$. There is a natural grading on differential forms, and ϕ^a is assigned degree 2 in order for d_V to raise the degree by 1. When $H = U(1)$, we do not need to carry around $\mathbb{C}[\mathfrak{h}]$.

since $\mathcal{M}$ does not have a boundary.

The integration of an equivariantly closed form $\alpha \in \Omega_V^\bullet(\mathcal{M})$ is defined in the same way:

$$\alpha \to \int_{\mathcal{M}} \alpha \ , \tag{3.8}$$

where it is understood to pick up the top form component on the right hand side. Also, since the top form component of an equivariantly exact form is exact in the ordinary de Rham sense, the integration of an equivariantly closed form $\alpha \in \Omega_V^\bullet(\mathcal{M})$, $d_V\alpha = 0$, depends only on the cohomology class:

$$\int_{\mathcal{M}} \alpha = \int_{\mathcal{M}} (\alpha + d_V\beta) \ . \tag{3.9}$$

As we will show below, an equivariantly closed differential form α is actually equivariantly exact *outside* the set of fix points of the group action on $\mathcal{M}$. This set of points is determined by the zeros of the vector field V, and it will be denoted by $\mathcal{M}_V$:

$$\mathcal{M}_V = \{x \in \mathcal{M} \mid V(x) = 0\} \ . \tag{3.10}$$

The power of this fact is that the integration of α can be reduced to an integration over $\mathcal{M}_V$, since by Stokes' theorem the integral vanishes outside this space. This phenomenon is called *the equivariant localization principle*, since the integral over $\mathcal{M}$ is localized to $\mathcal{M}_V$.

In order to show that α can be written as $\alpha = d_V\beta$ on $\mathcal{M} - \mathcal{M}_V$, we will explicitly construct β such that this holds. This construction requires the introduction of a metric g on $\mathcal{M}$ with the property that V is a Killing vector field with respect to this metric. Since the action on the manifold comes from a compact group, such a metric can always be constructed by averaging any metric over G. To construct β, we first consider the metric dual one-form to V, g(V). We have

$$d_V g(V) = dg(V) + |V|^2 \ , \tag{3.11}$$

where $|V|^2 = g_{\mu\nu}V^\mu V^\nu$. We define the inverse to a differential form in analogy with the with the formula

$$\frac{1}{(1+x)} = \sum_k (-x)^k \ , \tag{3.12}$$

where 1 represents the zero-degree component of the differential form, and x higher degrees. We see that the zero-degree component of $d_V g(V)$ is non-vanishing away from the fixed points of the group action, and $d_V g(V)$ is therefore invertible on the space $\mathcal{M} - \mathcal{M}_V$. Since V is a Killing vector field, it follows that the differential form

$$\rho = \frac{g(V)}{d_V g(V)} \tag{3.13}$$

fulfills $\mathcal{L}_V \rho = 0$, and we also have that $d_V \rho = 1$. Hence, 1 can be written as d_V of an equivariant differential form, and since α is d_V-closed we have

$$\alpha = d_V \rho\ \alpha = d_V(\rho\alpha)\ . \tag{3.14}$$

In this thesis, we will only consider the situation where $\mathcal{M}_V$ is a discrete set of points, which we from now on will assume. The more general situation will be commented on below. In order to find the contribution to the integral from the points in $\mathcal{M}_V$, we will consider the differential form $\alpha_s = \alpha\ e^{-s d_V g(V)}$, where $s \in \mathbb{R}$. The integral over α_s gives the same result for any value of s, since

$$\begin{aligned} \frac{d}{ds}\int_{\mathcal{M}} \alpha_s &= -\int_{\mathcal{M}} \alpha\ d_V g(V)\ e^{-s d_V g(V)} \\ &= -\int_{\mathcal{M}} d_V \left(\alpha\ g(V)\ e^{-s d_V g(V)}\right) = 0\ , \end{aligned} \tag{3.15}$$

where we have used that α is equivariantly closed, $\mathcal{L}_V(g(V)) = 0$ since V is a Killing vector field and that the integral of an equivariantly exact form vanishes. Taking the limit $s \to 0$ we get the integral of α over $\mathcal{M}$, whereas, since $d_V g(V) = |V|^2 + dg(V)$, taking the limit $s \to \infty$, the integral will localize to the fixed point set of V, in accordance with the discussion above. To show what the contributions from the fixed points are, we will follow [79] and introduce a more algebraic description of $\Omega^\bullet(\mathcal{M})$. This description is close to the field theory considerations that we will encounter below. Let us introduce a set of odd (fermionic) variables η^μ, $\mu = 1, 2, \ldots, n$. We identify η^μ with the basis dx^μ of $\Omega^1(\mathcal{M})$, and the degree k part of a differential form is written as

$$\alpha^{(k)}(x,\eta) = \alpha_{\mu_1\mu_2\ldots\mu_k}(x)\eta^{\mu_1}\eta^{\mu_2}\ldots\eta^{\mu_k}. \tag{3.16}$$

From this point of view, differential forms are functions on a supermanifold with local coordinates (x,η). The operator d_V acts on these local coordinates as

$$d_V x^\mu = \eta^\mu, \qquad d_V \eta^\mu = V^\mu(x) = \mathcal{L}_V x\ . \tag{3.17}$$

The integration of a differential form α is now written as

$$\int d^n x d^n \eta\ \alpha(x,\eta)\ , \tag{3.18}$$

where $d^n x d^n \eta$ is a coordinate independent measure since dx^μ and $d\eta^\mu$ transforms inversely under a change of coordinates on $\mathcal{M}$. We find that the integral of α_s is written as

$$\int\limits_{\mathcal{M}} \alpha_s = \int d^n x d^n \eta \; \alpha(x,\eta) \; e^{-sg_{\mu\nu}V^\mu(x)V^\nu(x) - sg_{\mu\lambda}\nabla_\nu V^\lambda \eta^\mu \eta^\nu} \; . \tag{3.19}$$

Here, ∇_μ is the covariant derivate constructed using the Levi-Civita connection, and we have used that V is a Killing vector field. Taking the limit $s \to \infty$ can be performed by using the following representations of delta-functions

$$\begin{aligned} \delta(V) &= \lim_{s\to\infty} \left(\frac{s}{\pi}\right)^{n/2} \sqrt{\det g} \; e^{-sg_{\mu\nu}V^\mu V^\nu} \\ \delta(\eta) &= \lim_{s\to\infty} \left(-\frac{s}{2}\right)^{-n/2} \frac{1}{\text{Pfaff } g\nabla V} e^{-sg_{\mu\lambda}\nabla_\nu V^\lambda \eta^\mu \eta^\nu} \; , \end{aligned} \tag{3.20}$$

where $g\nabla V$ denotes the antisymmetric matrix† $(g\nabla V)_{\mu\nu} = g_{\mu\lambda}\nabla_\nu V^\lambda$, and Pfaff A is the Pfaffian of an antisymmetric matrix A, defined by $(\text{Pfaff } A)^2 = \det A$. Inserting these two expressions in

$$\lim_{s\to\infty} \int\limits_{\mathcal{M}} \alpha \; e^{-sd_V g(V)} \; , \tag{3.21}$$

it is shown in [79], pages 40-41, that the integral over an equivariantly closed form α can be written as

$$\int\limits_{\mathcal{M}} \alpha = (-2\pi)^{n/2} \sum_{p\in\mathcal{M}_V} \frac{\alpha^{(0)}(p)}{(\det L_p)^{1/2}} \; , \tag{3.22}$$

where $\alpha^{(0)}(p)$ is the zero-form component of α evaluated at the point $p \in \mathcal{M}_V$, and L_p denotes the action of the Lie derivative $\mathcal{L}_V$ on the tangent space of $\mathcal{M}$ at the point $p \in \mathcal{M}_V$. In the field theory considerations in the following chapters, the coordinates x^μ and η^μ will correspond to fields in our theory. We will refer to the determinant $\det L_p$ as the *one-loop determinant*, since the exponent in the integrand in (3.19) will correspond to the action of the field theory and taking the $s \to \infty$ limit corresponds to calculating the one-loop determinant in a field theory.

The formula (3.22) was first derived in the special case of a Hamiltonian group action on a symplectic manifold by Duistermann and Heckman in 1982 [29]. The localization property was then understood as a general property of equivariant cohomology by Atiyah and Bott [4], and the general localization formula was derived by Berline and Vergne at about the same time in [15].

† $(g\nabla V)_{\mu\nu}$ is anti-symmetric since $g_{(\mu|\lambda}\nabla_{\nu)}V^\lambda = 0$ for a Killing vector field V^μ.

It sometimes goes by the name the Atiyah-Bott-Vergne-Berline localization formula.

Below, when we will apply a path integral version of (3.22), the manifold $\mathcal{M}$ will be a supermanifold with both even (bosonic) and odd (fermionic) coordinates. A generalization of the formula (3.22) to the case of a supermanifold has been derived in [76]. The difference compared to the case of an ordinary manifold is that in the case of a supermanifold the factor $(\det L_p)^{-1/2}$ should be understood as the ratio of determinants of the action of $\mathcal{L}_V$ on the tangent space of the fermionic subspace of $\mathcal{M}$ and the action of $\mathcal{L}_V$ on the tangent space of the bosonic subspace of $\mathcal{M}$, respectively.

In the case of non-isolated zeros of the vector field V generating the action on $\mathcal{M}$, the subspace $\mathcal{M}_V$ will have non-zero dimensionality and the sum in (3.22) is replaced by an integral over $\mathcal{M}_V$. The integrand is in this case given by a generalization of the Euler form to the equivariant setting. Namely, (3.22) is replaced by

$$\int_{\mathcal{M}} \alpha = \int_{\mathcal{M}_V} \frac{\alpha}{eu(N_{\mathcal{M}_V})} \,, \tag{3.23}$$

where $eu(N_{\mathcal{M}_V})$ denotes the equivariant Euler form of the normal bundle of $\mathcal{M}_V$. As mentioned above, only the case of isolated zeros of V will be considered in this thesis, and we refer to [16] for more details of the general case.

3.3 Localization of path integrals

Quantum field theories can be formulated in terms of path integrals, which can be thought of as infinite dimensional integrals. Let us consider the situation of a theory defined on a manifold M with both bosonic and fermionic fields, and an action denoted by S. If there is an odd symmetry in the theory, that is, a transformation Q of the fields such that $QS = 0$, the path integral can sometimes be reduced to a finite dimensional integral. This can be understood as an analog of the Atiyah-Bott-Vergne-Berline localization formula for finite dimensional integrals. The typical situation when such a dramatic reduction of the space of integration from something infinite dimensional to something finite dimensional occurs is when the quantum field theory has (enough) supersymmetries.

The correspondence between the geometric quantities in section 3.2 and the field theory data is the following. The manifold $\mathcal{M}$ corresponds to the space of fields of the theory. The group acting on $\mathcal{M}$ is usually a combination of the gauge group and some isometry of the underlying manifold M. Q has a natural interpretation as an equivariant differential, and the fact that it is a symmetry means that e^{iS} can be considered as an equivariantly closed differential form on the space of fields. The partition function Z of the theory, that is the path integral

$$Z = \int e^{iS} , \tag{3.24}$$

can thus be interpreted as an integral of an equivariantly closed form, and it can be reduced to an integral over the fixed points of the Q-action. If we can find observables which are supersymmetric, that is, Q-closed, their expectation values can be calculated with the same method.

The idea of localization of the path integral in supersymmetric quantum field theories has a long and very successful history. Let us describe some of its highlights, whereas for a more complete list of historical references we refer to [79, 20]. The idea was first used in quantum mechanical systems in the beginning of the 1980's, suggested in [82] and applied in [7]. In a quantum field theory setting, the idea has its origin in topological field theories, see for example [84, 86]. An important application of the method in recent years has occurred for *physical* supersymmetric gauge theories. It has been used by Nekrasov to derive the Seiberg-Witten solution of $\mathcal{N} = 2$ supersymmetric four-dimensional Yang-Mills theory from first principles in [68]. Even more recently, it has been applied to physical gauge theories on spheres. First by Pestun [72] for supersymmetric theories on S^4, and later by Kapustin, Willett and Yaakov [47] for supersymmetric Chern-Simons-matter theories on S^3. Since both of these types of theories have gravity duals, one reason that these applications are important is in the context of gauge/gravity dualities, as the AdS/CFT correspondence [56, 1]. Using the exact results on the gauge theory side obtained from localization, some very precise tests of the AdS/CFT correspondence have been made. For a pedagogical review of these advances in the three-dimensional case, see [62].

Our goal now is to write the path integral in (2.5) in a cohomological, or supersymmetric, form. In order to do this, we must introduce a geometric structure on the three-manifold known as a contact structure, and this is the subject of the next chapter.

4. Contact structures

In the chapter that follows we will write topological field theories in cohomological form using a geometric structure on the manifold where the theory is defined. The structure we need is a contact structure, and in this chapter we will introduce this notion. We will be quite brief and only state the main properties of contact structures that we need in the chapters that follows. We will mostly follow the books [18, 35], to which we refer for more details on the subject.

4.1 Definition of a contact structure

Let M be a $2n+1$ dimensional differentiable manifold. M is called a *contact manifold* if it admits a one-form κ such that $\kappa \wedge (d\kappa)^n$ is everywhere non-vanishing. This means that $\kappa \wedge (d\kappa)^n$ can serve as a volume element on M. This is equivalent to saying that the distribution $\xi := \ker\, \kappa$ is maximally non-integrable. Here, $\ker\, \kappa = \{X \in TM | \kappa(X) = 0\}$, where TM is the tangent bundle of M. The pair (M, ξ) is called a *contact structure*, and the one-form κ used to define ξ is a called a *contact form*. In fact, we can rescale $\kappa \to \tilde{\kappa} = e^f \kappa$, where $f : M \to \mathbb{R}$ is some function on M, and $\tilde{\kappa}$ will still define the same contact structure ξ. Given a contact form κ, there is an associated vector field v, known as the *Reeb vector field*. It is uniquely defined by the conditions

$$\begin{aligned} \iota_v d\kappa &= 0 \ , \\ \iota_v \kappa &= 1 \ . \end{aligned} \tag{4.1}$$

A final notion we will need is that of a regular contact form. A vector field X is called *regular* if every point on the manifold M has a neighbourhood such that any integral curve of X passing through that neighbourhood passes through only once. Consequently, a contact form κ is called a *regular contact form* if the associated Reeb vector field is regular.

4.2 Examples

4.2.1 $\mathbb{R}^{2n+1}$

Let $(x^1, x^2, \ldots x^n, y^1, y^2, \ldots y^n, z)$ be coordinates on $\mathbb{R}^{2n+1}$. A contact form is given by

$$\kappa = dz - \sum_{i=1}^{n} y^i dx^i \tag{4.2}$$

and the Reeb vector field is given by

$$v = \frac{\partial}{\partial z} \,. \tag{4.3}$$

In fact, there is an analog of Darboux's theorem for contact manifolds. Given any manifold M and contact form κ, locally on M there exists a choice of coordinates such that κ takes the above form.

4.2.2 *Compact orientable three-manifolds*

Any compact orientable three-manifold admits a contact structure. This was proven by Martinet in 1971 [63].

4.2.3 *Circle fibrations over a symplectic manifold*

An important example of a contact manifold M_{2n+1} is that of a circle fibration over a symplectic manifold Σ_{2n} of integral class:

$$\begin{array}{ccc} S^1 & \longrightarrow & M_{2n+1} \\ & & \pi \big\downarrow \\ & & \Sigma_{2n} \end{array} \quad . \tag{4.4}$$

That the symplectic manifold Σ_{2n} is of integral class means that the symplectic form ω is in $H^2(\Sigma_{2n}, \mathbb{Z})$, where $H^2(\Sigma_{2n}, \mathbb{Z})$ is the second de Rham cohomology group of Σ_{2n} with coefficients in $\mathbb{Z}$. In fact, there is a celebrated theorem by Boothby and Wang [22] which states that *any* compact regular contact manifold is of this type. The connection of the circle fibration is given by the contact form. We will refer to this construction as the *Boothby-Wang fibration*.

4.3 Metrics and contact structures

In the field theory considerations to follow, we will need to introduce a metric on our contact manifold. A metric g on a contact manifold M with contact form κ and Reeb vector field v is called an *associated metric* if two conditions are fulfilled. Firstly,

$$\kappa(X) = g(X, v), \qquad \forall X \in TM\ , \tag{4.5}$$

and secondly, if there exists a $(1,1)$ tensor field ϕ such that

$$\phi^2 = -I + \kappa \otimes v, \quad d\kappa(X,Y) = g(X, \phi(Y)), \qquad \forall X, Y \in TM. \tag{4.6}$$

Given a contact form κ and Reeb vector field v, such a metric can always be constructed. For the construction we refer to [18], proof of theorem 4.4. The collection (ϕ, v, κ, g) is called a *contact metric structure*. Finally, a contact metric structure with the property that v is a Killing vector field, that is $\mathcal{L}_v g = 0$, is called a *K-contact structure*. We now give examples of K-contact structures.

4.3.1 K-contact structure on $\mathbb{R}^{2n+1}$

Let us on $\mathbb{R}^{2n+1}$ choose coordinates and contact form as in example 4.2.1. Then the metric

$$g = \kappa \otimes \kappa + \sum_{i=1}^{n} \left(dx^i \otimes dx^i + dy^i \otimes dy^i \right) \tag{4.7}$$

gives a K-contact structure on $\mathbb{R}^{2n+1}$.

4.3.2 K-contact structure on circle fibrations over a symplectic manifold

Given a symplectic manifold (Σ_{2n}, ω), there exists a metric G and an almost complex structure J such that $G(X, JY) = \omega(X,Y)$, $\forall X, Y \in T\Sigma_{2n}$. For a proof of this statement, see [18], theorem 4.3. The metric G is called a metric associated to the symplectic structure. Let us choose such a metric G on the base of the Boothby-Wang fibration in example 4.2.3. Then the metric

$$g = \kappa \otimes \kappa + \pi^* G\ , \tag{4.8}$$

gives a K-contact structure on M_{2n+1}.

There is a generalization of the Boothby-Wang fibration to the situation where the base manifold is allowed to have orbifold points, and again such a manifold admits a K-contact structure. In fact, for a compact manifold, any K-contact manifold will a given by a Boothby-Wang fibration, where the base is allowed to be an orbifold, see for example theorem 7.1.3 in [24]. In the case of three-dimensional manifolds, such manifolds are called Seifert manifolds.

4.3.3 Relation between volume forms

Given a metric on an orientable manifold, there is a natural volume form Vol associated to this metric. As we have seen, given a contact manifold, there is a volume form associated to the contact structure as well. On a manifold with a contact metric structure, the relation between these two volume forms are given by ([18], theorem 4.6)

$$\mathrm{Vol} = \frac{(-1)^n}{2^n n!} \kappa \wedge (d\kappa)^n \ . \tag{4.9}$$

4.3.4 Reduction of the structure group

Another way of understanding a contact manifold M_{2n+1} is that the structure group of its tangent bundle can be reduced to $U(n) \times 1$ (see chapter 4 in [18]). This aspect of contact manifolds will be important when we consider the twisting of supersymmetric theories below.

5. The Atiyah-Singer index theorem

When localizing path integrals, the main part of the calculation is the computation of the so called 1-loop determinant. In this chapter we will describe an important tool when performing these calculations, namely the *Atiyah-Singer index theorem* [5, 6]. The description below is based on [30, 66, 67], and we refer to these works for more details.

5.1 Elliptic operators and elliptic complexes

We begin by describing the notion of elliptic operators. We follow the exposition in [30] closely. Let M be a compact manifold without a boundary. Let E and F be vector bundles over M, and let D be a first order differential operator mapping sections of E to sections of F:

$$D:\ \Gamma(M,E) \to \Gamma(M,F)\ . \tag{5.1}$$

Let x_j denote local coordinates on M. The operator D can then be written as

$$D = a_j(x)\frac{\partial}{\partial x_j} + b\ , \tag{5.2}$$

where $a_j(x)$ and b are matrix valued. Let $f(x)$ be a section of E. Let $\tilde{f}(k)$ denote the Fourier transform of $f(x)$. We then have

$$Df(x) = a_j(x)\frac{\partial f(x)}{\partial x_j} + bf(x) = \int (ia_j(x)k_j + b)\,\tilde{f}(k)e^{ik\cdot x}dk\ . \tag{5.3}$$

We define the *leading symbol* $\tilde{D}(x,k)$ of D to be

$$\tilde{D}(x,k) = ia_j(x)k_j\ . \tag{5.4}$$

If the matrix $\tilde{D}(x,k)$ is invertible for $k \neq 0$, the operator D is said to be an *elliptic operator*. One criterion for ellipticity is therefore that $\dim E = \dim F$. The definition of ellipticity for higher order operators is similar to the one above.

