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Abstract

We propose a new test for invertibility of moving average processes. The test is based on an explicit local approximation of the likelihood ratio. In a simulation study, we compare the power of the test with a score type test of Tanaka (1990) and a numerical likelihood ratio test suggested by Davis, Cheng and Dunsmuir (1995). Local to the null of noninvertibility, our test is seen to have better power properties than the score type test and its power is only slightly below that of the numerical likelihood ratio test. Moreover, we extend our test to an ARMA(p,1) framework. A simulation study compares size properties of the methods under an ARMA(1,1) model.

Key words: Moving average process, Invertibility, Likelihood ratio test.

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1 Introduction

We will study the moving average (MA) model

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1}, \tag{1}$$

where $\theta \leq 1$, all ε_t are $NID\left(0, \sigma^2\right)$ for $t \geq 0$ and we have observations at $t = 1, ..., T$. Our focus is the test of nonivertibility, i.e. to test $H_0 : \theta = 1$ versus $H_1 : \theta < 1$. Traditionally, such tests are used to check for overdifferencing, see e.g. Saikkonen and Luukkonen (1993), Leyborne and McCabe (1994) and the discussion in Davies and Dunsmuir (1996).

Tanaka (1990) suggested a score type test, which he proved to be locally best invariant and unbiased (LBIU). He argued that Likelihood Ratio or Wald tests are not feasible since the maximum likelihood estimator (MLE) is not explicit in MA models, see e.g. Cryer and Ledolter (1981). In spite of this, Davis, Cheng and Dunsmuir (1995) and Davies and Dunsmuir (1996) exploited likelihood methods. They showed the interesting result that the limiting distribution of the MLE differs from that of the local maximizer (LM), which is the estimator obtained when maximizing the likelihood under the parametrization $\theta = 1 - \gamma/T$. Davis et al (1995) also derived the asymptotic distribution of the Generalized Likelihood Ratio (GLR) test. Moreover, they compared finite sample and limiting powers of the score and GLR tests, as well as tests directly based on the LM and ML estimators. They found that GLR outperforms the other tests except for in the region $\gamma < 5$, where the score test is slightly better. Among the other two, the LM estimator based test uniformly outperforms the ML counterpart in the studied range of γ values. However, Tanaka presented a generalization of the score test to the general ARMA case, while Davis et al did not provide such a generalization of the GLR test.

More recent work includes Davis and Song (2011), who extend the GLR test to the MA(2) case. Yabe (2012) generalized the asymptotic distribution results of Tanaka (1990), which were obtained for $\theta = 1 - \gamma/T$, to the moderate deviation case, where $\theta = 1 - \gamma/T^{\alpha}$ for some $\alpha \in (0, 1)$. Vougas (2008) proposes a new ML estimation method that avoids the pile-up phenomenon (the estimator equals one with positive probability). A test for invertibility is based on the suggested estimator.

In the present paper, we suggest an approximation of the Likelihood Ratio (LR) test which, as in LM estimation, is based on a local approximation of the parameter around 1. Simulations indicate that the power of our test is close to the power of the GLR test, and larger than the power of the other explicit one, the score test. We also generalize our test to invertibility testing in the ARMA$(p, 1)$ case, and derive the asymptotic distribution. A simulation study in the case $p = 1$ investigates the size performances of the three tests. It is found that our proposed test works well when the AR parameter is close to zero, but that it is undersized in general. For small T, the score type test is oversized for negative values of the AR parameter and undersized for positive values. For large T, the score type test works fairly well, although it may be severely oversized for values of the AR parameter close to -1. The numerical likelihood ratio test appears rather robust except for when the AR parameter is close to one, in which instance it may be severely oversized. Our test is non similar with respect to the AR parameters, but incorporating a correction of its critical values, it is seen to perform very well in terms of size.

The rest of the paper is as follows. In section 2, we derive the approximative LR test as well as some of its limiting properties. A simulation study is conducted in section 3. Generalization to the ARMA$(p, 1)$ model is discussed in section 4. Under this framework, section 5 contains a simulation comparison of small sample sizes of the studied tests, while section 6 concludes.

2 An approximative LR test

Writing $\mathbf{y} = (y_1, ..., y_T)'$, $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, ..., \varepsilon_T)'$, (1) is equivalent to

$$\mathbf{y} = \mathbf{H}'\boldsymbol{\varepsilon}, \tag{2}$$

where $\mathbf{H}$ is the $(T+1) \times T$ matrix

$$\mathbf{H} \equiv \begin{pmatrix} -\theta & 0 & \cdots & 0 \\ 1 & -\theta & \ddots & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & -\theta \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence, the log likelihood is

$$l\left(\theta, \sigma^2\right) = -\frac{T}{2}\log\left(2\pi\sigma^2\right) - \frac{\mathbf{y}'\boldsymbol{\Omega}^{-1}\mathbf{y}}{2\sigma^2} - \frac{1}{2}\log\left(\det\boldsymbol{\Omega}\right). \tag{3}$$

where $\boldsymbol{\Omega} = \mathbf{H}'\mathbf{H}$ is the covariance matrix of $\mathbf{y}$. Defining

$$\boldsymbol{\Omega}_1 = \left(\mathbf{I} - \theta\mathbf{L}\right)\left(\mathbf{I} - \theta\mathbf{L}'\right), \tag{4}$$

where $\mathbf{I}$ and $\mathbf{L}$ are the identity and lag matrices of dimension T, we may write