Let M again be a compact manifold without boundary, let $\dim M = n$ and let now $\{E_p\}$ be a sequence of vector bundles over M. Let $\Gamma(M,E_p)$ denote the space of sections of E_p. Let D_p be a differential operator mapping $\Gamma(M,E_p)$ to $\Gamma(M,E_{p+1})$:

$$D_p: \ \Gamma(M,E_p) \to \Gamma(M,E_{p+1}) \ . \tag{5.5}$$

Let $\{E_p\}$ be endowed with fibre metrics, denoted by $(\ ,\ )_{E_p}$. Using this fibre metric, we can define the adjoint operator $D_p^\dagger$ to D_p in the usual way

$$(s, D_p s')_{E_{p+1}} = (D_p^\dagger s, s')_{E_p} \ . \tag{5.6}$$

If the composition $D_p \circ D_{p-1}$ is zero for all p, and if $D_p^\dagger D_p + D_{p-1} D_{p-1}^\dagger$ is elliptic for all p, the sequence $\{E_p, D_p\}$ is called an *elliptic complex*. An example of an elliptic complex is the de Rham complex. In this case, $E_p = \Omega^p(M)$ and $D_p = d$ is the exterior derivative acting on differential p-forms.

5.2 The theorem

We define the cohomology groups $H^p(E,D)$ for an elliptic complex $\{E_p, D_p\}$ by

$$H^p(E,D) = \frac{\ker D_p}{\operatorname{Im} D_{p-1}} \ , \tag{5.7}$$

and we define the *index* of the elliptic complex $\{E_p, D_p\}$ to be

$$\operatorname{ind}(D,E) = \sum_p (-1)^p \dim H^p(E,D) \ . \tag{5.8}$$

The Atiyah-Singer index theorem states that the above defined index of an elliptic complex can instead be calculated by integrating certain characteristic classes over M. For the elliptic complexes that we will need in this thesis, the Atiyah-Singer index theorem states that

$$\operatorname{ind}(D,E) = (-1)^{n(n+1)/2} \int_M \operatorname{ch}\left(\bigoplus_p (-1)^p E_p \right) \frac{\operatorname{Td}(TM^{\mathbb{C}})}{\mathrm{e}(TM)} \ . \tag{5.9}$$

Above, $\operatorname{ch}(E)$ denotes the Chern character of the vector bundle E, $\operatorname{Td}(TM^{\mathbb{C}})$ denotes the Todd class of the complexified tangent bundle of M, and e(TM) is the Euler class of the tangent bundle of M. It is understood that we pick up the top form of the integrand. For a definition and properties of characteristic classes, see for example chapter 11 in [66]. For the applications of the formula (5.9) that we will encounter in this thesis, we need to use the following properties of the characteristic classes introduced above. Both the Chern character ch(E) and the Todd class Td(E) can be expanded in terms of another set of characteristic classes, the Chern classes $c_j(E)$, $j = 1,2\ldots,\dim E = k$, (again, for a definition, see chapter 11 in [66]). Namely, we have (equation 6.12-13 in [30]):

$$\begin{aligned} \mathrm{ch}(E) &= k + c_1(E) + \frac{1}{2}\left(c_1^2 - 2c_2\right)(E) + \ldots \\ \mathrm{Td}(E) &= 1 + \frac{1}{2}c_1(E) + \frac{1}{12}\left(c_1^2 + c_2\right)(E) + \ldots \\ \mathrm{Td}(E^*) &= (-1)^k \left(1 - \frac{1}{2}c_1(E) + \frac{1}{12}\left(c_1^2 + c_2\right)(E) + \ldots\right) . \end{aligned} \tag{5.10}$$

In the last expression, E^* denotes the dual bundle to E. All these expressions can be derived from their definitions, using the splitting principle. Finally, if L and L' are line bundles, the first Chern class fulfills (page 300 in [30]):

$$c_1(L \otimes L') = c_1(L) + c_1(L') . \tag{5.11}$$

5.2.1 Index of twisted Dolbeault operators

When we apply the Atiyah-Singer index theorem in the localization computations of path integrals, M will be a complex manifold with $\dim M = n$, n even. E_p will be given by $\bigwedge^{(0,p)} T^*M \otimes V$ or $\bigwedge^{(p,0)} T^*M \otimes V$, where V is a line bundle, and D_p will be the twisted $\bar{\partial}$- or ∂-operator, respectively. For these two complexes, let us abbreviate the index defined in (5.8) to ind $(\bar{\partial}_V)$ and ind (∂_V), respectively. The integral over characteristic classes in the Atiyah-Singer index theorem, equation (5.9), can in this case be reduced to (see for example page 117 in [67] for the derivation)

$$\begin{aligned} \text{ind}\,(\bar{\partial}_V) &= \int_M \mathrm{ch}(V) \wedge \mathrm{Td}(TM^+) \\ \text{ind}\,(\partial_V) &= (-1)^{n/2} \int_M \mathrm{ch}(V) \wedge \mathrm{Td}(TM^-) , \end{aligned} \tag{5.12}$$

where $TM^\pm$ denotes the holomorphic and anti-holomorphic part of $TM^{\mathbb{C}}$. Since $TM^+ = \overline{TM}^-$, where the bar denotes the conjugate bundle, and since for a complex bundle E, the dual and conjugate bundles are isomorphic (see for example page 82 in [67]), these two indices can be written in terms of Chern classes, using (5.10). In the case $n = 2$, M_2 a Riemann surface with genus g, we get

$$\begin{aligned} \text{ind}\,(\bar{\partial}_V) &= \int_{M_2} \mathrm{ch}(V) \wedge \mathrm{Td}(TM^+) = (1-g) + \int_{M_2} c_1(V) \\ \text{ind}\,(\partial_V) &= (-1)^{n/2} \int_M \mathrm{ch}(V) \wedge \mathrm{Td}(TM^-) = (1-g) - \int_{M_2} c_1(V) . \end{aligned} \tag{5.13}$$

In the case of a Riemann surface M_2, the above formulas are also called the Riemann-Roch theorem.

For $n = 4$, we get

$$
\begin{aligned}
\text{ind}\,(\bar{\partial}_V) &= \int_{M_4} \text{ch}(V) \wedge \text{Td}(TM^+) \\
&= \int_{M_4} \left(\frac{1}{12}[c_1(TM^+)^2 + c_2(TM^+)] + \frac{1}{2}c_1(TM^+) \wedge c_1(V) + \frac{1}{2}c_1(V)^2 \right) \\
\text{ind}\,(\partial_V) &= (-1)^{n/2} \int_{M_4} \text{ch}(V) \wedge \text{Td}(TM^-) \\
&= \int_{M_4} \left(\frac{1}{12}[c_1(TM^+)^2 + c_2(TM^+)] - \frac{1}{2}c_1(TM^+) \wedge c_1(V) + \frac{1}{2}c_1(V)^2 \right) .
\end{aligned}
\tag{5.14}
$$

6. Localization of Chern-Simons theory on K-contact three-manifolds

In this chapter, we will explain how to recast Chern-Simons theory into a form so that the method of localization can be applied. We will then describe how, in this new formulation of Chern-Simons theory, to calculate the path integral on Seifert manifolds using localization. This chapter is based on paper I. It aims to summarize the main points of that paper without going into too much technical detail.

6.1 Twisting of supersymmetric Chern-Simons theory using a contact structure

As explained in chapter 3, path integrals of supersymmetric quantum field theories can in many cases be understood as infinite dimensional analogs of integration of equivariantly closed differential forms, and they can be localized to a finite dimensional space. Chern-Simons theory, as formulated in chapter 2, is not supersymmetric as it stands, since there are only bosonic fields in the theory. However, as explained in for example [77], Chern-Simons theory *is* secretly supersymmetric on $\mathbb{R}^3$. The meaning of this statement is the following. Take the standard $\mathcal{N}=2$ vector multiplet in three dimensions, consisting of a gauge field A, two real scalar fields σ and D, and a two component complex spinor λ. The fields σ, D, λ all take value in the Lie algebra $\mathfrak{g}$ and they transform in the adjoint representation of the gauge group G. Then the action

$$S_{\mathcal{N}=2} = \frac{k}{4\pi}\mathrm{Tr}\int\limits_{\mathbb{R}^3}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A - \lambda^\dagger\lambda - 2\sigma D\right) \tag{6.1}$$

is invariant under the standard supersymmetry transformations. However, the fields σ, D, λ are all non-dynamical, meaning that the path integral over all fields A, σ, D, λ with the action (6.1) gives the same result (up to some uninteresting overall factor) as the path integral over the field A with the Chern-Simons action (2.2). In [47], it was shown how to modify the supersymmetry transformations in [77] in order to write the supersymmetric version of Chern-

Simons theory on S^3. Indeed, not only the Chern-Simons action but a large class of supersymmetric Chern-Simons theories coupled to matter was considered in [47], and it was shown how to apply a version of the method described in chapter 3 to calculate the path integrals in these theories using localization. The end results are given by matrix models and, as mentioned above, the importance of these results lies in the fact that these theories have gravity duals. Using the matrix model description of these theories tests of the AdS/CFT conjecture can be performed. However, ignoring the matter sector, the supersymmetric version of Chern-Simons theory is interesting in its own, since it gives a new way to calculate the partition function of Chern-Simons theory on the three-sphere.

In paper I, we address the problem of extending the formulation of Chern-Simons theory as a supersymmetric theory in [47] to a broader class of three-manifolds. As always for supersymmetric theories, there can be obstructions of doing this due to the presence of spinors in the theory. For theories with enough supersymmetry, there is a standard method of modifying the theory so that it can be defined on general manifolds. The method is called *topological twisting*, and it was introduced by Witten in his 1988 paper [84]. In essence, given a supersymmetric theory defined on $\mathbb{R}^n$, the topological twist redefines the rotation group of the theory into a mix of the rotation group and the R-symmetry group. Under this new rotation group, the spinors have become differential forms, and there is no obstruction to define the theory on a general manifold. At least one of the supercharges in the twisted theory becomes a scalar. For a detailed explanation of this procedure, see for example chapter 5 in [50].

The R-symmetry group of $\mathcal{N}=2$ supersymmetric theories in three dimensions is $U(1)$, and this is not enough to perform the twist in the usual way. One way to twist the theory anyway is to choose a suitable geometrical structure on the underlying manifold, since this reduce the rotation group (structure group of the tangent bundle). As mentioned in the end of chapter 4, the choice of a contact structure on the three-manifold M reduces the structure group to $U(1)\times 1$, and now the mixture of R-symmetry and rotational symmetry is possible and the twist can be performed. This fact was used in paper I [†]to introduce a twisted version of supersymmetric Chern-Simons theory, defined with the choice of a contact structure. This twisted version of supersymmetric Chern-Simons theory can be defined on any compact, orientable three-manifold, since any such manifold admits a choice of contact structure.

As explained in paper I, the field content of the twisted $\mathcal{N}=2$ supersymmetric Chern-Simons theory is given by the following set of fields: (A,Ψ,α,σ,D). The connection A and the even scalars σ and D are not affected by the twist,

[†] The same idea of twisting $\mathcal{N}=1$ supersymmetric Chern-Simons theories using a contact structure has appeared in the work by Thompson [80].

whereas the spinor λ is redefined into an odd one-form Ψ with values in $\mathfrak{g}$: $\Psi \in \Omega^1(M,\mathfrak{g})$, an odd scalar α with values in $\mathfrak{g}$: $\alpha \in \Omega^0(M,\mathfrak{g})$. As before, all the fields σ, D, Ψ, α transform in the adjoint representation of the gauge group. Let us call the supersymmetry transformation which becomes a scalar after the twist by δ. It acts on the fields as

$$\begin{aligned}
&\delta A = \Psi \\
&\delta\Psi = \iota_v F + i\, d_A\sigma \\
&\delta\sigma = -i\,\iota_v\Psi \\
&\delta\alpha = -\frac{\kappa\wedge F}{\kappa\wedge d\kappa} + \frac{i}{2}(\sigma + D) \\
&\delta D = -2i\mathcal{L}_v^A\alpha - 2[\sigma,\alpha] - 2i\frac{\kappa\wedge d_A\Psi}{\kappa\wedge d\kappa} + i\,\iota_v\Psi\,.
\end{aligned} \tag{6.2}$$

Above, v denotes the Reeb vector field, associated with the contact form κ. F is the curvature of the connection A, defined by $F = dA + A\wedge A$. $[\ ,\ ]$ denotes the Lie algebra bracket, and d_A is the de Rham differential twisted by the connection, $d_A = d + [A,\ ]$. We have also introduced the notation $\mathcal{L}_v^A := \mathcal{L}_v + [\iota_v A,\ ]$. Finally, since $\kappa\wedge d\kappa$ is a non-vanishing top form, any three form, for example $\kappa\wedge F$, can be written as $\kappa\wedge F = \gamma\,\kappa\wedge d\kappa$ for some $\gamma\in\Omega^0(M,\mathfrak{g})$, and by the notation $\frac{\kappa\wedge F}{\kappa\wedge d\kappa}$ we mean this γ. The transformation δ is a symmetry of the action

$$\begin{aligned}
S_{tw} = \frac{k}{4\pi}\mathrm{Tr}\int\limits_{M_3}\Big(&A\wedge dA + \frac{2}{3}A\wedge A\wedge A \\
&- \kappa\wedge\Psi\wedge\Psi - 2d\kappa\wedge\Psi\alpha + \kappa\wedge d\kappa\, D\sigma\Big)\,.
\end{aligned} \tag{6.3}$$

As in the untwisted case, the fields σ, D, Ψ, α are auxiliary, meaning that they can be integrated out in the path integral. However, now we have a theory with an odd symmetry, which is an essential ingredient if we want to apply the method of equivariant localization to the Chern-Simons path integral.

The set of fields (A,σ,D,Ψ,α), the transformations δ in (6.2) and the action (6.3) is what we end up with after twisting the $\mathcal{N}=2$ supersymmetric Chern-Simons theory. However, this way of writing things is not the most transparent. Instead, if we make the field redefinition

$$D \to H = -\frac{\kappa\wedge F}{\kappa\wedge d\kappa} + \frac{i}{2}(\sigma + D)\ , \tag{6.4}$$

the transformations (6.2) is given by

$$\begin{aligned}
&\delta A = \Psi && \delta\alpha = H \\
&\delta\Psi = \mathcal{L}_v A + d_A(i\sigma - \iota_v A) && \delta H = \mathcal{L}_v H - [i\sigma - \iota_v A, H] \\
&\delta\sigma = -i\,\iota_v\Psi &&
\end{aligned} \tag{6.5}$$

and the action (6.3) is given by

$$S_{tw} = \frac{k}{4\pi}\mathrm{Tr}\int\limits_{M_3}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A$$
$$-\kappa\wedge\Psi\wedge\Psi - 2d\kappa\wedge\Psi\alpha - 2i\sigma\kappa\wedge F - \kappa\wedge d\kappa\left(2iH\sigma+\sigma^2\right)\Big)\ . \tag{6.6}$$

The transformations (6.5) should now be compared with the standard action of an equivariant differential on a finite dimensional manifold, equation (3.17). Comparing (6.5) with (3.17), we see that δ has a natural interpretation as an equivariant differential acting on the space of fields, where we interprete A,α as coordinates and Ψ,H as the de Rham differentials of the coordinates. The transformation δ fulfills

$$\delta^2 = \mathcal{L}_v + G_\Phi\ , \tag{6.7}$$

where G_Φ denotes a gauge transformation with parameter $\Phi = i\sigma - \iota_v A$. The gauge transformations acts as $G_\Phi A = d_A\Phi$ on the connection, and $G_\Phi X = -[\Phi, X]$ on any other field X.

In the chapter below, we will introduce higher dimensional theories, constructed using a contact structure. The action S_{tw} defined in equation (6.6) is not yet written in the form which most simply admits generalizations to higher dimensions. According to standard arguments, since S_{tw} is δ-closed, adding δ-exact terms δV to S_{tw} will not affect the theory, as long as $\delta^2 V = 0$. We can take advantage of this freedom in order to write the action S_{tw} in a form which makes higher dimensional generalizations straightforward. Let us therefore add

$$\delta V = \delta\left(\frac{k}{4\pi}\mathrm{Tr}\int\limits_{M_3} 2i\alpha\sigma\kappa\wedge d\kappa\right)\ . \tag{6.8}$$

The new action can be written as

$$S_{SCS_3} = S_{CS}(A - i\kappa\sigma) - \frac{k}{4\pi}\mathrm{Tr}\int\limits_{M_3}\kappa\wedge\Psi\wedge\Psi\ . \tag{6.9}$$

where S_{CS} is the Chern-Simons action defined in (2.2).

6.2 Aspects of localization of Chern-Simons theory

Our goal is to calculate the path integral

$$Z(M_3,k) = \int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}H\mathcal{D}\sigma\mathcal{D}\alpha\ e^{iS_{tw}}\ , \tag{6.10}$$

which, as we argued above, is equivalent to the Chern-Simons partition function, equation (2.5). We will do this by using localization.

6.2.1 *Metric on the space of fields*

As we saw in chapter 3 when we described the equivariant localization formula for finite dimensional integrals, in order to localize we have to pick up a metric on the manifold which we are integrating over. In order to localize the path integral, we also have to pick a metric, this time on the space of fields. Since the tangent space at a point A in the space of connections is modelled by $\Omega^1(M,\mathfrak{g})$, and similarly for the other fields in the theory, we can construct a metric as follows. Pick up a metric g on M, and let $*$ denote the Hodge star operator constructed from this metric. Then, if X and Y are two tangent vectors, we define a metric $(\ ,\ )$ by

$$(X,Y) = \mathrm{Tr}\int\limits_M X \wedge *Y \ . \tag{6.11}$$

In order to localize, we will need to pick up a metric, compatible with the contact structure, such that the Reeb vector field v generates an isometry. In the language of section 4.3, we will therefore have to assume that M can be endowed with a K-contact structure. In the case of a compact manifold, such manifolds are the total space of a Boothby-Wang fibration, described in section 4.2.3. Another name for such manifolds are Seifert manifolds.

6.2.2 *Fixing the gauge symmetry*

Since Chern-Simons theory has a gauge invariance, we must gauge fix the theory. The standard way to do this is to introduce a set of ghost fields and a BRST-symmetry, and using this BRST symmetry to impose the condition

$$d^\dagger A = 0 \tag{6.12}$$

on the gauge field. Here $d^\dagger$ is the adjoint operator to the de Rham differential, defined with respect to the inner product defined using the metric. As shown in paper I, this can be done in a way compatible with the transformations δ, and we refer to paper I for more details on this point.

6.2.3 *Localization locus*

Combining all the above described ingredients, it is shown in paper I that, for M being a Seifert manifold, the path integral of Chern-Simons theory localizes to the solutions of the equations

$$\begin{aligned} F &= 0 \ , \\ d_A\sigma &= 0 \ , \end{aligned} \tag{6.13}$$

modulo gauge invariance, and the rest of the fields are zero.

The trivial connection, that is A=0, is always a solution to the first equation in (6.13). For manifolds M_3 with trivial first homology group, that is $H_1(M_3,\mathbb{R}) = 0$, this solution is isolated. This follows from the following reasoning. If A is flat, that is $F(A) = 0$, we can perturbe $A \to \tilde{A} = A + \delta A$, and determine the condition for $\tilde{A}$ to be flat, modulo gauge transformations, as well. Requiring

$$F(\tilde{A}) = 0 \tag{6.14}$$

leads, to first order in δA, to

$$0 = F(A) + d_A \delta A. \tag{6.15}$$

Hence, if A is zero, we have to find solutions to the equations

$$\begin{aligned} d\delta A &= 0 \ , \\ d^\dagger \delta A &= 0 \ . \end{aligned} \tag{6.16}$$

This means that δA is a harmonic one-form. On manifolds with trivial first homology group, there are no harmonic one-forms, and hence $A = 0$ is an isolated solution. When $A = 0$, the second equation in (6.13) is solved by σ being a constant.

6.2.4 *Calculation of the one-loop determinant*

We will now demonstrate the main steps in calculating the contribution to the path integral (6.10) coming from the solution

$$\begin{aligned} &A = 0 \ , \\ &\sigma = \text{constant} =: \sigma_0 \end{aligned} \tag{6.17}$$

on a Seifert manifold with $H_1(M_3,\mathbb{R}) = 0$. For more details, see paper I. The final answer will be given by an integral over the constant field σ_0. Since σ_0 is Lie algebra valued, this will be an integral over the Lie algebra $\mathfrak{g}$. Such integrals are also called matrix models, and we will therefore get a matrix model after the localization is performed. The integrand of this matrix model is given by two parts. First, it is the action (6.6) evaluated at the point (6.17). Secondly, it is the 1-loop determinant, which is given by the square root of the determinant of the operator

$$L_\phi := \mathcal{L}_v + G_\phi \ , \tag{6.18}$$

acting on the tangent space of the space of fields†. We will, as in paper I, denote this determinant by $h(\phi)$. The operator L_ϕ generates an action of the

† We have denoted $\phi = i\sigma_0$

group $U(1) \times G$, where G is the gauge group, and ϕ is a parameter for the action of G. Since the space of fields is given by a supermanifold, it will be the ratio of determinants of L_ϕ on the fermionic part and the bosonic part. As described in paper I, after introduction of the ghosts in the theory, this two tangent spaces are given by $\Omega^0(M_3, \mathfrak{g}) \oplus \Omega^0(M_3, \mathfrak{g}) \oplus \Omega^0(M_3, \mathfrak{g})$ for the fermionic part and $\Omega^1(M_3, \mathfrak{g}) \oplus H^0(M_3, \mathfrak{g}) \oplus H^0(M_3, \mathfrak{g})$ for the bosonic part. $H^0(M_3, \mathfrak{g})$ denotes the space of harmonic zero-forms with values in $\mathfrak{g}$. We therefore need to calculate

$$h(\phi) = \sqrt{\frac{\det_{\Omega^0(M_3,\mathfrak{g})} L_\phi}{\det_{\Omega^1(M_3,\mathfrak{g})} L_\phi} \cdot \frac{\det_{\Omega^0(M_3,\mathfrak{g})} L_\phi}{\det_{H^0(M_3,\mathfrak{g})} L_\phi} \cdot \frac{\det_{\Omega^0(M_3,\mathfrak{g})} L_\phi}{\det_{H^0(M_3,\mathfrak{g})} (L_\phi)}} \,. \tag{6.19}$$

As described in paper I, we can make a partial cancellation in the above determinants. By decomposing differential one-forms into those along the contact form κ, called vertical one-forms, and those not along κ, called horizontal one-forms, we can cancel the part in the denominator coming from vertical one-forms. Hence, the above ratio of determinants can be written entirely in terms of determinants of L_ϕ acting on horizontal differential forms (defining a zero-form to be horizontal). Moreover, following [11, 21], we can decompose horizontal forms into representations of the $U(1)$ action generated by $\mathcal{L}_v$. That is, let $\xi \in \Omega^\bullet_H(M_3)$. We can write ξ as

$$\xi = \sum_{t=-\infty}^{\infty} \xi_t, \qquad \mathcal{L}_v \xi_t = 2\pi i t \cdot \xi_t \,. \tag{6.20}$$

As explained in [11, 21], the geometrical interpretation of ξ_t is the following. Let L denote the line bundle associated with the $U(1)$ fibration which describes the Seifert manifold M_3. Then ξ_t can be identified with sections of L^t, where L^t is defined by

$$L \otimes L \ldots \otimes L, \qquad \text{if } t > 0 \tag{6.21}$$

$$L^0 = I, \qquad \text{(the trivial line bundle)} \tag{6.22}$$

$$L^* \otimes L \ldots \otimes L^*, \qquad \text{if } t < 0 \,, \tag{6.23}$$

where L^* is the line bundle dual to L. Hence we can write

$$\Omega^\bullet_H(M_3) = \bigoplus_{t-\infty}^{\infty} \Omega^\bullet(\Sigma_2, L^t) \,, \tag{6.24}$$

where Σ_2 denotes the base of the Seifert fibration. That is, horizontal differential forms can be "Fourier expanded", and the "Fourier coefficients" take value in the line bundle L^t. Finally, using a complex structure on Σ_2, the space of horizontal one-forms can be decomposed into $(1,0)$- and $(0,1)$-forms. Using all of the above ingredients, our determinant (6.19) can be written as

$$h(\phi) = \prod_t \left(\sqrt{\frac{\det_{\Omega^0(\Sigma_2, L^t\otimes\mathfrak{g})}(L_\phi)}{\det_{\Omega_H^{1,0}(\Sigma_2, L^t\otimes\mathfrak{g})}(L_\phi)} \cdot \frac{\det_{\Omega^0(\Sigma_2, L^t\otimes\mathfrak{g})}(L_\phi)}{\det_{\Omega_H^{0,1}(\Sigma_2, L^t\otimes\mathfrak{g})}(L_\phi)}} \right) \cdot \frac{1}{\det_{H^0(M_3,\mathfrak{g})}(L_\phi)}, \tag{6.25}$$

where the determinant is now for the adjoint action on the Lie algebra $\mathfrak{g}$. For each value of t in the above product, the operator L_ϕ acts with the same eigenvalue on the space in the denominator and the numerator. Most of the terms in this product will cancel out between the denominator and the numerator, since a zero-form and a horizontal $(1,0)$-form (or a $(0,1)$-form) has almost the same number of degrees of freedom. The difference of degrees of freedom can be determined using the Atiyah-Singer index theorem, which we described in chapter 5.

The Atiyah-Singer index theorem comes into play in the following way. The space of (0,1)-forms on a two-dimensional manifold Σ_2 can be decomposed into $\bar{\partial}$-exact (where $\bar{\partial}$ is the Dolbeault operator) one-forms and harmonic harmonic $(0,1)$-forms. Hence, the difference of number of elements in $\Omega^{(0,1)}(\Sigma_2)$ and $\Omega^{(0,0)}(\Sigma_2)$ is given by the difference of the dimension of the space of harmonic (0,0)-forms and harmonic $(0,1)$-forms. From chapter 5 and equation (5.8), we know that this number is calculated using the Atiyah-Singer index theorem, equation (5.9). In (6.25), the differential forms take value in a vector bundle, so it is the index of the Dolbeault complex twisted by a vector bundle which we need to calculate. The relevant formulas are written in chapter 5, equation (5.13), where V in those formulas are understood as L^t. After the application of the index theorem, the resulting products over t can be calculated using ζ-function regularization. In paper I, we are performing the above described calculation in a situation where the base of the Boothby-Wang fibration is an orbifold. In this case, there is a generalization [49] of the Atiyah-Singer index theorem to a vector bundle over an orbifold.†

The details of the above described calculation can be found in paper I, and the end result is given by equation (6.25) in that paper. After the dust settles, we are left with a matrix model. This result, that the contribution to the partition function originating from fluctuations around an isolated trivial flat connection can be expressed as a matrix model, was first found in [53, 59], so it is not a new result. However, the derivation outlined above is new, and this way of writing Chern-Simons theory using a contact structure can be generalized to higher dimensional Chern-Simons like theories, which we will now turn to.

† An older name for such a bundle is V-bundle, which explains the title of [49].

7. Higher dimensional generalizations

The reformulation of Chern-Simons theory using a contact structure described in the previous chapter has a natural generalization to higher dimensional manifolds. In this chapter we will describe this generalization. In many ways, the ideas presented below can be viewed as a generalization of the construction by Baulieu, Losev and Nekrasov in [10] to a contact manifold. We will first write down a set of transformations which can be defined on any manifold using a vector field v. We will then specialize to contact manifolds, with v the Reeb vector field, and define a set of observables. The focus will be on the theory in five dimensions. When the five-manifold is K-contact, we will describe the set of equations that the theory localizes to. The outline below is based on paper II, and as in the previous chapter the aim is summarize the main points without going into too many technical details.

7.1 The cohomological multiplet and transformations

Let v be a vector field on some manifold M. Let, as above, A denote a connection one-form on a principal G-bundle over M, Ψ an odd one-form and σ an even zero-form. Let χ^a denote a set of odd differential forms on M and let H^a denote a set of even differential forms on M. The index a enumerates the differential forms. All fields take values in $\mathfrak{g}$. Using the above data, let us define the transformations

$$\begin{aligned}
\delta A &= \Psi \\
\delta\Psi &= \iota_v F + i\, d_A\sigma \\
\delta\sigma &= -i\,\iota_v\Psi \\
\delta\chi^a &= H^a \\
\delta H^a &= \mathcal{L}_v^A\chi^a - i[\sigma,\chi^a]\ .
\end{aligned} \tag{7.1}$$

These transformations fulfill $\delta^2 = \mathcal{L}_v + G_\Phi$, where G_Φ denotes a gauge transformation with parameter $i\sigma - \iota_v A$, on all fields. These transformations are a generalization of the ones defined in (6.5) for a three-dimensional manifold. In that case, the index a took only one value, and χ and H were zero-forms.

We refer to the set of fields $(A, \Psi, \sigma, \chi^a, H^a)$ and transformation δ as a cohomological multiplet using the vector field v. It can be defined on any manifold M with a choice of vector field v.

In order to have an interesting theory we must also define observables, and this is what we will do next.

7.2 Observables

In order to define observables, we give M more structure. We let M be a $2n+1$ dimensional contact manifold, and let v be a Reeb vector field. In three dimensions, the action (6.6) is invariant under the transformations (6.5). We therefore call this action a non-trivial observable of the theory, since it is δ-closed but not δ-exact. Once we have the set of transformations (7.1) for an arbitrary $2n+1$ dimensional contact manifold, the natural question is if there are any interesting observables in this theory. Indeed there is. In paper II we construct observables for the theory defined on contact manifolds with dimension five, seven and nine. Let us illustrate the method to construct observables in the case of a five-dimensional contact manifold. The expressions for the observables on higher dimensional manifolds can be found in appendix B in paper II.

In order to illustrate the procedure, we start by looking more closely at the three-dimensional observable S_{SCS_3}, given by (6.9). We see that only the fields A, σ, Ψ appears. Moreover, the combination $A - i\kappa\sigma$ transforms as

$$\delta(A - i\kappa\sigma) = \Psi - \kappa\, \iota_v \Psi = \iota_v(\kappa \wedge \Psi) \tag{7.2}$$

and the transformation for Ψ can be written as

$$\delta\Psi = \iota_v F(A - i\kappa\sigma) + i\,\kappa \mathcal{L}_v^A \sigma\ . \tag{7.3}$$

Varying the action (6.9), we find

$$\begin{aligned}
\delta S_{SCS_3} &= \frac{k}{4\pi}\mathrm{Tr}\int_{M_3} \left(2\delta(A - i\kappa\sigma) \wedge F(A - i\kappa\sigma) - 2\kappa \wedge \delta\Psi \wedge \Psi\right) \\
&= \frac{k}{4\pi}\mathrm{Tr}\int_{M_3} \left(2\iota_v(\kappa \wedge \Psi) \wedge F(A - i\kappa\sigma) + 2\kappa \wedge \Psi \wedge \iota_v F(A - i\kappa\sigma)\right) \\
&= \frac{k}{4\pi}\mathrm{Tr}\int_{M_3} \left(-2\kappa \wedge \Psi \wedge \iota_v F(A - i\kappa\sigma) + 2\kappa \wedge \Psi \wedge \iota_v F(A - i\kappa\sigma)\right) = 0\ .
\end{aligned} \tag{7.4}$$

Above, we have in the first line used that M does not have a boundary in order to drop boundary terms, and in the passing from the second to the third line we have used an analog of integration by parts for the operator ι_v: we have $0 = \iota_v(\kappa \wedge \Psi \wedge F)$ since a four form is identically zero on a three-dimensional manifold. Hence, $\iota_v(\kappa \wedge \Psi) \wedge F = -\kappa \wedge \Psi \wedge \iota_v F$.

7.2.1 Five-dimensional observables

Going up to five dimensions, we can define two different observables. First, we can immediately lift the observable constructed in the three-dimensional case to five dimensions. Namely,

$$S_{SCS_{3,2}} = \mathrm{Tr} \int_{M_5} \left(d\kappa \wedge CS_3(A - i\kappa\sigma) - d\kappa \wedge \kappa \wedge \Psi \wedge \Psi \right) , \tag{7.5}$$

where $CS_3(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$ is the three-dimensional Chern-Simons form, automatically fulfills $\delta S_{SCS_{3,2}} = 0$.

Also, using the five-dimensional Chern-Simons form $CS_5(A) = A \wedge dA \wedge dA + \frac{3}{2} A \wedge A \wedge A \wedge dA + \frac{3}{5} A \wedge A \wedge A \wedge A \wedge A$, we can construct the observable

$$S_{SCS_5} = \frac{k}{24\pi^2} \mathrm{Tr} \int_{M_5} \left(CS_5(A - i\kappa\sigma) - 3\kappa \wedge \Psi \wedge \Psi \wedge F(A - i\kappa\sigma) \right) . \tag{7.6}$$

In the same way as for the three-dimensional case, we can show that the above action fulfills $\delta S_{SCS_5} = 0$. For this observable, also in the same way as for the three-dimensional case, k must be an integer, $k \in \mathbb{Z}$, in order for the exponential of S_{SCS_5} to be gauge invariant.

7.3 Localization locus of the five-dimensional theory

Now that we have constructed observables for the five-dimensional theory, we want to localize the path integral. In order to apply localization, we have to require M_5 to be a five-dimensional K-contact manifold, that is a contact manifold with an associated metric such that the Reeb vector field generates an isometry. With these assumptions, we will now describe to which set of equations the five-dimensional theory localizes.

In order to localize the theory, we need to determine the (χ, H) multiplet in the transformations (7.1) for a five-dimensional K-contact manifold. To do this, we will first describe how to decompose the space of differential two-forms on a contact manifold with an associated metric. First, we can decompose a two-form ω into one part along κ (the vertical part) and the rest (the horizontal part):

$$\omega = \kappa \wedge \iota_v \omega + \iota_v(\kappa \wedge \omega) =: \omega_V + \omega_H . \tag{7.7}$$

We can therefore write the space of two-forms on M_5 as

$$\Omega^2(M_5) = \Omega^2_V(M_5) \oplus \Omega^2_H(M_5) \ . \tag{7.8}$$

One of the properties of an associated metric, namely $g(v) = \kappa$, implies the following property for the Hodge star when acting on a two-form ω:

$$\iota_v(*\omega) = *(\kappa \wedge \omega) \ . \tag{7.9}$$

Using the above relations, the operators

$$P_\pm = \frac{1}{2}(1 \pm \iota_v *) \tag{7.10}$$

are projectors when acting on $\Omega^2_H(M_5)$, that is, they fulfill

$$\begin{aligned} P_\pm P_\mp &= 0 \ , \\ P_\pm^2 &= P_\pm \ . \end{aligned} \tag{7.11}$$

Using these projectors, we can decompose $\Omega^2_H(M_5)$ further:

$$\Omega^2_H(M_5) = \Omega^{2+}_H(M_5) \oplus \Omega^{2-}_H(M_5) \ . \tag{7.12}$$

Let now $\chi^+_H \in \Omega^{2+}_H(M_5)$ and $H^+_H \in \Omega^{2+}_H(M_5)$. These are the fields that will enter in the transformations (7.1) in the five-dimensional theory. An interesting point is that the field content in the five-dimensional theory can be identified with the field content of five-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, see paper II for more details.

As explained in paper II, this multiplet allows us to localize the path integral of δ-closed observables in the five-dimensional theory to the solutions of the equations

$$\begin{aligned} \kappa \wedge F &= - * F \ , \\ d_A \sigma &= 0 \ , \end{aligned} \tag{7.13}$$

modulo gauge invariance, and the rest of the fields are set to zero. In analogy with the instanton equations in four dimensions, we call a field configuration fulfilling the first equation above a *contact instanton*. For a discussion of analogies between instantons on four-dimensional manifolds and five-dimensional contact instantons, see section 3 in paper II. Recently, other aspects of the contact instanton equation have been studied in [88]. The contact instanton equation has also appeared in a slightly different context in [38].

7.4 Partition function of the five-dimensional theory

We will now describe the main steps in the calculation of the partition function of the five-dimensional theory on a K-contact manifold, that is, how to calculate the path integral

$$Z_5(M_5,k,w) = \int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\sigma\; e^{iS_{SCS_5}+iwS_{SCS_{3,2}}} \tag{7.14}$$

using localization. Above, $k \in \mathbb{Z}$ is included in the definition of S_{SCS_5}, equation (7.6), and w is another parameter, not restricted to be an integer. As explained above, for M_5 being a K-contact manifold, the partition function $Z_5(M_5,k,w)$ will be localized to the solution of the equations (7.13). As in the three-dimensional calculation, we restrict ourselves to five-manifolds with $H_1(M_5,\mathbb{R}) = 0$. A flat connection is a solution to the first equation in (7.13), and for the trivial flat connection, the second equation in (7.13) gives that the scalar field σ is a constant. As in the three-dimensional case, since $H_1(M_5,\mathbb{R}) = 0$, the trivial flat connection is an isolated point in the space of solutions to (7.13). We call the part of the partition function which comes from fluctuations around this solution to (7.13) the perturbative part of the partition function. The calculation of the exact perturbative partition function can be performed in exactly the same way as the analogues calculation in three dimensions, and it is outlined in appendix C in paper II. The result is again given by a matrix model. For M_5 being the five-sphere, the matrix model is given by equation (4.15) in paper II. The main difference between the three-dimensional and five-dimensional calculations is that the Atiyah-Singer index theorem gives extra terms in the higher dimensional case, as seen by comparing equations (5.13) and (5.14) in chapter 5. This difference makes the 1-loop determinant different in the two cases, and the matrix model for the theory on S^5 is a little bit more complicated than the corresponding one on S^3. Obviously, it would be very interesting to understand and study the matrix model arising in the five-dimensional theory further.

8. Discussion

In this final chapter we will provide some speculations about the five-dimensional theory that we study in paper II. We will discuss both a possible mathematical interpretation of the model and also the relation to physical five-dimensional Yang-Mills theory.

8.1 Possible invariant of contact five-manifolds

We start with a speculation about a mathematical interpretation. Let us first determine which of the fields we are integrating in (7.14) that are auxiliary. We remember that in the three-dimensional theory, all fields but the gauge field were auxiliary, that is, they could be trivially integrated out from the path integral. Since both of the observables $S_{SCS_{3,2}}$, equation (7.5), and S_{SCS_5}, equation (7.6), have the shift symmetry

$$\tilde{\delta}A = \xi\kappa, \qquad \tilde{\delta}\sigma = -i\xi, \qquad \xi \in C^\infty(M_5) \,, \tag{8.1}$$

we can argue that the path integral defined in (7.14) will be the same as the path integral over the connection A and the odd one-form Ψ. Hence, σ is an auxiliary field in the five-dimensional theory as well. However, since the fields A and Ψ couple to each other in the observable (7.6), we can not trivially integrate out Ψ, as we could in the three-dimensional case. Therefore, we cannot integrate out the term involving the contact form κ, and we therefore expect that the choice of contact structure enters the story in a more drastic way in the five-dimensional theory, as compared to three dimensions. This can also be argued from the fact that the localization equations of the theory are defined using κ, see equation (7.13). Formally, since the observables (7.5) and (7.6) are defined without the use of any metric on M_5, we expect the partition function of the five-dimensional theory $Z_5(M_5,k,w)$ to give a topological invariant of contact five-manifolds, in analogy with the partition function of three-dimensional Chern-Simons theory. What the nature of this possible invariant of a contact five-manifold is we leave for future work.

8.2 Relation to physical Yang-Mills theory on the five-sphere

Recently, in [41], a five-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory has been constructed on the five-sphere, analogously with the constructions of supersymmetric theories on S^4 and S^3 in [72] and [47]. The construction involves both a vector multiplet and a hypermultiplet. Using localization, the authors in [41] show that the fields in the vector multiplet localize to contact instantons. However, the calculation of the one-loop determinants for the theory on the five-sphere have, in the time of writing, not been calculated.

We believe that the theory constructed in [41] is closely related to the theory described above, in the special case of the contact manifold being S^5. As explained in paper II, the field content of the five-dimensional cohomological multiplet can be mapped to the field content of the five-dimensional $\mathcal{N} = 1$ vector multiplet. Moreover, as shown in equation (3.15) in paper II, the Yang-Mills action is an observable in our five-dimensional theory when the contact manifold is S^5 (or more generally, any K-contact manifold). We therefore conjecture that the action which defines the theory in [41] is some combination of the observables defined in paper II, and that the one-loop determinant for the vector multiplet in [41] is given by the one-loop determinant calculated in paper II, equation (4.15). We also believe that the one-loop determinant for the hypermultiplet in [41] can be calculated with techniques similar to the ones used for the vector multiplet.

The calculation of the partition function on the five-sphere raises a puzzle, since by standard arguments Yang-Mills theory in five dimensions is perturbatively non-renormalizable, and at the same time we are able to calculate the full perturbative answer. However, there have been recent discussions in the literature about the consistency of supersymmetric five-dimensional Yang-Mills theory, see for example [28, 52]. Perhaps the matrix model obtained in this thesis can be a valuable hint towards a better understanding of supersymmetric Yang-Mills theories in five dimensions.

With these speculations, we end the first part of the thesis.

Part II:

Vertex algebras and sigma models

9. Introduction

This part of the thesis is about sigma models and vertex algebras. A sigma model is a field theory in which the fields are interpreted as coordinates on some manifold. They appear in many different parts of physics. For example, they are fundamental building blocks in string theory. The sigma models which are relevant for string theory "live" in two dimensions; they are called two-dimensional sigma models. In this thesis, we will study such two-dimensional sigma models in a quantum mechanical setup. We will study them using the so called *Chiral de Rham complex*, introduced by Malikov, Schechtman and Vaintrob in [57]. Let us explain the origin of this construction. In the introduction to the first part of this thesis, we explained the notion of topological field theories. There are also topological sigma models, originally introduced by Witten in [83, 86]. Topological sigma models compute topological invariants and they are interesting from a purely mathematical perspective. However, they are usually formulated using methods which are difficult to define rigorously mathematically. With the aim of understanding one of the topological sigma models (the so called A-model) more rigorously from a mathematical perspective, Malikov, Schechtman and Vaintrob introduced the Chiral de Rham complex. Technically, the Chiral de Rham complex is a sheaf of vertex algebras; what this means will be explained in following chapters. For an explanation of the name "the Chiral de Rham complex", see section 11.10.

In this thesis, we will argue that the Chiral de Rham complex can be understood as a framework for formal canonical quantization of a large class of sigma models. The Chiral de Rham complex can therefore be used to understand a large class of sigma models quantum mechanically, not only topological sigma models which was its original usage†.

Armed with this interpretation of the Chiral de Rham complex, we will apply it to the following problem. If the sigma model has a symmetry, there is an associated quantity which generates this symmetry; the quantity is called a *current*. Classically, the currents typically form a closed algebra using the Poisson bracket. We will call such an algebra a *symmetry algebra*. In the quantum theory, the Poisson bracket is replaced by the equal time commutator and

† A viewpoint similar to ours is advocated in [58].

the currents should now be understood as operators. In this thesis, we will construct quantum versions of the currents within the framework of the Chiral de Rham complex. This is in itself a non-trivial problem. Once they are constructed, we will calculate the equal time commutator between the quantum versions of the currents and see if they still form a closed algebra.

In general, if we have currents which fulfill a symmetry algebra classically but not quantum mechanically, we say that we have an anomaly. An interesting result coming out from the above outlined interpretation of the Chiral de Rham complex is the following. We will introduce below a sigma model which classically has $N = (2,2)$ superconformal symmetry when the target manifold is Kähler. However, the quantum version of the currents do not fulfill the $N = (2,2)$ superconformal symmetry algebra when the target manifold is only Kähler; we have an anomaly. If the target manifold is Calabi-Yau, with a Ricci-flat metric, the anomalous term in the algebra vanishes. This was first calculated in [13, 40]. We will comment further about this result in the last chapter.

Classically, the symmetries of a sigma model are determined by geometrical structures on the target space. This was first realized in [90, 3], and it has since then been further explored in many different directions. In [42, 43], Howe and Papadopoulos found an interesting connection between symmetries of the sigma model and covariantly constant differential forms on the target manifold. Under certain conditions, manifolds which admit covariantly constant forms can be classified, see section 10.3.4 for the classification. In this thesis, we will "lift" the construction of Howe and Papadopoulos to the Chiral de Rham complex framework. The main calculation in this thesis is that of the so called *Odake algebra* [71]. In this calculation, we find no anomaly.

Other works in the physics literature which in some way or another use the Chiral de Rham complex include [32–34], where the Chiral de Rham complex appeared when discussing the so called infinite volume limit of sigma models, and [48, 87], where it has appeared in the discussion of half-twisted sigma models.

This part of the thesis is organized as follows. The first two chapters serve as background for the papers III and IV. In chapter 10, we give an introduction to the classical supersymmetric sigma model. We will also describe Poisson vertex algebras, which is the mathematical structure we use to describe sigma models in the Hamiltonian framework. In chapter 11, we will give an introduction to vertex algebras and also outline the construction of the Chiral de Rham complex. In chapter 12, we will summarize the main results of paper III and IV, and in chapter 13 we will discuss some open problems.

10. Classical sigma models and Poisson vertex algebras

A sigma model is a theory of maps from a d-dimensional manifold Σ, the *source manifold*, to a D-dimensional manifold M, the *target space*. Let us denote the map from Σ to M by X:

$$X:\ \Sigma \to M\ . \tag{10.1}$$

Choosing local coordinates ξ^a, $a = 1,2,\dots d$, on Σ, and local coordinates x^i, $i = 1,2,\dots D$ on M, we identify the field $X^i(\xi)$ with the coordinate x^i on M. Let us now specialize to a two-dimensional source manifold. In this case, Σ is usually called the *worldsheet*. We will consider the case when Σ is a cylinder, $\Sigma = \mathbb{R}\times S^1$, where S^1 is the unit circle. We denote the local coordinates on this cylinder by (t,σ). We refer to $t\in\mathbb{R}$ as the time coordinate, and σ as the space coordinate. Below, we will introduce certain aspects of two-dimensional sigma models which serve as a background for the papers III and IV. We will also introduce a for us convenient algebraic structure, that of a *Poisson vertex algebra*. For a more extensive introduction to two-dimensional sigma models, see for example the string theory text books [36, 73, 12]. For a review of the relation between symmetries and geometry of the target space of supersymmetric sigma models, see for example [54].

10.1 Bosonic sigma model

10.1.1 Lagrangian formulation

We now have a two-dimensional field theory, with D fields X^i, which we interprete as local coordinates on a manifold M. Given this data, we want to write down an action for our field theory. This requires introduction of additional geometrical structures on M. Choosing a metric g_{ij} on M, we can write down the action

$$S = \frac{1}{2}\int\limits_{\Sigma} g_{ij}(X)dX^i\wedge *dX^j\ . \tag{10.2}$$

Above, d is the de Rham differential on Σ, and $*$ is the Hodge star operator defined using a metric on Σ. On Σ we will work with a flat metric with Minkowski signature, and we can write (10.2) equivalently as

$$S = \frac{1}{2}\int_{\Sigma} d\sigma\, dt\, g_{ij}(X)\left(\frac{\partial X^i}{\partial\sigma}\frac{\partial X^j}{\partial\sigma} - \frac{\partial X^i}{\partial t}\frac{\partial X^j}{\partial t}\right) . \tag{10.3}$$

If the metric g_{ij} on the target manifold is a constant, flat metric, the model defined by the action (10.3) is called a linear sigma model, otherwise it is called a *non-linear sigma model.*

For our considerations below, it is convenient to introduce a new set of coordinates on Σ, denoted by $\sigma^{\pm}$:

$$\sigma^{\pm} = \frac{1}{2}(t \pm \sigma) , \tag{10.4}$$

and derivatives $\partial_{\pm}$, defined by

$$\partial_{\pm} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial\sigma} . \tag{10.5}$$

The coordinates $\sigma^{\pm}$ are known as light-cone coordinates. The equations of motion derived from (10.3) are written in terms of light-cone coordinates as

$$\partial_+\partial_- X^i + \Gamma^i_{jk}\partial_+ X^j \partial_- X^k = 0 , \tag{10.6}$$

where Γ^k_{ij} is the Christoffel symbol of the Levi-Civita connection constructed using the metric g_{ij}, defined by

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(g_{lk,j} + g_{lj,k} - g_{jk,l}\right) , \tag{10.7}$$

where $g_{ij,k} := \partial_k g_{ij}$.

10.1.2 Hamiltonian formulation

The sigma model (10.3) can equally well be formulated in the Hamiltonian framework. The conjugate momenta P_i is defined by

$$P_i = \frac{\delta S}{\delta X^i} = g_{ij}(X)\dot{X}^j , \tag{10.8}$$

where we have defined $\dot{X} := \partial X/\partial t$. Using the above relation, we can write (10.3) as

$$S = \int_{\mathbb{R}} dt \left(\oint_{S^1} P_i \dot{X}^i d\sigma - H\right) , \tag{10.9}$$

where

$$H = \frac{1}{2} \oint_{S^1} \left(g^{ij}(X) P_i P_j + g_{ij}(X) \partial_\sigma X^i \partial_\sigma X^j \right) d\sigma \tag{10.10}$$

is the Hamiltonian in the theory. Above, we have defined $\partial_\sigma := \partial/\partial\sigma$. The configuration space of the theory is given by the loop space LM, that is, the space of smooth maps from S^1 into M:

$$LM = \{S^1 \to M\} . \tag{10.11}$$

The phase space of the model is the cotangent bundle T^*LM of the loop space. From (10.9), we read off that the phase of the sigma model comes equipped with a symplectic form ω given by

$$\omega = \int_{S^1} \delta X^i \wedge \delta P_i \, d\sigma , \tag{10.12}$$

where δ denotes the de Rham differential on T^*LM. The symplectic form (10.12) gives rise to a Poisson bracket $\{\ ,\ \}$ on T^*LM, generated by the relations

$$\begin{gathered} \{P_i(\sigma), X^j(\sigma')\} = \delta_j^i \delta(\sigma - \sigma') \\ \{X^i(\sigma), X^j(\sigma')\} = 0, \quad \{P_i(\sigma), P_j(\sigma')\} = 0 . \end{gathered} \tag{10.13}$$

In the Hamiltonian formalism, the equations of motion are given by

$$\dot{X}^i = \{H, X^i\}, \qquad \dot{P}_i = \{H, P_i\} . \tag{10.14}$$

With the Hamiltonian (10.10), we find

$$\dot{X}^i(\sigma) = g^{ij}(X(\sigma)) P_j(\sigma) \tag{10.15}$$

and

$$\begin{aligned} \dot{P}_i(\sigma) = -\Big(&\frac{1}{2} g^{jk}_{,i}(X(\sigma)) P_j(\sigma) P_k(\sigma) + \frac{1}{2} g_{jk,i}(X(\sigma)) \partial_\sigma X^j \partial_\sigma X^k \\ &+ g_{ik,j}(X(\sigma)) \partial_\sigma X^k \partial_\sigma X^j - g_{ik}(X(\sigma)) \partial_\sigma^2 X^k(\sigma) \Big) . \end{aligned} \tag{10.16}$$

Combining (10.15) and (10.16), we find

$$\ddot{X}^i + \Gamma^i_{jk} \dot{X}^j \dot{X}^k = \partial_\sigma^2 X^i + \Gamma^i_{jk} \partial_\sigma X^j \partial_\sigma X^k , \tag{10.17}$$

which are the same as the equations derived in the Lagrangian approach, equation (10.6).

10.2 Supersymmetric sigma model

10.2.1 Lagrangian formulation

The sigma model described above only involves bosonic fields X^i. We will now introduce fermionic fields $\Psi^i_\pm$, and extend the sigma model to a supersymmetric sigma model. Under a change of coordinates on the target manifold M, $\Psi^i_\pm$ transforms as a section of the tangent bundle TM. A convenient way to introduce the fields $\Psi^i_\pm$ in a supersymmetric way is to extend the surface Σ to a supersurface $\Sigma^{2,2}$. In addition to the even coordinates $\sigma^\pm$, the surface $\Sigma^{2,2}$ also have two odd coordinates $\theta^\pm$. Under a change of coordinates $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$, the odd coordinates $\theta^\pm$ transform as $\theta^\pm \to \tilde{\theta}^\pm = \sqrt{\frac{\partial\tilde{\sigma}^\pm}{\partial\sigma^\pm}}\theta^\pm$.

We introduce an even superfield $\Phi^i(\sigma, t, \theta^+, \theta^-)$, which is a function on $\Sigma^{2,2}$. Since $(\theta^\pm)^2 = 0$, the Taylor expansion in the odd coordinates ends quickly. The coefficients in front of each θ-term are the fields in our theory. We have

$$\Phi^i = X^i + \theta^+\Psi^i_+ + \theta^-\Psi^i_- + \theta^+\theta^- F^i \ . \tag{10.18}$$

Above, F is a field which will turn out to be auxiliary; it can be integrated out from the theory.

We also introduce odd derivatives $D_\pm$, defined by

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\pm\partial_\pm \ . \tag{10.19}$$

$D_\pm$ fulfills the algebra

$$D_\pm^2 = \partial_\pm, \quad D_\pm D_\mp + D_\mp D_\pm = 0 \ . \tag{10.20}$$

Using these odd derivatives and superfields Φ^i, the supersymmetric extension of the model (10.3) is given by action

$$S = \int dt\ d\sigma\ d\theta^+\ d\theta^- g_{ij}(\Phi) D_+\Phi^i D_-\Phi^j \ . \tag{10.21}$$

The above action is manifestly supersymmetric, since it is written in terms of superfields. The equations of motion derived from the above action is given by

$$D_- D_+\Phi^i + \Gamma^i_{jk} D_-\Phi^j D_+\Phi^k = 0 \ . \tag{10.22}$$

10.2.2 Hamiltonian formulation

Similarly as for the bosonic sigma model, we can work with the supersymmetric model formulated in terms of superfields in the Hamiltonian framework. As shown in [89], the Hamiltonian formulation of the model is in terms of two superfields, ϕ^i and S_i. These fields are functions of σ and one odd coordinate, which we denote by θ^1. θ^1 is a linear combination of the original $\theta^{\pm}$. Under a change of coordinates $\sigma \to \tilde{\sigma}(\sigma)$ on S^1, the odd coordinate θ^1 transforms as $\theta^1 \to \tilde{\theta}^1 = \sqrt{\frac{\partial\tilde{\sigma}}{\partial\sigma}}\theta^1$. The fields ϕ^i are interpreted as coordinates on the target manifold M, whereas S_i transforms as a one-form on M. Under a change of coordinates $\sigma \to \tilde{\sigma}(\sigma)$ on S^1, ϕ^i is a scalar whereas S_i transforms as $S_i \to \tilde{S}_i = \sqrt{\frac{\partial\sigma}{\partial\tilde{\sigma}}}S_i$. See paper III for how the fields ϕ^i and S_i are related to the original superfield Φ^i.

The configuration space of the supersymmetric model is the superloop space $\mathcal{L}M$, that is smooth maps from the supercircle $S^{1|1}$ (with coordinates σ, θ^1) to M:

$$\mathcal{L}M = \{S^{1|1} \to M\} \ . \tag{10.23}$$

The phase space is the cotangent bundle of $\mathcal{L}M$, denoted by $T^*\mathcal{L}M$. This space comes equipped with a natural symplectic structure $\omega_{1|1}$, defined by

$$\omega_{1|1} = \int d\sigma\, d\theta^1\ \delta S_i \wedge \delta\phi^i \ . \tag{10.24}$$

This symplectic structure defines a super-Poisson bracket, denoted by $\{\ ,\ \}$, on $T^*\mathcal{L}M$, generated by the relations

$$\{\phi^i(\sigma,\theta^1), S_j(\sigma',\theta'^1)\} = \delta^i_j\delta(\sigma-\sigma')\delta(\theta^1-\theta'^1) \ . \tag{10.25}$$

The Hamiltonian H of the supersymmetric theory is given by

$$H = \frac{1}{2}\int d\sigma\, d\theta^1\ \left(g_{ij}\partial\phi^i D_1\phi^j + g^{ij}S_i D_1 S_j + S_k D_1\phi^i S_l g^{jl}\Gamma^k_{ij}\right) \ , \tag{10.26}$$

where $\partial := \partial_\sigma$, and $D_1 = \frac{\partial}{\partial\theta^1} + \theta^1\partial$.

10.3 Symmetries, currents and geometrical structures

When there is a global symmetry of a model, there is always an associated Nöther current, whose divergence vanishes on-shell, that is with the use of the equations of motion (10.6). Calculating the Poisson bracket between the different currents we expect to get a closed algebra, that is that the Poisson bracket between two currents can be written in terms of currents. We call such a closed algebra between currents a *symmetry algebra*. As we will see, for a sigma model, the existence of global symmetries, apart from the ones that has to do with world sheet translations, is usually associated with additional geometrical structures on the target space.

10.3.1 The bosonic sigma model and the Virasoro algebra

We begin with a symmetry that any sigma model defined by the action (10.2) has. A translation

$$\delta\xi^a = \epsilon v^a \tag{10.27}$$

on the world sheet, where ϵ is some infinitesimal parameter, induces a transformation

$$\delta X^i = \epsilon v^a \partial_a X^i \tag{10.28}$$

on the fields in the theory, and this is a symmetry of the sigma model (10.2). It gives rise to the Nöther current L_{ab}, also known as the energy-momentum tensor. The indices ab are world sheet tensor indices, which in light-cone coordinates (10.4) take the values $+$ or $-$. In these coordinates, the only non-vanishing components of L_{ab} are L_{++} and L_{--}, and we will denote these components by $L_\pm$. They are given by

$$L_\pm = \frac{1}{4} g_{ij}(X)\partial_\pm X^i \partial_\pm X^j \; . \tag{10.29}$$

With the use of the equations of motion (10.6), we find that the divergence of the currents $L_\pm$ are given by

$$\partial^a L_{a\mp} = \frac{1}{2}\left(\partial_+ L_{-\mp} + \partial_- L_{+\mp}\right) = \frac{1}{2}\partial_\pm L_\mp = 0 \; . \tag{10.30}$$

In the equation above, we lower the index a with the Minkowski metric, which in light-cone coordinates only have off-diagonal components, and we have used that the off-diagonal components of L_{ab} vanishes. We call a current whose divergence vanishes a *conserved current*. From the conservation of $L_\pm$, it follows that $L_\pm$ are functions of only $\sigma^\pm$, respectively. We call a current which is a function of only $\sigma^\pm$ a *left/right-moving current*, respectively. The fact that $\partial_\mp L_\pm = 0$ means that we have infinitely many conserved currents, since we can multiply $L_\pm$ by any function $f_\pm(\sigma^\pm)$, that is a function $f_\pm$ such that $\partial_\mp f_\pm = 0$, and we still have $\partial_\mp\left(f_\pm L_\pm\right) = 0$. This is related to the fact that the transformation (10.28) is a symmetry of the sigma model (10.2) not only when v^a is a constant, but also when $\partial_\mp v^\pm = 0$. Such a transformation of the fields X^i is induced by a transformation $\delta\sigma^\pm = \epsilon v^\pm(\sigma^\pm)$ on the world sheet. The finite form of this transformation is reparametrizations $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$. Such transformations of the two-dimensional world sheet are called *conformal* transformations, and the two-dimensional sigma model (10.2) has a conformal symmetry. For more information about two-dimensional conformal field theories, see for example the review [75] or the string theory texts books mentioned in the beginning of this chapter.

In the Hamiltonian formalism, the phase space expression for the currents (10.29) are

$$L_\pm = \frac{1}{4}\left(g^{ij}(X)P_iP_j \pm 2P_i\partial_\sigma X^i + g_{ij}(X)\partial_\sigma X^i \partial_\sigma X^j\right) \tag{10.31}$$

Below, we will introduce a very convenient tool to calculate Poisson brackets between currents, the λ-bracket. The Poisson brackets between $L_\pm$ are given by

$$\begin{aligned}\{L_\pm(\sigma), L_\pm(\sigma')\} &= \pm\left(2\partial'_\sigma\delta(\sigma-\sigma') + \delta(\sigma-\sigma')\partial'_\sigma\right)L_\pm(\sigma') \\ \{L_\pm(\sigma), L_\mp(\sigma')\} &= 0\ ,\end{aligned} \tag{10.32}$$

which can be calculated using the λ-bracket.

The algebra that the currents $L_\pm$ fulfill is called the classical Virasoro algebra.

10.3.2 The supersymmetric sigma model and superconformal symmetry

The supersymmetric sigma model (10.21) has, in addition to the stress energy tensor, conserved currents associated with the supersymmetry of the model. We will call these currents *supercurrents*. The stress energy tensor and the supercurrents can be packed together in a superfield, which is given by

$$T_\pm = g_{ij}(\Phi)D_\pm\Phi^i\partial_\pm\Phi^j\ . \tag{10.33}$$

As shown in paper IV, using the equation of motion derived from the action (10.21), equation (10.22), $T_\pm$ are functions of only $\sigma^\pm$ and $\theta^\pm$, so also in the supersymmetric model we have left/right moving currents. As is also shown in paper IV, the Poisson brackets between the currents $T_\pm$ generate two commuting copies of the $N=1$ superconformal algebra, and we therefore say that the model has an $N=(1,1)$ superconformal symmetry.

10.3.3 Additional symmetries and geometrical structures

If the target space M has additional geometric structures, the supersymmetric sigma model (10.21) in general have more symmetries. The first example of such a connection between symmetry and geometry is given in [90], where it was shown that if the target manifold M is Kähler, then the sigma model has $N=(2,2)$ superconformal symmetry. Moreover, in [3] it was shown that if M is hyperkähler, the sigma model has $N=(4,4)$ superconformal symmetry.

In [42, 43], it was shown that the above two examples are special cases of the following general construction. Let M admit a differential n-form ω which is covariantly constant with respect to the Levi-Civita connection ∇, that is $\nabla\omega=0$. Then the following set of transformations are symmetries of the action (10.21):

$$\delta_\pm\Phi^i = \epsilon_\pm g^{ii_1}\omega_{i_1i_2\ldots i_n}D_\pm\Phi^{i_1}D_\pm\Phi^{i_2}\ldots D_\pm\Phi^{i_n}\ . \tag{10.34}$$

Holonomy	*dim M*	*Name of manifold*
$\mathrm{SO}(n)$	n	Orientable
$\mathrm{U}(n)$	$2n$	Kähler
$\mathrm{SU}(n)$	$2n$	Calabi–Yau
$\mathrm{Sp}(n)$	$4n$	Hyperkähler
$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$	$4n$	Quaternionic Kähler
G_2	7	G_2-manifold
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)$-manifold

Table 10.1: Berger's list of possible holonomy groups.

Above, g^{ij} is the inverse of the metric g_{ij}, so that $g^{ik}g_{kj} = \delta^i_j$, and ϵ are some parameters fulfilling $D_\mp\epsilon_\pm = 0$. The currents associated with these two symmetries we denote by $J_\pm^{(n)}$, and they are given by

$$J_\pm^{(n)} = \omega_{i_1 i_2 \dots i_n} D_\pm\Phi^{i_1} D_\pm\Phi^{i_2} \dots D_\pm\Phi^{i_n} \ . \tag{10.35}$$

We will now discuss a set of examples of manifolds which admits covariantly constant differential forms.

10.3.4 Berger's list

Let (M,g) be a Riemannian manifold of dimension m. There exists a unique torsion free connection on TM, the Levi-Civita connection. The existence of a covariantly constant tensor on M is reflected by the fact that the holonomy group of the Levi-Civita connection on TM is reduced to a subgroup of $O(m)$. Under certain conditions, Berger [14] classified which subgroups of $O(m)$ that are possible as holonomy groups of the Levi-Civita connection. The list of possible groups we will refer to as *Berger's list*. The classification is reviewed in [45], and we refer to this book for detailed explanations of the following stated facts. The conditions which give Berger's list is that (M,g) is required to be a simply-connected, non-symmetric, irreducible manifold. Under these conditions, the possible holonomy groups are given by table 10.1. As mentioned above, for each of these seven cases, there exists covariantly constant forms. For example, on a Kähler manifold, the covariantly constant form is the Kähler form. In each case, for each covariantly constant form we can write down an associated current and the Poisson brackets between the different currents will give a closed algebra. This was first done in [43], and we list the different algebras in section 4 in paper IV.

10.4 Calculating Poisson brackets using the λ- and Λ-bracket

Given the Poisson bracket (10.13) between X^i and P_i, we want to be able to calculate the Poisson bracket between functions of X^i and P_i. For example, we would like to calculate the Poisson bracket between the currents defined in (10.31). Let us introduce the notation $f(\sigma)$ for expressions of the form $f(X(\sigma), \partial X(\sigma), \ldots, P(\sigma), \partial P(\sigma), \ldots)$, that is, f is a function of $X^i(\sigma)$ and $P_i(\sigma)$ and a finite number of their derivatives with respect to σ. The Poisson bracket between the functions $f(\sigma)$ and $g(\sigma')$ can be written in the form

$$\{f(\sigma), g(\sigma')\} = \sum_{n\geq 0} \partial^n_{\sigma'} \delta(\sigma - \sigma') h_n(\sigma') \ , \tag{10.36}$$

for some functions h_n. Following [9], we will now introduce a very convient tool for doing calculations, namely the λ-bracket. Given a Poisson bracket $\{f(\sigma), g(\sigma')\}$ between two different functions $f(\sigma)$ and $g(\sigma')$, we define the λ-bracket $\{\ {}_\lambda\ \}$ by

$$\{f\,{}_\lambda\, g\} := \int e^{\lambda(\sigma-\sigma')} \{f(\sigma), g(\sigma')\} d\sigma \ . \tag{10.37}$$

So instead of writing the Poisson bracket between two functions $f(\sigma)$ and $g(\sigma')$, we write the λ-bracket. The λ-bracket between two functions will give a polynomial in λ. When translating back to the Poisson bracket, λ^n corresponds to $\partial^n_{\sigma'} \delta(\sigma - \sigma')$. This can be derived from the definition (10.37).

The Poisson bracket $\{\ ,\ \}$ has the following properties: it is *anti-symmetric*:

$$\{f(\sigma), g(\sigma')\} = -\{g(\sigma'), f(\sigma)\} \ , \tag{10.38}$$

and for some given functions f, g and h, it fulfills the *Leibniz rule*

$$\{f(\sigma), g(\sigma')h(\sigma')\} = \{f(\sigma), g(\sigma')\}h(\sigma') + g(\sigma')\{f(\sigma), h(\sigma')\} \ , \tag{10.39}$$

and the *Jacobi identity*:

$$\{f(\sigma),\{g(\sigma'),h(\sigma'')\}\} + \{g(\sigma'),\{h(\sigma''),f(\sigma)\}\} + \{h(\sigma''),\{f(\sigma),g(\sigma')\}\} = 0 \ . \tag{10.40}$$

These properties translate into the following properties for the λ-bracket:

Leibniz rule :

$$\{f\,{}_\lambda\, gh\} = \{f\,{}_\lambda\, g\}h + g\{f\,{}_\lambda\, h\} \tag{10.41}$$

Anti-symmetry :

$$\{f\,{}_\lambda\, g\} = -\{g\,{}_{-\partial-\lambda} f\} \tag{10.42}$$

Jacobi identity :

$$\{f_\lambda \{g_\mu h\}\} - \{g_\mu \{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\} \ . \tag{10.43}$$

Here, the bracket on the right hand side in rule for anti-symmetry in computed as follows. First calculate the bracket $\{g_\mu f\}$, then replace μ by $-\partial-\lambda$. The last term in the Jacobi identity is computed in the same way. In addition, one can derive the following property, which is called *sesquilinearity*:

$$\begin{aligned}&\text{Sequilinearity :}\\&\{\partial f_\lambda g\}=-\lambda\{f_\lambda g\},\ \ \{f_\lambda \partial g\}=(\partial+\lambda)\{f_\lambda g\}\,.\end{aligned}\qquad(10.44)$$

Using these algebraic rules for the λ-bracket simplifies calculations of Poisson brackets. The Poisson bracket between X^i and P_i, defined in (10.13), is in λ-bracket notation given by

$$\{P_{i\,\lambda}X^j\}=\delta_i^j\,. \qquad (10.45)$$

10.5 Lie conformal algebra and Poisson vertex algebra

Let us now formally define two algebraic structures, namely a *Lie conformal algebra* and a *Poisson vertex algebra*. We here follow [9]. Let us denote the phase space coordinates X^i and P_i collectively by u_k, $k=1,2,\dots 2D$, so that $X^i=u_i$ and $P_i=u_{D+i}$. Let us also define $u_k^{(n)}:=\partial^n u_k$. In this new notation, the functions $f(\sigma)$ considered above are polynomials in the variables $u_k^{(n)}$, where $k=1,2,\dots 2D$ and n is some finite number. The derivative operator ∂ acts on these polynomials as

$$\partial=\sum_{k,n}u_k^{(n+1)}\frac{\partial}{\partial u_k^{(n)}}\,. \qquad (10.46)$$

The algebra of functions $f(u_k^{(n)})$ together with the derivative (10.46) is called an *algebra of differentiable functions*, and we denote this algebra by $\mathcal{V}$. The λ-bracket, with the properties sesquilinearity, anti-symmetry and Jacobi identity, together with $\mathcal{V}$ and ∂ gives a structure of a *Lie conformal algebra* [25, 46]. The formal definition of a Lie conformal algebra is the following:

Definition 10.1. (Lie conformal algebra [26]) A *Lie conformal algebra* is a $\mathbb{C}[\partial]$-module $\mathcal{V}$ (∂ acts on $\mathcal{V}$), endowed with a $\mathbb{C}$-linear λ-bracket $\mathcal{V}\otimes\mathcal{V}\to\mathbb{C}[\lambda]\otimes\mathcal{V}$ (where $\mathbb{C}[\lambda]$ denote a polynomial in λ with coefficients in $\mathbb{C}$), denoted $\{\ {}_\lambda\ \}$, such that the following axioms hold:

$$\begin{aligned}
&\text{Sequilinearity :}\\
&\{\partial f\,{}_\lambda g\} = -\lambda\{f\,{}_\lambda g\},\ \ \{f\,{}_\lambda \partial g\} = (\partial+\lambda)\{f\,{}_\lambda g\}\\
&\text{Anti-symmetry :}\\
&\{f\,{}_\lambda g\} = -\{g_{-\partial-\lambda} f\} &&(10.47)\\
&\text{Jacobi identity :}\\
&\{f_\lambda\{g_\mu h\}\} - \{g_\mu\{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\}\,. &&(10.48)
\end{aligned}$$

•

If we define a commutative and associative product between the elements in $\mathcal{V}$, then the Leibniz rule, equation (10.41), tells us how the λ-bracket and the product between elements in $\mathcal{V}$ are related. The algebraic structure we obtain is that of a *Poisson vertex algebra*, which is formally defined as follows:

Definition 10.2. (Poisson vertex algebra [26]) A *Poisson vertex algebra* is a tuple $(\mathcal{V},|0\rangle,\partial,\{\ {}_\lambda\ \},\cdot)$ where

- $(\mathcal{V},\partial,\{\ {}_\lambda\ \})$ is a Lie conformal algebra
- $(\mathcal{V},|0\rangle,\partial,\cdot)$ is a unital associative (that is, $f\cdot(g\cdot h)$=$(f\cdot g)\cdot h$), commutative (that is, $f\cdot g = g\cdot f$) differential (that is, ∂ is a derivation) algebra
- the operations $\{\ {}_\lambda\ \}$ and $\cdot$ are related by the Leibniz rule

$$\{f\,{}_\lambda g\cdot h\} = \{f\,{}_\lambda g\}\cdot h + g\cdot\{f\,{}_\lambda h\}$$

•

The strange name “Poisson vertex algebra” comes from the fact that it arises in the so called quasi-classical limit of vertex algebras; see the next chapter.

10.6 SUSY Poisson vertex algebra

For simplicity, in the previous section we only considered the case of a bosonic sigma model. In this case, $\mathcal{V}$ consists of only even objects. In the papers III and IV, we are considering the supersymmetric sigma model, and in this case $\mathcal{V}$ will consist of both even and odd objects. Moreover, in these articles, we work almost exclusively with superfields. There is an analogous construction of the bracket (10.37) for superfields, called the Λ-bracket, denoted by $\{\ {}_\Lambda\ \}$. Let us briefly state the algebraic rules that the Λ-bracket fulfills, since these are the rules we are using when calculating the Poisson brackets in paper IV. To define the Λ-bracket, we introduce together with the even formal variable λ

an odd variable χ, such that $\chi^2 = -\lambda$. In the superfield formulation we have an even derivative ∂ and an odd derivative D, fulfilling $D^2 = \partial$. The commutation relations between (D, ∂) and (χ, λ) are the following:

$$[D, \partial] = 0 \,, \quad [D, \chi] = 2\lambda \,, \quad [\partial, \chi] = 0 \,, \quad [\partial, \lambda] = 0 \,. \tag{10.49}$$

For an element $a \in \mathcal{V}$ with degree $\Delta_a \in \mathbb{Z}_2$ (that is, Δ_a is either 0 or 1), we will denote $(-1)^{\Delta_a}$ by $(-1)^a$. The Λ-bracket fulfills the following algebraic rules:

Sesquilinearity:

$$\{Da\,{}_\Lambda\, b\} = \chi\{a\,{}_\Lambda\, b\}, \quad \{a\,{}_\Lambda\, Db\} = -(-1)^a(D+\chi)\{a\,{}_\Lambda\, b\} \tag{10.50}$$

$$\{\partial a\,{}_\Lambda\, b\} = -\lambda\{a\,{}_\Lambda\, b\}, \quad \{a\,{}_\Lambda\, \partial b\} = (\partial+\lambda)\{a\,{}_\Lambda\, b\} \tag{10.51}$$

Anti-symmetry:

$$\{b\,{}_\Lambda\, a\} = (-1)^{ab}\{a\,{}_{-\Lambda-\partial}\, b\} \tag{10.52}$$

Jacobi identity:

$$\{a_\Lambda\{b_\Gamma c\}\} - (-1)^{(a+1)(b+1)}\{b_\Gamma\{a_\Lambda c\}\} = -(-1)^a\{\{a_\Lambda b\}_{\Lambda+\Gamma} c\} \tag{10.53}$$

Leibniz rule :

$$\{a\,{}_\Lambda\, bc\} = \{a\,{}_\Lambda\, b\}c + (-1)^{(a+1)b} b\{a\,{}_\Lambda\, c\} \,. \tag{10.54}$$

The above described structure defines a *SUSY Poisson vertex algebra* [39]. When translating a Λ-bracket expression to a Poisson bracket expression, $\lambda^n\chi^N$ corresponds to $(-1)^N\partial^n_{\sigma'}D^N_{\sigma'\theta'}\delta(\sigma-\sigma')(\theta-\theta')$. The Poisson bracket between ϕ^i and S_i, defined in (10.25), is written in Λ-bracket notation as

$$\{\phi^i\,{}_\Lambda\, S_j\} = \delta^i_j \,. \tag{10.55}$$

10.7 Summary of the classical part

This concludes the introduction to the classical supersymmetric non-linear sigma model, its symmetries and Poisson vertex algebras. As a summary, for each covariantly closed form in the target space, we can define a conserved current. The Hamiltonian treatment of the two-dimensional sigma model can be formalized mathematically using Poisson vertex algebras. Using the λ-bracket, we have shown how to calculate symmetry algebras between the conserved currents.

We end with a discussion about coordinate transformations on the target manifold M and the Hamiltonian formulation of the supersymmetric non-linear sigma model. The fields in the model are ϕ^i and S_i, which are local coordinates on the phase space $T^*\mathcal{L}M$, the cotangent bundle of the superloop space of M. Every object in the theory, for example Poisson brackets and the currents, are written in terms of these local coordinates together with delta functions. If we perform a change of coordinates on M, with ϕ^i transforming as a coordinate and S_i transforming as a one-form:

$$\tilde{\phi}^a = f^a(\phi), \quad \tilde{S}_a = \frac{\partial g^i}{\partial \tilde{\phi}^a} S_i, \qquad g = f^{-1} \ , \tag{10.56}$$

all the structures remain the same. For example, the Poisson bracket between the new fields $\tilde{\phi}^a$ and $\tilde{S}_a$ is the same as between the old fields ϕ^i and S_i:

$$\{ \tilde{\phi}^a {}_\wedge \tilde{S}_b \} = \{ f^a(\phi) {}_\wedge \frac{\partial g^i}{\partial \tilde{\phi}^b} S_i \} = \frac{\partial g^i}{\partial \tilde{\phi}^b} \frac{\partial f^a}{\partial \phi^i} = \delta^a_b, \tag{10.57}$$

where we have used the Poisson bracket for the fields ϕ^i and S_i. Also, the currents written in terms of phase space coordinates (see paper IV, equation (3.24) for the expressions) are invariant under a change of coordinates on M.

If we formally pass from the classical setup to the quantum mechanical setup, the fields ϕ^i and S_i are promoted to operators, and the Poisson bracket becomes the equal time commutator between operators. If we still, in the quantum mechanical setup, interprete ϕ^i and S_i as local coordinates on $T^*\mathcal{L}M$, we can ask if the equal time commutator between ϕ^i and S_i are invariant under the change of coordinates (10.56), and if the quantum counterpart of the conserved currents are invariant as well. In order to address such questions, we will use the construction known as the *Chiral de Rham Complex*, introduced by Malikov, Schechtman and Vaintrob in [57]. This construction is based on *vertex algebras*, which is the topic of the next chapter.

11. Vertex algebras and the Chiral de Rham complex

In this chapter we will describe the notion of vertex algebras. Vertex algebras were first introduced by Borcherds in [23]. They are a rigorous mathematical description of two-dimensional conformal field theories, and its axioms are inspired by the Wightman axioms [81] for quantum field theories. In this chapter, we will define vertex algebras and describe some of its features, mainly following [46]. We will leave out many technical details, and we refer to [46] to fill in the gaps. After the introduction to vertex algebras, we will describe the construction of the Chiral de Rham Complex, which plays a central role in the papers III and IV.

11.1 Formal distributions

In the following we will consider expressions of the form

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^n \,, \tag{11.1}$$

where a_n are elements of some vector space $\mathcal{U}$, and z is a formal parameter, also called an indeterminate. We call expressions of the form (11.1) a *formal distribution*. We will not be concerned with the convergence of such series, z should be understood as a formal book keeping variable. We can add formal distributions and multiply them by elements in $\mathbb{C}$; they form a vector space over $\mathbb{C}$ and we will denote this vector space by $\mathcal{U}[[z,z^{-1}]]$. However, in general, we can not multiply two formal distributions and get another formal distribution. For example, consider the case of $\mathcal{U} = \mathbb{C}$, and the formal distribution

$$d(z) = \sum_{n\in\mathbb{Z}} z^n \,. \tag{11.2}$$

Multiplying $d(z)$ with itself, we find

$$(d(z))^2 = \sum_{n,m\in\mathbb{Z}} z^{m+n} \,, \tag{11.3}$$

and it is hard to make sense of the coefficient in front of z^k; it is not an element in $\mathbb{C}$.

We define differentiation with respect to z of a formal distribution $a(z) \in \mathcal{U}[[z,z^{-1}]]$ as follows:

$$\partial_z a(z) = \sum_{n\in\mathbb{Z}} n a_n z^{n-1} \ , \tag{11.4}$$

that is, we differentiate term by term in the series. Given a formal distribution $a(z)$ of the form (11.1), we define the *residue* Res_z by

$$\mathrm{Res}_z a(z) = a_{-1} \ . \tag{11.5}$$

More generally, we can consider formal distributions with more than one indeterminate, that is formal expressions of the form

$$a(z,w,\ldots) = \sum_{m,n,\ldots\in\mathbb{Z}} a_{m,n,\ldots} z^m w^n \ldots \ . \tag{11.6}$$

We define the *formal delta-function* $\delta(z-w)$ as a formal distribution in the variables z and w:

$$\delta(z-w) = \frac{1}{z}\sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n \ . \tag{11.7}$$

For a distribution $f(z,w)$, let $i_w f(z,w)$ denote the expansion in positive powers of w. For example, for $f(z,w) = (z-w)^{-1}$, we have

$$i_w \frac{1}{z-w} = \frac{1}{z}\sum_{n\geq 0}\left(\frac{w}{z}\right)^n \ . \tag{11.8}$$

Similarly, let us denote the expansion of $f(z,w)$ in positive powers of z by $i_z f(z,w)$. For example, for $f(z,w) = (z-w)^{-1}$, we have

$$i_z \frac{1}{z-w} = -\frac{1}{z}\sum_{n<0}\left(\frac{w}{z}\right)^n \ . \tag{11.9}$$

An alternative way of writing $\delta(z-w)$ is thus

$$\delta(z-w) = i_w \frac{1}{z-w} - i_z \frac{1}{z-w} \ . \tag{11.10}$$

The properties of the formal delta function $\delta(z-w)$ that we will need are

$$\begin{aligned} (z-w)\partial_w^{j+1}\delta(z-w) &= j\, \partial_w^j \delta(z-w), \quad j \in \mathbb{Z}_+ \\ (z-w)^{j+1}\partial_w^j \delta(z-w) &= 0, \quad j \in \mathbb{Z}_+ \ , \end{aligned} \tag{11.11}$$

which can be derived from the representation (11.10).

Below, we will consider the case when $\mathcal{U}$ is the space of endomorphisms of a vector superspace $V, \mathcal{U} = \mathrm{End}(V)$. V consists of both even and odd elements, and we write $V = V_0 \oplus V_1$. An element $a \in \mathrm{End}(V)$ has *degree* Δ_a if $a(v_\alpha) \in V_{\Delta_a+\alpha}$, where $v_\alpha \in V_\alpha$, and $\alpha, \Delta_a \in \mathbb{Z}_2$. We define the supercommutator $[\ ,\ ]$ between elements in $\mathrm{End}(V)$ as

$$[a,b] = a \circ b - (-1)^{(ab)} b \circ a \,, \tag{11.12}$$

where, as in the previous chapter, we denote $(-1)^{\Delta_a}$ by $(-1)^a$.

Using the supercommutator (11.12), two fields $a(z), b(z) \in \mathcal{U}[[z,z^{-1}]]$ are called *mutually local* if there exists a $N \geq 0$ such that

$$(z-w)^N [a(z), b(w)] = 0 \,. \tag{11.13}$$

The above condition on $a(z), b(z) \in \mathcal{U}[[z,z^{-1}]]$ is equivalent to that the bracket between the two distributions can be written as (Theorem 2.3 (i) [46])

$$[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^j \delta(z-w) c_j(w) \,, \tag{11.14}$$

where $c_j(z) \in \mathcal{U}[[z,z^{-1}]]$.

11.2 Definition of a vertex algebra

Let V be a vector superspace, that is V consists of both even and odd elements. We write $V = V_0 \oplus V_1$. Let $a_{(n)} \in \mathrm{End}(V)$, $n \in \mathbb{Z}$. We define a *field* as a formal distribution $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ where for each $v \in V$, there is an $n \geq 0$ such that

$$a_{(n)} v = 0 \,. \tag{11.15}$$

We call the coefficients $a_{(n)}$ the *Fourier modes* of the field $a(z)$.

We are now ready for the definition of a vertex algebra. Since we are considering a vector superspace V, what we will define below is sometimes called a vertex superalgebra. However, we will drop the word super.

Definition 11.1. (Vertex algebra) A *vertex algebra* consists of the data $(V, |0\rangle, \partial, Y)$, where V is a vector superspace, $|0\rangle$ is an even element in V (called the vacuum), ∂ is an even endomorphism (the translation operator) and Y is a map $a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ from V to fields (the state-field correspondence).

This data should fulfill the following set of axioms:

$$Y(|0\rangle, z) = \mathrm{Id} \tag{11.16}$$
$$Y(a,z)|0\rangle = a + \mathcal{O}(z) \tag{11.17}$$
$$\partial|0\rangle = 0 \tag{11.18}$$
$$[\partial, Y(a,z)] = \partial_z Y(a,z) \tag{11.19}$$
$$(z-w)^N [Y(a,z), Y(b,w)] = 0, \quad N \gg 0 \; . \tag{11.20}$$

Above, Id denotes the identity operator in $\mathrm{End}(V)$, $\mathcal{O}(z)$ denotes a power series in z without a constant term.

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Let us comment on some of the axioms. We will usually denote the field associated to an element $a \in V$ by $a(z)$, that is, $Y(a,z) = a(z)$. Given a field $a(z)$, we get back the corresponding element in V from the second axiom above by acting with $a(z)$ on the vacuum state and sending $z \to 0$. From the second axiom we also see that for a field $a(z)$, $a_{(-1)}|0\rangle = a$, and $a_{(n)}|0\rangle = 0$ for $n > 0$.

The definition of a vertex algebra is inspired by structures that appear in two-dimensional conformal quantum field theories. For example, V is usually referred to as the space of states. The state $|0\rangle$, the vacuum, is translationally invariant by the third axiom, and all fields are mutually local by the last axiom.

11.3 Normally ordered product

As we mentioned in section 11.1, given two formal distributions $a(z)$ and $b(z)$, their naive product $a(z)b(z)$ is in general not a well defined distribution. When $a(z)$ and $b(z)$ are fields, we will now define a product between them, the *normally ordered product*, which again gives a field. In order to do so, we define $a(z)_+$ and $a(z)_-$ by

$$a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-1-n} \; , \qquad a(z)_+ = \sum_{n<0} a_{(n)} z^{-1-n} \; . \tag{11.21}$$

For two fields $a(z)$ and $b(z)$, we define the normally ordered product $:a(z)b(z):$ by

$$:a(z)b(z): \quad := a(z)_+ b(z) + (-1)^{(ab)} b(z) a(z)_- \; . \tag{11.22}$$

Writing $:a(z)b(z):$ as $\sum_{n\in\mathbb{Z}} :ab:_{(n)} z^{-1-n}$, we find that the Fourier modes $:ab:_{(n)}$ are given by

$$:ab:_{(n)} = \sum_{m<0} a_{(m)} b_{(n-m-1)} + (-1)^{(ab)} \sum_{m\geq 0} b_{(n-m-1)} a_{(m)} \; . \tag{11.23}$$

Given any element $v \in V$, there exists $N, M, K \in \mathbb{Z}$ such that $a_{(j)}v = 0$ for $j \geq N$, $b_{(s)}v = 0$ for $s \geq M$, and $b_{(s)}a_{(j)}v = 0$ (where $1 \leq j \leq N$) for $s \geq K$. We find that $:ab:_{(n)} v = 0$ for $n \geq N + M + K$, and we see that $:a(z)b(z):$ is a field. We can therefore multiply fields together and get another field using the normally ordered product. Looking at its definition, it does not come as a surprise that the normally ordered product is not associative nor commutative. Below we will, after the introduction of the λ-bracket, write down useful formulas for changing the order of multiplication when using the normally ordered product.

Remark 11.1. In paper III, the normally ordered product is defined in a different, but equivalent, way, which we now describe. Using the state field correspondence Y, we can alternatively define the normally ordered product $:a(z)b(z):$ between the two fields $a(z)$ and $b(z)$ as the field whose corresponding element in V is given by $a_{(-1)}b$. In order to show that this definition is equivalent to the one given above, we have to introduce some notation. For two fields $a(w)$ and $b(w)$, we define $a(w)_{(n)}b(w)$ by

$$a(w)_{(n)}b(w) := \mathrm{Res}_z \left(i_w(z-w)^n a(z)b(w) - i_z(z-w)^n b(w)a(z) \right) . \tag{11.24}$$

From [46], Proposition 4.4, we have the relation

$$Y(a_{(n)}b, w) = a(w)_{(n)}b(w) , \tag{11.25}$$

and hence we find

$$\begin{aligned} Y(a_{(-1)}b, w) &= a(w)_{(-1)}b(w) \\ &= \mathrm{Res}_z \Big(i_w(z-w)^{-1} a(z)b(w) - (-1)^{(ab)} i_z(z-w)^{-1} b(w)a(z) \Big) \\ &= \mathrm{Res}_z \left(\frac{1}{z} \sum_{n\geq 0} \left(\frac{w}{z}\right)^n a(z)b(w) + (-1)^{(ab)} \frac{1}{z} \sum_{n<0} \left(\frac{w}{z}\right)^n b(w)a(z) \right) \\ &= a(w)_+ b(w) + (-1)^{(ab)} b(w)a(w)_- , \end{aligned} \tag{11.26}$$

which coincides with (11.22). •

11.4 The λ-bracket

Given a formal distribution $a(z, w)$ in two indeterminates z and w, we define the *formal Fourier transform* $F^{\lambda}_{z,w}$ by

$$F^{\lambda}_{z,w} a(z, w) := \mathrm{Res}_z e^{\lambda(z-w)} a(z, w) . \tag{11.27}$$

For the formal delta function $\delta(z-w)$, using the properties (11.11), we find that

$$F^{\lambda}_{z,w} \partial^n_w \delta(z-w) = \lambda^n . \tag{11.28}$$

For two mutually local fields $a(z)$ and $b(z)$, we define the λ-bracket $[a_\lambda b]$, first introduced in [25], by

$$[a_\lambda b] := F^{\lambda}_{z,w}[a(z), b(w)] \ . \tag{11.29}$$

In λ-bracket notation, the bracket defined in (11.14) is written as

$$[a_\lambda b] = \sum_{j=0}^{N-1} \lambda^j c_j \ , \tag{11.30}$$

where we have used (11.28).

The λ-bracket fulfills the following relations [46]:

Anti-symmetry :

$$[a_\lambda b] = -(-1)^{(ab)}[b_{-\partial-\lambda} a]$$

Jacobi identity :

$$[a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = (-1)^{(ab)}[[a_\lambda b]_{\lambda+\mu} c]$$

Sequilinearity :

$$[\partial a_\lambda b] = -\lambda [a_\lambda b], \ \ [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b] \ .$$

Comparing with definition 10.1, we see that a vertex algebra is a Lie conformal algebra, with the above defined λ-bracket.

11.5 Quasi-commutativity, quasi-associativity and quasi-Leibniz

From now on, we will write all the formulas in terms of λ-brackets and we will usually not write out any indeterminates. Moreover, since the only product between fields that we will use is the normally ordered product, we will usually not write out the sign : : explicitly, that is, for two fields a and b we define

$$ab := \ \ :ab: \tag{11.31}$$

As we mentioned above, the normally ordered product is neither associative, nor commutative. Using the λ-bracket, we can write down the rule for changing the order of multiplication of two fields a and b [46]:

$$ab - (-1)^{(ab)} ba = \int_{-\partial}^{0} [a_\lambda b] d\lambda \ . \tag{11.32}$$

The expression on the right hand side is calculated as follows. First we compute the λ-bracket, that will give us a polynomial in λ. Place all the λ's on the left and compute the formal integral, then replace the λ's by the limits 0 and $-\partial$. The above rule is called *quasi-commutativity*.

Since the normally ordered product is non-associative, we have to specify in which order we multiply when we multiply three fields or more. We will indicate the order of multiplication by a parenthesis. When changing the order of multiplication, the following formula is useful [46]:

$$(ab)c - a(bc) = \left(\int_0^{\partial} d\lambda a \right) [b_\lambda c] + (-1)^{(ab)} \left(\int_0^{\partial} d\lambda b \right) [a_\lambda c] \,. \tag{11.33}$$

The terms on the right hand side is computed as follows. First compute the λ-bracket, which will give a polynomial in λ. Then set all the λ's inside the integral, and compute the integral as described above. The operator ∂ only hits the field inside the parenthesis. The above rule is called *quasi-associativity*.

In this chapter, we have defined two products, the λ-bracket and the normally ordered product. The two products fulfill the following relation [46]:

$$[a_\lambda bc] = [a_\lambda b]c + (-1)^{(ab)} b[a_\lambda c] + \int_0^{\lambda} [[a_\lambda b]_\mu c] d\mu \,. \tag{11.34}$$

The above rule is called *quasi-Leibniz* or the *non-commutative Wick formula*.

11.6 Alternative definition of a vertex algebra

The axioms of a vertex algebra that we presented in definition 11.1 have a fairly clear physical meaning. After the introduction of the λ-bracket, we have stated a number of rules for the λ-bracket, which follows from definition 11.1. In fact, there is an alternative definition of a vertex algebra, which is equivalent to the one given in definition 11.1, in which the λ-bracket plays a central role. The definition is the following:

Definition 11.2. (Alternative definition of a vertex algebra [26]) A *vertex algebra* is a tuple $(V, |0\rangle, \partial, [\ _\lambda\ ], :\ :)$ where

- $(V, \partial, [\ _\lambda\ ])$ is a Lie conformal superalgbra (see definition 10.1)

- $(V, |0\rangle, \partial, :\,:)$ is a unital differential superalgebra, satisfying

  Quasi-commutativity:

  $$:ab: -(-1)^{(ab)} :ba:= \int_{-\partial}^{0} [a_\lambda b] d\lambda \,.$$

  Quasi-associativity:

  $$:(:ab:)c: - :a(:bc:) :=: \left(\int_0^{\partial} d\lambda a \right) [b_\lambda c] : +(-1)^{(ab)} : \left(\int_0^{\partial} d\lambda b \right) [a_\lambda c] : \,. \tag{11.35}$$

- the λ-bracket and the product $:\,:$ are related by

  Quasi-Leibniz:

  $$[a_\lambda :bc:] =: [a_\lambda b]c: +(-1)^{(ab)} :b[a_\lambda c]: + \int_0^{\lambda} [[a_\lambda b]_\mu c] d\mu \,. \tag{11.36}$$

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This definition of a vertex algebra is the most convenient one when it comes to calculating symmetry algebras, which is what we are interested in in this thesis. Moreover, it has the same structure as the definition of a Poisson vertex algebra, definition 10.2. The difference between the two definitions is that the product between elements in a vertex algebra is non-commutative and non-associative, and the Leibniz rule looks different.

11.7 Examples of λ-bracket calculations

11.7.1 The $\beta\gamma$-system and the Virasoro algebra

The $\beta\gamma$-system consists of two even fields, $\beta(z)$ and $\gamma(z)$. In λ-bracket notation, their brackets are given by

$$[\beta_\lambda \gamma] = 1 \,, \quad [\beta_\lambda \beta] = [\gamma_\lambda \gamma] = 0 \,. \tag{11.37}$$

where 1 represents the identity operator. Let us define field

$$L = \beta \partial \gamma \,, \tag{11.38}$$

and calculate the bracket $[L_\lambda L]$. We begin by calculating

$$\begin{aligned}
[\beta_\lambda \beta\partial\gamma] &= [\beta_\lambda\beta]\partial\gamma + \beta[\beta_\lambda \partial\gamma] + \int_0^\lambda [[\beta_\lambda\beta]_\mu \partial\gamma] d\mu \\
&= \beta(\lambda+\partial)1 = \lambda\beta\,, \\
&\Rightarrow \\
&[L_\lambda\beta] = (\lambda+\partial)\beta\,.
\end{aligned} \tag{11.39}$$

In the above calculation, we have first used the quasi-Leibniz rule, then sesquilinearity, and finally the anti-symmetry property of the λ-bracket. In the same way, we can calculate

$$[L_\lambda \partial\gamma] = (\lambda+\partial)\partial\gamma\,. \tag{11.40}$$

Using the above two relations, we find

$$\begin{aligned}
[L_\lambda L] = [L_\lambda\beta\partial\gamma] = (\lambda+\partial)\beta\partial\gamma + \beta(\lambda+\partial)\partial\gamma + \int_0^\lambda [(\lambda+\partial)\beta_\mu\partial\gamma] \\
= (2\lambda+\partial)L + \lambda\int_0^\lambda \mu d\mu - \int_0^\lambda \mu^2 d\mu = (2\lambda+\partial)L + \frac{2\lambda^3}{12}\,.
\end{aligned} \tag{11.41}$$

A formal distribution $L(z)$ with $[L_\lambda L]$ given by

$$[L_\lambda L] = (2\lambda+\partial)L + \frac{c}{12} \tag{11.42}$$

is called a *Virasoro formal distribution*, or *Virasoro field*, with *central charge* c ([46], page 31). Hence, we see that $L = \beta\partial\gamma$ is a Virasoro field for the $\beta\gamma$-system, with central charge 2.

11.7.2 The bc-system, the $\beta\gamma$-bc-system and supersymmetry

The bc-system is consists of two odd fields $b(z)$ and $c(z)$, with the brackets

$$[b_\lambda c] = 1\,, \quad [b_\lambda b] = [c_\lambda c] = 0\,. \tag{11.43}$$

The Virasoro field is given by $L = \frac{1}{2}((\partial c)b + (\partial b)c)$, and it gives rise to a Virasoro algebra with central charge 1.

We can combine the $\beta\gamma$- and bc-system into the $\beta\gamma$-bc-system by declaring that the only non-zero brackets are given by

$$[\beta_\lambda\gamma] = 1\,, \qquad [b_\lambda c] = 1\,. \tag{11.44}$$

The Virasoro field of the combined system is given by

$$L = \beta\partial\gamma + \frac{1}{2}\left((\partial c)b + (\partial b)c\right) , \tag{11.45}$$

and in addition to L, we can define the field $G = c\beta + (\partial\gamma)b$. Calculating the λ-brackets, we find that the fields L and G give rise to

$$\begin{aligned}
[L_\lambda L] &= (2\lambda + \partial)L + \frac{3\lambda^3}{12} \\
[L_\lambda G] &= \left(\partial + \frac{3}{2}\lambda\right) G \\
[G_\lambda G] &= 2L + \lambda^2 .
\end{aligned} \tag{11.46}$$

The above algebra is called the $N = 1$ superconformal algebra with central charge 3.

11.8 The quasi-classical limit

We can relate a vertex algebra to a Poisson vertex algebra in the following way. Say that the vertex algebra is generated by a set of relations $[a^i{}_\lambda b^j] = f^{ij}$. For example, the $\beta\gamma$-system is generated by the relations (11.37). We can introduce a parameter $\hbar$ in the game by rescaling the generating bracket of the vertex algebra: $[a^i{}_\lambda b^j]_\hbar = \hbar f^{ij}$, where we have indicated the dependence on $\hbar$. We can then consider a vertex algebra which depends on a parameter $\hbar$, denoted by $V_\hbar$. Looking at the rules for quasi-commutativity, quasi-associativity and quasi-Leibniz (see definition 11.2), we see that the terms which prevents the vertex algebra from being commutative, associative and fulfilling the Leibniz rule (henceforth called the "extra" terms) all involve an application of the λ-bracket. Using the rescaled bracket $[\ {}_\lambda\ ]_\hbar$, these terms will have extra factors of $\hbar$ in them compared to the other terms. Therefore, if we send $\hbar \to 0$ and consider $V_0 = \lim_{\hbar\to 0} V_\hbar$, and introduce a bracket $\{\ {}_\lambda\ \}$ in V_0 by the relation

$$\lim_{\hbar\to 0} \frac{[\ {}_\lambda\ ]_\hbar}{\hbar} = \{\ {}_\lambda\ \} , \tag{11.47}$$

V_0 together with the bracket $\{\ {}_\lambda\ \}$ will fulfill the axioms of a Poisson vertex algebra (see definition 10.2). This limit of a vertex algebra is called the *quasi-classical limit*. It is natural to interprete the "extra" terms as quantum corrections to a classical calculation. For example, taking the quasi-classical limit of the $\beta\gamma$-system, the algebra fulfilled by the Virasoro field L defined in (11.38) descends to the algebra

$$\{L_\lambda L\} = (2\lambda + \partial)L \tag{11.48}$$

when taking the quasi-classical limit, which is the classical Virasoro algebra (see equation (10.32)).

11.9 SUSY vertex algebras

As for a Poisson vertex algebra, we can introduce a supersymmetric version of vertex algebras, called SUSY vertex algebras. They are introduced in [8], and thoroughly studied in [39]. The most useful definition for us is in terms of the supersymmetric analog of the λ-bracket, namely the Λ-bracket, denoted by $[\ {}_\Lambda\ ]$. Most structures are exactly the same as for the SUSY Poisson vertex algebra, described in section 10.5. Again we introduce both an odd derivation D and an even derivation ∂, with $D^2 = \partial$. We also introduce an odd formal variable χ and an even formal variable λ, such that $\chi^2 = -\lambda$. The commutation relations between (D,∂) and (χ,λ) are the same as in equation (10.49). Let us denote $\Lambda = (\lambda,\chi)$ and $\nabla = (\partial,D)$. Using this structure, a SUSY vertex algebra is defined by the following relations:

- Sesquilinearity:

$$[Da\,{}_\Lambda\, b] = \chi[a\,{}_\Lambda\, b], \quad [a\,{}_\Lambda\, Db] = -(-1)^a(D+\chi)[a\,{}_\Lambda\, b] \tag{11.49}$$

$$[\partial a\,{}_\Lambda\, b] = -\lambda[a\,{}_\Lambda\, b], \quad [a\,{}_\Lambda\, \partial b] = (\partial+\lambda)[a\,{}_\Lambda\, b] \tag{11.50}$$

- Anti-symmetry:

$$[b\,{}_\Lambda\, a] = (-1)^{ab}[a\,{}_{-\Lambda-\partial}\, b] \tag{11.51}$$

- Jacobi identity:

$$[a_\Lambda\,[b_\Gamma\, c]] - (-1)^{(a+1)(b+1)}[b_\Gamma\,[a_\Lambda\, c]] = -(-1)^a[[a_\Lambda\, b]_{\Lambda+\Gamma}\, c] \tag{11.52}$$

- Quasi-commutativity:

$$ab - (-1)^{ab}ba = \int_{-\nabla}^{0} [a\,{}_\Lambda\, b]d\Lambda, \tag{11.53}$$

  where the integral term is computed as follows. First compute the Lambda bracket. Second take the derivative with respect to χ to obtain a polynomial in λ. Compute the formal integral with respect to $d\lambda$. Finally evaluate the results replacing λ by the limits: zero and $-\partial$.

- Quasi-associativity:

$$(ab)c - a(bc) = \left(\int_0^\nabla d\Lambda a\right)[b_\Lambda c] + (-1)^{ab}\left(\int_0^\nabla d\Lambda b\right)[a_\Lambda c]\,, \tag{11.54}$$

where the integrals are interpreted as follows: expand the Lambda bracket and put the Λ terms inside of the integral; then take the definite integral as in the previous item.

- Quasi-Leibniz

$$[a_\Lambda bc] = [a_\Lambda b]c + (-1)^{(a+1)b} b[a_\Lambda c] + \int_0^\Lambda [[a_\Lambda b]_\Gamma c] d\Gamma \ . \tag{11.55}$$

As for a (non-SUSY) vertex algebra, there is a quasi-classical limit, recovering the SUSY Poisson vertex algebra described in section 10.5, with bracket $\{\ _\Lambda\ \}$. Some calculations in vertex algebras simplifies considerably when (if it is possible) performed in a supersymmetric version. The main reason for this is that the "quantum" terms involve integration over the odd formal variable χ, and if there are no χ-terms, the integral vanishes.

In a SUSY vertex algebra, if we reintroduce the indeterminates, the fields will depend on two formal parameters, one odd, usually denoted by θ, and one even, which we usually denote by z. Hence, a field is written as $a(z,\theta)$. The translation between (χ,λ) and delta-functions is given by

$$\lambda^n \chi^N \sim (-1)^N \partial^n_{\sigma'} D^N_{w\theta'} \delta(z-w)(\theta-\theta') \ , \tag{11.56}$$

where the odd derivation $D_{w\theta'}$ is given by

$$D_{w\theta'} = \frac{\partial}{\partial\theta'} + \theta' \partial_w \ . \tag{11.57}$$

11.9.1 The $\beta\gamma$-bc-system using superfields

Let us consider a SUSY vertex algebra with two fields, one even field ϕ and one odd field S, with the generating Λ-brackets given by

$$[S_\Lambda \phi] = 1 \ , \quad [\phi_\Lambda \phi] = [S_\Lambda S] = 0 \ . \tag{11.58}$$

Let us define the field

$$T = \partial\phi S + D\phi DS \ , \tag{11.59}$$

and calculate the Λ-bracket $[T_\Lambda T]$. Using the calculation rules above we find

$$[T_\Lambda T] = (2\partial + \chi D + 3\lambda) T + \lambda^2 \chi \ . \tag{11.60}$$

The above SUSY vertex algebra is the $\beta\gamma$-bc-system written in terms of superfields. Let us show this. Reintroducing the indeterminate θ, we can expand ϕ and S in θ:

$$\begin{aligned} \phi &= \gamma + \theta c \\ S &= b + \theta\beta \ . \end{aligned} \tag{11.61}$$

The Λ-brackets decompose to the following λ-brackets:

$$\begin{aligned}
&[S_\Lambda \phi] = 1 \sim [b_\lambda \gamma] + \theta[\beta_\lambda \gamma] - \theta'[b_\lambda c] + \theta\theta'[\beta_\lambda c] = (\theta - \theta') \\
&[S_\Lambda S] = 0 \sim [b_\lambda b] + \theta[\beta_\lambda b] + \theta'[b_\lambda \beta] + \theta\theta'[\beta_\lambda \beta] = 0 \\
&[\phi_\Lambda \phi] = 0 \sim [\gamma_\lambda \gamma] + \theta[\gamma_\lambda c] + \theta'[c_\lambda \gamma] + \theta\theta'[c_\lambda c] = 0 \, .
\end{aligned} \tag{11.62}$$

Comparing the right hand side with the left hand side, we recover the λ-brackets defining the $\beta\gamma$-bc-system, described in section 11.7.2. Decomposing the field $T = G + 2\theta L$, the Λ-bracket $[T_\Lambda T]$ decomposes to

$$[T_\Lambda T] \sim [G_\lambda G] + 2\theta[L_\lambda G] - 2\theta'[G_\lambda L] + 4\theta\theta'[L_\lambda L] \, . \tag{11.63}$$

Using the correspondence (11.56) and the odd derivative (11.57), we find

$$\lambda^n \chi \sim \lambda^n + \theta\theta' \lambda^{n+1} \, . \tag{11.64}$$

Using (11.56) and (11.57), together with the decomposition $T = G + 2\theta L$ we find

$$\begin{aligned}
2\partial T &\sim 2\theta\partial G - 2\theta'\partial G + 4\theta\theta'\partial L \\
\chi D T &\sim 2L + \theta'\partial G + 2\theta\theta'\lambda L \\
3\lambda T &\sim 3\theta\lambda G - 3\theta'\lambda G + 6\theta\theta'\lambda L \, .
\end{aligned} \tag{11.65}$$

Equating the right hand sides of (11.63) and (11.65) we recover the λ-brackets $N = 1$ superconformal algebra defined in (11.46).

11.10 The Chiral de Rham complex

Consider a D-dimensional manifold M. M can be covered with coordinate patches $U_\alpha \simeq \mathbb{R}^D$, such that $M = \cup_\alpha U_\alpha$. For two patches U_α and U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, the local coordinates x^i on U_α and the local coordinates $\tilde{x}^a$ on U_β are related as $\tilde{x}^a = f^a(x)$, for some invertible function f. We say that we use the relations $\tilde{x}^a = f^a(x)$ to "glue" the local coordinate patches together to form a global object. The relations $\tilde{x}^a = f^a(x)$ induce the relation $d\tilde{x}^a = \frac{\partial f^a}{\partial x^i} dx^i$ between the basis of differential one-forms on U_α and U_β, and similarly for other tensors. We denote the inverse of $f^a(x)$ by $g^i(\tilde{x})$, and we will often use the short hand notation

$$f^a_{,i} := \frac{\partial f^a}{\partial x^i} \qquad\qquad g^i_{,a} := \frac{\partial g^i}{\partial \tilde{x}^a} \, . \tag{11.66}$$

In [57], the authors did the following construction. They assigned D copies of the $\beta\gamma$-bc-system to each local coordinate patch U_α. That is, they considered a vertex algebra generated by the fields $(\gamma^i, \beta_i, b_i, c^i)$, $i = 1, 2 \dots D$, with the non-vanishing λ-brackets given by

$$[\beta_{j\,\lambda}\gamma^i] = \delta^i_j \qquad\qquad [b_{j\,\lambda}c^i] = \delta^i_j \,. \tag{11.67}$$

In order to get a global object, we have to say how the fields in the vertex algebra transform when going from patch to patch. Let us denote the fields assigned to the patch U_α by $(\gamma^i, \beta_i, b_i, c^i)$ and the fields assigned to the patch U_β by $(\tilde{\gamma}^a, \tilde{\beta}_a, \tilde{b}_a, \tilde{c}^a)$. On overlapping patches, the fields are related by

$$\begin{aligned} \tilde{\gamma}^a &= f^a(\gamma) \\ \tilde{\beta}_a &= \beta_i g^i_{,a}(f(\gamma)) + \left(\left(g^i_{,ab}(f(\gamma)) f^b_{,j}\right) c^j\right) b_i \\ \tilde{c}^a &= f^a_{,i} c^i \\ \tilde{b}_a &= g^i_{,a}(f(\gamma)) b_i \,, \end{aligned} \tag{11.68}$$

where f is the same function as the one which we use to relate the coordinates in the different patches. Note that we have to specify in which order we multiply the fields, since a vertex algebra is not commutative nor associative. The amazing fact is that, given the λ-brackets of the fields $(\gamma^i, \beta_i, b_i, c^i)$, equation (11.67), the λ-brackets between the fields $(\tilde{\gamma}^a, \tilde{\beta}_a, \tilde{b}_a, \tilde{c}^a)$ are the same. Hence, we can use the relations (11.68) to "glue" together the different vertex algebras assigned to each coordinate patch, and thereby construct a global object.

Remark 11.2. It is a non-trivial fact that the above described construction works. If we would attempt a similar construction for the purely bosonic $\beta\gamma$-system, with transformations

$$
\begin{aligned}
\tilde{\gamma}^a &= f^a(\gamma) \\
\tilde{\beta}_a &= \beta_i g^i_{,a}(f(\gamma)) ,
\end{aligned}
\tag{11.69}
$$

the λ-bracket $[\tilde{\beta}_a{}_\lambda \tilde{\beta}_a]$ would *not* give zero. See [69] for a discussion of how to modify the transformations for β such that the purely bosonic $\beta\gamma$-system can be glued for a certain type of manifolds. For the supersymmetric $\beta\gamma$-bc-system, with the above defined transformations, the construction works for any manifold M.

•

In [57], a number of technical details related to the above construction are addressed. For example, they show how to make sense of the "functions" $f(\gamma)$, where γ^i are fields in a vertex algebra. For a discussion of such technical details, we refer to [57]. The structure which is obtained from the construction in [57] is that of a *sheaf of vertex algebras*. Roughly speaking, a sheaf is a collection of local data attached to local patches on a manifold M, such that they can be consistently glued to define a global object. This is a more general notion than, for example, a fiber bundle. In particular, a sheaf is not required to be a manifold. Since sheaf-theoretical aspects play a minor role to us in this thesis, we content ourselves with this loose description of what a sheaf is and refer to for example [37] for a formal definition. The authors in [57] called the above constructed sheaf of vertex algebras *the Chiral de Rham Complex*. This choice of name can be explained as follows. The word "chiral" comes from the fact that a vertex algebra is usually said to define the chiral part of a two-dimensional conformal field theory. The words "de Rham Complex" comes from the fact that the main motivation of the paper [57] was to put the study of the topological A-model [83, 86], which is a topological sigma model whose "physical" operators can be identified with elements in the de Rham cohomology of the target manifold, on a firmer mathematical footing. As will be explained in the next chapter, in paper III we suggest the use of the Chiral de Rham Complex for a much broader class of sigma models. Therefore, we find the name "the Chiral de Rham Complex" somewhat misleading, and we will from now on usually refer to it as the CDR.

As described above, the $\beta\gamma$-bc-system can formulated in terms of the superfields ϕ^i and S_i, with the only non-vanishing Λ-bracket given by

$$
[S_j{}_\Lambda \phi^i] = \delta^i_j \, . \tag{11.70}
$$

As noticed in [13], the transformation rules (11.68) and the derivation of the Λ-brackets for the new fields $(\tilde{\gamma}^a, \tilde{\beta}_a, \tilde{b}_a, \tilde{c}^a)$ can be easily understood when written in terms of the superfields ϕ^i and S_i. We define new fields $\tilde{\phi}^a$ and $\tilde{S}_a$ from the relations

$$\begin{aligned}\tilde{\phi}^a &= f^a(\phi)\\ \tilde{S}_a &= g^i_{,a}(f(\phi))S_i\ ,\end{aligned} \tag{11.71}$$

and the Λ-bracket between the fields $\tilde{\phi}^a$ and $\tilde{S}_a$ is given by

$$[\tilde{\phi}^a{}_\Lambda \tilde{S}_b] = [f^a(\phi)_\Lambda g^i_{,b}S_i] = g^i_{,b}f^a_{,i} = \delta^a_b\ , \tag{11.72}$$

where we used the quasi-Leibniz rule. Notice that, when written in terms of superfields, the order of multiplication between the factors in the product $g^i_{,b}(f(\phi))S_i$ does not matter, since the Λ-bracket between $g^i_{,b}(f(\phi))$ and S_i has no χ-term. Due to the absence of χ-terms, the above calculation essentially reduces to the one performed in the Poisson vertex algebra in the end of chapter 10. The analogous calculation for the fields $(\tilde{\gamma}^a, \tilde{\beta}_a, \tilde{b}_a, \tilde{c}^a)$ using the λ-bracket will give the same result, but the calculations are quite tedious. This illustrates the advantage of formulating the vertex algebra in terms of superfields: many calculations greatly simplifies.

12. The Chiral de Rham complex and non-linear sigma models

This chapter is a summary of the content in paper III and IV. In paper III, we argue that the Chiral de Rham Complex (henceforth CDR) is a natural framework for understanding the supersymmetric non-linear sigma model in a quantum mechanical setup. In paper IV, we use this idea to extend the calculations of symmetry algebras, which we described classically in chapter 10, to this quantum mechanical setup. The main results of paper IV are the embedding of the currents associated to the manifolds on Berger's list in the CDR and the calculation of a symmetry algebra associated to a Calabi-Yau manifold, the *Odake algebra* [71].

12.1 Formal Hamiltonian quantization of non-linear sigma models

In chapter 10, we described the Hamiltonian formulation of the classical supersymmetric non-linear sigma model, and we showed that the mathematical structure which describes such systems is that of a Poisson vertex algebra. Even though the Poisson brackets between coordinates and momenta are written in terms of local coordinates on the target manifold, we can do a change of coordinates on the target manifold, and the Poisson brackets between the new coordinates and momenta is the same as in the old coordinates, see equation (10.57). Formally, we can therefore define a sheaf of Poisson vertex algebras. In paper III, we argue that a natural framework to make sense of the canonical quantization of such non-linear sigma models is that of CDR, which we described in section 11.10. The Λ-brackets in the $\beta\gamma$-bc-system, equation (11.58), has a natural interpretation as an equal-time commutator between coordinate and momenta in the operator formalism. The key property of the CDR is that even though the equal-time commutator is written in terms of local coordinates on the target manifold, we can consistently glue between different patches and construct a global object, a sheaf of vertex algebras. Moreover, taking the quasi-classical limit of the CDR, described in section 11.8, we get back the sheaf of Poisson vertex algebras which describes the classical non-linear sigma model.

In the framework we have described, we only capture "small" loops, that is, loops which can be contracted to a point. An extension of the CDR framework to a situation where the first homotopy group $\pi_1(M)$ of the target manifold is non-trivial, and therefore all loops cannot be contracted to a point, is considered in [2].

12.2 Mapping differential forms to sections of the CDR

In section 10.3, we showed that for any manifold on Berger's list, we could construct a current associated with a symmetry of the non-linear sigma model. The currents, given by (10.35), are written in terms of the phase space coordinates ϕ^i and S_i as (paper IV, equation (3.24)):

$$\begin{aligned} J_+^{(n)} &= \frac{1}{n!}\omega_{i_1\dots i_n}(\phi)e_+^{i_1}\dots e_+^{i_n} \\ J_-^{(n)} &= \frac{i^n}{n!}\omega_{i_1\dots i_n}(\phi)e_-^{i_1}\dots e_-^{i_n} , \end{aligned} \tag{12.1}$$

where $e_\pm^i$ are defined in terms of ϕ^i and S_i by

$$e_\pm^i := \frac{g^{ij}S_j \pm D\phi^i}{\sqrt{2}} . \tag{12.2}$$

The factor of i is inserted in $J_-^{(n)}$ for computational convenience. These are the "classical" expressions for the currents. Since $e_\pm^i$ transforms as a vector under a change of coordinates on the target manifold, the currents $J_\pm^{(n)}$ are classically invariant under a change of coordinates. Formally, they are well defined sections of the sheaf of Poisson vertex algebras.

In paper III, we show how to "lift" these currents into the CDR, that is how to make them well defined sections of the CDR. First off, since ϕ^i and S_i do not commute, we have to choose an order of multiplication between the different terms. After this is done, we have to check if the current is invariant under the transformations

$$\begin{aligned} \tilde{\phi}^a &= f^a(\phi) \\ \tilde{S}_a &= g^i_{,a}(f(\phi))S_i , \end{aligned} \tag{12.3}$$

where ϕ^i and S_i are now understood as fields in a SUSY vertex algebra. Below, we will call $g^i_{,a}$ and $f^a_{,i}$ the *transition functions*.

For $n \geq 2$, the currents (12.1) are *not* invariant under a of change of coordinates on the target manifold in the quantum mechanical setup. Let us show this for $n = 2$. We will need to use that the Λ-brackets between $e^i_\pm$ are given by (paper IV, equation (4.1)-(4.3))

$$[e^i_\pm {}_\Lambda e^j_\pm] = \pm\chi g^{ij} + \frac{1}{\sqrt{2}}\left(g^{kj}\Gamma^i_{mk}e^m_\mp - g^{ki}\Gamma^j_{mk}e^m_\pm\right) , \tag{12.4}$$

$$[e^i_+ {}_\Lambda e^j_-] = \frac{1}{\sqrt{2}}\left(g^{kj}\Gamma^i_{mk}e^m_+ - g^{ki}\Gamma^j_{mk}e^m_-\right) , \tag{12.5}$$

and we also have

$$[e^i_\pm {}_\Lambda f(\phi)] = \frac{1}{\sqrt{2}}g^{ij}f_{,j} , \tag{12.6}$$

for any smooth function f on M, where $f_{,j} = \partial_j f$.

The current $J^{(2)}_+$ is written as

$$J^{(2)}_+ = \frac{1}{2}\omega_{ij}(\phi)\left(e^i_+ e^j_+\right) , \tag{12.7}$$

where we have specified the order of multiplication. Changing coordinates, we find †

$$\tilde{J}^{(2)}_+ = \frac{1}{2}\tilde{\omega}_{ab}\left(\tilde{e}^a_+\tilde{e}^b_+\right) = \frac{1}{2}\left(g^i_{,a}g^j_{,b}\omega_{ij}\right)\left(\left(f^a_{,k}e^k_+\right)\left(f^b_{,l}e^l_+\right)\right) . \tag{12.8}$$

In order to see if this expression is the same as $J^{(n)}_\pm$, we want to put all transition functions together so that we can cancel them out. Using the quasi-associativity rule (11.54) and the Λ-brackets (12.4) and (12.6) we find ‡

$$\tilde{J}^{(2)}_+ = J^{(2)}_+ + \frac{1}{2}g^i_{,a}\, f^a_{,kl}\, g^{kj}\omega_{ij}\partial\phi^l . \tag{12.9}$$

We see that $\tilde{J}^{(2)}_+ \neq J^{(2)}_+$ and therefore (12.7) is not a well defined section of the CDR. However, from the structure of the "quantum"-term in the expression above, we see that we can cancel it out by adding a term to $J^{(2)}_+$ involving the Levi-Civita connection. Indeed, the current

$$J^{(2)}_{+q} = \frac{1}{2}\omega_{ij}(\phi)\left(e^i_+ e^j_+\right) + \frac{1}{2}\Gamma^i_{jk}g^{jl}\omega_{il}\partial\phi^k \tag{12.10}$$

is a well defined section of the CDR. Here we have introduced a label q to specify that these are the "quantum" currents. With the same added term, $J^{(2)}_{-q}$

† When the Λ-brackets between the factors in a term lacks a χ-term, we do not need to specify the order of multiplication, since in this case the normally ordered product is both associative and commutative.

‡ Note that $g^i_{,a}$ is a transition function, whereas g^{kj} is the inverse of the metric on the target manifold.

is a well defined section of the CDR. This modification for the currents $J_{\pm}^{(2)}$ into well-defined sections of the CDR was first found in [13]. In paper IV, we generalize this construction to work for any n, not only $n = 2$. For $n > 2$, we have to repeatably use the quasi-associativity rule in order to put the transition functions together. In order to cancel out the terms which arise from the use of the quasi-associativity rule, we have to add more than one term to the "classical" expression; the number of terms we have to add depends on n. The expression for $J_{\pm q}^{(n)}$, for general n, is given in paper IV, theorem 6.1. Since we take a differential n-form $\omega_{i_1 \ldots i_n}$ and construct sections of the CDR, we say that we map differential forms into the CDR. It is a non-trivial fact that it can be done in this way.

For other tensors, for example the metric g_{ij}, the construction outlined above to define sections of the CDR does not work. The reason is that we are heavily using the fact that an n-form $\omega_{i_1 \ldots i_n}$ is anti-symmetric in all its indices in the proof of theorem 6.1 in paper III. An important current for the classical supersymmetric sigma model is the combination of the Virasoro current and the supercurrent, (10.33), which generates the $N = (1,1)$ superconformal symmetry algebra. Classically, this current is written in terms of the phase space coordinates $e_{\pm}^{i}$ as (paper IV, equation (3.24)):

$$T_{\pm} = \pm \left(g_{ij} De_{\pm}^{i} e_{\pm}^{j} + g_{ij} \Gamma^{i}{}_{kl} D\phi^{k} e_{\pm}^{l} e_{\pm}^{j} \right) , \tag{12.11}$$

and we see that it involves the metric tensor g_{ij}. If the target space is a general Riemannian manifold, it is an open problem how to lift these two currents into well-defined sections of the CDR. If we write

$$T_{\pm} = \frac{P \pm H}{2} , \tag{12.12}$$

with P given by

$$P = D\phi^{i} DS_{i} + \partial\phi^{i} S_{i} , \tag{12.13}$$

it is shown in [13] that P *can* be lifted to a section of the CDR for a general Riemannian manifold; the section is given by

$$P_{q} = D\phi^{i} DS_{i} + \partial\phi^{i} S_{i} - \partial D \log \sqrt{\det g_{ij}} . \tag{12.14}$$

When the target space is Calabi-Yau, with a Ricci-flat metric, also H can be lifted to a section of the CDR. This was first shown in [40], and the explicit expression for H_q is given by

$$H_{q} = g_{ij} D\phi^{i} \partial\phi^{j} + g^{ij} S_{i} DS_{j} + \Gamma^{j}_{kl} g^{il} D\phi^{k} (S_{j} S_{i}) . \tag{12.15}$$

To check that this expression is well-defined by a straightforward calculation, as we did with $J_{\pm q}^{(n)}$, is very difficult. Instead, it can be shown indirectly, by using that it can be written as a Λ-bracket between two well-defined currents, see the discussion below.

12.3 Symmetry algebras in the CDR framework

Let us first recall some facts about the classical supersymmetric sigma model. Choosing a metric, we have the currents $T_\pm$. As explained in section 10.3, for every manifold on Berger's list, we have at least one covariantly constant differential form, and we can write down additional currents. Taking the Poisson brackets between the different currents we generate a symmetry algebra. For example, if the target manifold is Kähler, we have a covariantly constant two-form ω_{ij}, the Kähler form, and two associated currents, $J^{(2)}_\pm$. Calculating the Λ-brackets between $T_\pm, J^{(2)}_\pm$, we find the algebra (section 4, paper IV)

$$\begin{aligned}\{T_\pm\,{}_\Lambda\,T_\pm\} &= (2\partial + \chi D + 3\lambda)\,T_\pm \\ \{T_\pm\,{}_\Lambda\,J^{(2)}_\pm\} &= (2\partial + \chi D + 2\lambda)\,J^{(2)}_\pm \\ \{J^{(2)}_\pm\,{}_\Lambda\,J^{(2)}_\pm\} &= -T\pm \\ \{\text{any}_\mp\,{}_\Lambda\,\text{any}_\pm\} &= 0\,.\end{aligned} \tag{12.16}$$

The above is called the $N=(2,2)$ superconformal algebra, since we have two commuting copies of the $N=2$ superconformal algebra. From the above algebra, we notice that the currents $T_\pm$ are generated from the Λ-brackets between $J^{(2)}_\pm$ with itself. The algebras associated to the other manifolds on Berger's list are given in paper IV. We will now address the question whether the currents "lifted" to the CDR generate the same algebras, or if there are anomalous terms. This is what we mean when we talk about calculating symmetry algebras in the CDR framework.

For the case of the target space M being Kähler, this question was addressed, and answered, in [40] and [13]. There it was shown that the quantum counterparts of the currents $T_\pm, J^{(2)}_\pm$, given in (12.10), (12.14) and (12.15), generate the $N=(2,2)$ superconformal algebra with central charge $\frac{3}{2}\dim M$ *if the target space is Calabi-Yau, with a Ricci-flat metric*. For a general Kähler manifold, there are anomalous terms present in the algebra. These anomalous terms are due to the "quantum" correction term $\partial D \log\sqrt{\det g_{ij}}$ to the current P. We see that this "quantum" correction involves the determinant of the metric. The Ricci-form is given by derivatives on the determinant of the metric, and the anomalous terms in the algebra vanish if we require the Kähler metric on M to be Ricci-flat metric. For more details, see [40, 13].

If M is Calabi-Yau we have, in addition to the Kähler form, a covariantly constant holomorphic three-form Ω, and its complex conjugate $\bar{\Omega}$. Let us denote the associated currents by $X_\pm$ and $\bar{X}_\pm$. For M a flat, six-dimensional manifold, Odake calculated the algebra generated by the currents $T_\pm, J^{(2)}_\pm, X_\pm, \bar{X}_\pm$ in [71]. We call the algebra found in [71] the *Odake algebra*. In paper IV we extend this calculation to the *curved* case. In theorem 7.1 in paper IV, we show that the currents $T_{\pm q}, J^{(2)}_{\pm q}, X_{\pm q}, \bar{X}_{\pm q}$ generate two commuting copies of the Odake algebra on a six-dimensional Calabi-Yau manifold with Ricci-flat metric. This is the main theorem of paper IV.

For a flat target manifold, the symmetry algebras generated by the currents associated to a G_2-manifold and a $\mathrm{Spin}(7)$-manifold were calculated in [78]. It is still an open problem to calculate the corresponding algebras for a curved G_2- and $\mathrm{Spin}(7)$-manifold. In the case of a G_2-manifold, using theorem 6.1 in paper IV we conjecture which currents generate the symmetry algebra for a curved G_2-manifold, see conjecture 7.3 in that paper. At present, the calculations are too complicated to be carried out.

Due to the non-commutativity and non-associativity of vertex algebras, the calculations of symmetry algebras performed in paper IV are quite involved. Ekstrand has constructed a computer software [31] which is very helpful when computing Λ-brackets in vertex algebras. For many of the calculations in paper IV, we relied on this software.

13. Discussion

In this part of the thesis, we have considered supersymmetric non-linear sigma models and we have argued that a natural interpretation of the CDR, introduced in [57], is as a formal canonical quantization of non-linear sigma models. We have described how to derive “quantum” versions of currents associated to symmetries of the classical non-linear sigma model and we described how to compute the equal-time commutator between these “quantum” currents. In particular, for any closed form on the target space we have derived the corresponding “quantum" current. In the case of a six-dimensional Calabi-Yau manifold, we have calculated the symmetry algebra between the different currents, and shown that they satisfy the Odake algebra. We end the thesis by commenting on a few open problems related to this work.

The calculation of the Odake algebra for a curved target space is possible due to the existence of special types of coordinates, in which many of the quantities in the currents are constant, or vanishes. For example, we can choose coordinates such that Ω and $\bar{\Omega}$ are constant. Moreover, in these coordinates, all “quantum” corrections to the currents vanish. Still, even with this clever choice of coordinates, the calculation of the Λ-brackets are quite involved. These calculations are outlined in section 7 in paper IV. It would be interesting to perform similar calculations for the other manifolds on Berger’s list, in particular for a G_2- and Spin(7)-manifold. The main obstacle to extend these types of calculations to the G_2- and Spin(7)-manifolds is the lack of special types of coordinates which makes quantities in the currents constant, or vanishing. Even using the computer software presented in [31], the calculations quickly explodes in complexity.

An interesting puzzle which arises with the above described interpretation of the CDR as a framework for understanding the non-linear sigma model in a quantum mechanical setup is the following. With the interpretation of the Λ-bracket given in paper III as an equal-time commutator, the calculations in the vertex algebra formalism in [13, 40] suggest that the supersymmetric sigma model is conformally invariant when the target manifold is Calabi-Yau, with a Ricci-flat metric. This is not compatible with the four-loop calculation of the β-function in the path integral formalism, see [70] and references therein. In [70] it is argued that the supersymmetric sigma model is conformally invariant for a Calabi-Yau manifold, but for a non-Ricci-flat metric. Hopefully, future work will allow us to resolve this puzzle.

Acknowledgements

First and foremost I want to express my sincere thanks to my supervisor Maxim Zabzine. Thanks for your encouragement, your patient answers to all my questions and for giving me the opportunity to attend many inspiring workshops and conferences. I also want to thank my assistant supervisor Ulf Lindström for introducing me to the theoretical physics group at Uppsala, sharing your knowledge and for always being very helpful.

I would like to express my gratitude to the co-authors of the papers that are included in this thesis. I want to thank Joel Ekstrand for always being enthusiastic, coming up with clever ideas and for many interesting discussions. I would like to thank Reimundo Heluani for teaching me about the world of vertex algebras.

During my time at Uppsala University there have been many PhD-students and post-docs and I would like to thank you all for contributing to a fun and stimulating working atmosphere at the department. Specifically I want to thank Malin Göteman for, among other things, always taking the necessary initiatives when something has to be done and Joel Ekstrand for his generous help with just about everything that happens in life and in physics. I want to acknowledge Jian Qiu for many valuable discussions on topological field theories; they have provided me with valuable research contributions. I want to thank Kasper Larsen and Martin Lundgren for all the nice time spent together and interesting discussions. I also want to thank the (from my perspective) "older" generation of PhD-students, Valentina Gaingreco Pulleti, Niklas Johansson, Magdalena Larfors, Sergey Slizovskiy and Olof Ohlsson Sax for teaching me various aspects of physics and for making me feel welcome during my initial time as a PhD-student.

Thanks also to the other faculty members at the department, Ulf Danielsson, Lisa Freyhult, Joe Minahan, Antti Niemi, Rolf Paulsson, Staffan Yngve and Konstantin Zarembo, for sharing your knowledge and always answering my questions. I especially want to thank Staffan for enjoyable collaborations when we have taught together. Thanks also to all the department support staff for quickly helping with all practical matters.

I have received help from various people when writing this text and for that I am very grateful. Maxim and Joel, I thank you for your feedback and suggestions. I think they improved the text considerably. I also want to thank Hans Källén for giving me valuable comments.

Finally I would like to thank my family, and especially my parents Cecilia and Erland for inspiration and encouragement. Lastly, I want to thank Julie for your love, patience and support. You make me a happy man.

Summary in Swedish

Vridning och klistring

Om topologiska fältteorier, sigmamodeller och vertexalgebror

Genom historien har det alltid funnits en stark koppling mellan fysik och matematik. Det traditionella sambandet är att matematik är språket för att formulera fysikaliska teorier, som i sin tur kan användas för att beskriva och göra förutsägelser om naturen. Ibland har de fysikaliska teorierna inspirerat till utvecklandet av ny matematik, men fysiken har traditionellt inte givit konkreta bidrag till den rena matematiken. Under de senaste årtiondena har denna växelverkan mellan matematiker och fysiker fått en ny dimension, i och med upptäckten av *topologiska kvantfältteorier*. Låt mig försöka förklara detta.

Kvantfältteorier är ett teoretiskt ramverk som bland annat används för att beskriva de allra minsta beståndsdelarna som hittills har observerats i naturen, nämligen elementarpartiklar och dess växelverkningar. Kvantfältteorier har i allmänhet en väldigt rik och komplex struktur, men det saknas en rigorös matematisk definition. Fysiker är trots det övertygade om att det finns en underliggande rigorös matematisk formulering, eftersom ramverket kan användas för att göra förutsägelser om naturen.

Topologi är den gren av matematiken där man inte är intresserad av objekts exakta form, utan bara skiljer på objekt som är “drastiskt” olika. Till exempel så är en kaffekopp och en badring topologiskt samma objekt, eftersom de har lika många hål. Däremot är en fotboll och en badring topologiskt olika, eftersom fotbollen saknar hål. Antalet hål är ett enkelt exempel på en *topologisk invariant*. Om formen bara ändras lite grann på en kaffekopp så ändras inte antalet hål; antalet hål är alltså invariant. Ett av målen inom topologin är att hitta och förstå egenskaper hos topologiska invarianter.

Med en topologisk kvantfältteori fångas topologiska invarianter. Med hjälp av olika (origorösa) metoder som har utvecklats för kvantfältteorier kan förutsägelser göras för egenskaper hos, och olika samband mellan, dessa topologiska invarianter. Dessa egenskaper och samband är oftast mycket svåra att hitta med hjälp av rigorösa matematiska metoder. Genom att använda topologiska kvantfältteorier kan därför olika, ofta överraskande, matematiska förmodanden formuleras. Dessa förmodanden kan allt som oftast bevisas med hjälp av rigorösa matematiska metoder. Förutom att ge värdefulla bidrag till den rena matematiken kan studiet av topologiska kvantfältteorier även motiveras av att de är en typ av kvantfältteorier som är relativt enkla, och därför möjliga att förstå på djupet. Samtidigt är topologiska kvantfältteorier nära besläktade med fysikaliskt mer intressanta kvantfältteorier, företrädelsevis de kvantfältteorier som är *supersymmetriska*. Vissa egenskaper som först har förståtts hos topologiska kvantfältteorier har senare hjälp till att ge en ökad förståelse av de fysikaliskt mer intressanta supersymmetriska kvantfältteorierna.

Ett av de mest berömda exemplen på en topologisk kvantfältteori är den så kallade Chern-Simons-teorin. Chern-Simons-teori beskriver topologiska invarianter av tre-dimensionella rum och topologiska invarianter av knutar. En knut är en inbäddning av en cirkel i ett tre-dimensionellt rum: ta ett snöre och sammanfoga dess två ändar och du har en knut.

I första delen av denna avhandling (artikel I och II) studeras aspekter av tre-dimensionella Chern-Simons-teorier, och även högre-dimensionella generaliseringar.

I artikel I visar jag hur Chern-Simons-teori kan omformuleras på ett sådant sätt att en kraftfull metod för att räkna ut exakta resultat kan appliceras. Denna metod kallas för *lokalisering av vägintegralen*. I litteraturen finns det ett par olika metoder för att göra beräkningar i Chern-Simons-teori på så kallade Seifertmångfalder, och med metoden introducerad i artikel I återproducerar jag många kända resultat på ett nytt sätt. En intressant aspekt av denna formulering av Chern-Simons-teori är att så kallade *kontaktstrukturer* spelar en viktig roll. Inspirerade av formuleringen av Chern-Simons-teori i artikel I introducerar vi i artikel II formuleringar av Chern-Simons-lika teorier i högre dimensioner på sådant sätt att den kraftfulla metoden lokalisering kan appliceras. I formuleringen av dessa högre-dimensionella teorier spelar kontaktstrukturer en viktig roll. Vi studerar den fem-dimensionella teorin i detalj. De viktigaste resultaten i artikel II är följande: Vi visar att för fem-dimensionella versioner av Seifertmångfalder lokaliseras teorin på *kontaktinstantoner*. När teorin är formulerad på den fem-dimensionella sfären, visar vi att den perturbativa delen av partitionsfunktionen ges av en matrismodell. Utöver intressanta matematiska aspekter kan även resultaten i artikel II bidra till en bättre förståelse av fem-dimensionella supersymmetriska kvantfältteorier.

Andra delen av avhandlingen handlar om sigmamodeller. I en sigmamodell studeras avbildningar från en mångfald till en annan. Exempel på en mångfald är en linje eller en yta; i allmänhet är en mångfald en generalisering av dessa objekt till högre dimensioner. Mångfalden som avbildningen startar i kallas för *världsytan*, medan mångfalden som avbildningen landar i kallas för *målmångfalden*. I denna avhandling studerar vi sigmamodeller med en två-dimensionell världsyta. Dessa typer av sigmamodeller är fundamentala byggstenar inom strängteori. Precis som för kvantfältteorier finns det även sigmamodeller som är intressanta ur ett rent matematiskt perspektiv, så kallade topologiska sigmamodeller. Studiet av topologiska sigmamodeller har gett upphov till många intressanta resultat i ren matematik.

Sigmamodeller med en krökt målmångfald är komplicerade att beskriva kvantmekaniskt på grund av deras ickelinjära natur. I den andra delen av avhandlingen använder vi *vertexalgebror* för att förstå vissa kvantmekaniska aspekter av supersymmetriska sigmamodeller med krökta målmångfalder. Ramverket vi använder kallas för *det kirala de Rham-komplexet*, som introducerades av Malikov, Schechtman och Vaintrob i slutet av 1990-talet. Deras motivering var att matematiskt bättre förstå en av de topologiska sigmamodellerna, den så kallade A-modellen. Vi argumenterar för att det kirala de Rham-komplexet kan användas för att studera en större klass av sigmamodeller, inte bara de som är topologiska. Av tekniska skäl måste sigmamodellerna vara supersymmetriska.

I artikel III visar vi att det kirala de Rham-komplexet kan tolkas som ett ramverk för att förstå kanonisk kvantisering av supersymmetriska sigmamodeller. I artikel IV använder vi detta ramverk för att studera symmetrialgebror för supersymmetriska sigmamodeller med krökta målmångfalder i en kvantmekanisk formulering. Det objektet som genererar en symmetri kallas för en ström. För sigmamodeller med en platt målmångfald så är dessa strömmar kända. Ett av resultaten i artikel IV är att vi visar hur en klass av dessa strömmar måste modifieras för att de ska vara definerade för krökta målmångfalder. När målmångfalden är en platt sex-dimensionell *Calabi-Yau-mångfald*, så finns det en uppsättning strömmar som genererar den så kallade *Odake-algebran*. För en krökt sex-dimensionell Calabi-Yau-mångfald så visar vi att även de modiferade strömmarna genererar Odake-algebran. Den viktigaste aspekten av resultaten i artikel III och IV är att de bidrar till att studiet av vissa kvantmekaniska aspekter av supersymmetriska sigmamodeller kan läggas på en matematiskt stabilare grund.

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