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_1 + \theta^2\boldsymbol{\delta}_1\boldsymbol{\delta}_1', \tag{5}$$

where $\boldsymbol{\delta}_1 = (1, 0, ..., 0)'$. (In fact, if one would assume $\varepsilon_0 = 0$, then $\boldsymbol{\Omega}_1$ would be the covariance matrix of $\mathbf{y}$.) The MLE of σ^2, $\widehat{\sigma}^2$ say, fulfills

$$\widehat{\sigma}^2 = T^{-1}\mathbf{y}'\boldsymbol{\Omega}^{-1}\mathbf{y}. \tag{6}$$

Hence, (3) implies

$$l\left(\theta, \widehat{\sigma}^2\right) = -\frac{T}{2}\left\{\log\left(2\pi\right) + 1\right\} - \frac{T}{2}\log\left(\widehat{\sigma}^2\right) - \frac{1}{2}\log\left(\det\boldsymbol{\Omega}\right). \tag{7}$$

Moreover, we find from (5) that

$$\det \mathbf{\Omega} = \det \mathbf{\Omega}_1 \left(1 + \theta^2 \boldsymbol{\delta}_1' \mathbf{\Omega}_1^{-1} \boldsymbol{\delta}_1\right),$$

where $\det \mathbf{\Omega}_1 = 1$ and

$$\boldsymbol{\delta}_1' \mathbf{\Omega}_1^{-1} \boldsymbol{\delta}_1 = 1 + \theta^2 + ... + \theta^{2T-2} = \frac{1-\theta^{2T}}{1-\theta^2},$$

i.e.

$$\det \mathbf{\Omega} = 1 + \theta^2 \frac{1-\theta^{2T}}{1-\theta^2} = \frac{1-\theta^{2T+2}}{1-\theta^2}.$$

Now, to facilitate an approximation of the LR test close to $\theta = 1$, the idea is to write $\theta = 1 - \gamma$, which yields

$$\eta \equiv \det \mathbf{\Omega} = \frac{1-(1-\gamma)^{2T+2}}{1-(1-\gamma)^2}. \tag{8}$$

Thus, from (7),

$$l\left(\theta, \widehat{\sigma}^2\right) = -\frac{T}{2}\left\{\log(2\pi) + 1\right\} - \frac{T}{2}\log\left\{f(\gamma)\right\}, \tag{9}$$

where

$$f(\gamma) \equiv \widehat{\sigma}^2 \eta^{T^{-1}}. \tag{10}$$

Hence, in the following, we may confine ourselves to minimizing $f(\gamma)$. To this end, using Taylor expansion in γ to the fourth order, we may derive the following result.

Proposition 1

$$f(\gamma) = T^{-1}(T+1)^{T^{-1}}\left(f_{0T} - \gamma^2 f_{2T} - \gamma^3 f_{2T} + \gamma^4 f_{4T}\right) + O\left(\gamma^5\right),$$

where for $i = 0, 2, 4,\ f_{iT} = \mathbf{x}'\mathbf{D}_{iT}\mathbf{x}$ *with* $\mathbf{x} = (\mathbf{I} - \mathbf{L})^{-1}\mathbf{y}$ *and*

$$\begin{aligned}
\mathbf{D}_{0T} &\equiv \mathbf{I} - \frac{1}{T+1}\mathbf{11}', \\
\mathbf{D}_{2T} &\equiv \frac{T^2-2T+4}{3(T+1)}\mathbf{11}' + \mathbf{SS}' - \frac{T+2}{6}\mathbf{I} - \frac{1}{T+1}\left(\mathbf{11}'\mathbf{SS}' + \mathbf{SS}'\mathbf{11}'\right), \\
\mathbf{D}_{4T} &\equiv \frac{8T^4+14T^3-309T^2+529T-602}{360(T+1)}\mathbf{11}' - \frac{T+8}{6}\mathbf{SS}' \\
&\quad - \frac{(T+2)\left(2T^2-T-61\right)}{360}\mathbf{I} + \frac{T^2-2T+7}{3(T+1)}\left(\mathbf{11}'\mathbf{SS}' + \mathbf{SS}'\mathbf{11}'\right) \\
&\quad + \left(\mathbf{SS}'\right)^2 - \frac{1}{T+1}\mathbf{SS}'\mathbf{11}'\mathbf{SS}' - \frac{1}{T+1}\left\{\mathbf{11}'\left(\mathbf{SS}'\right)^2 + \left(\mathbf{SS}'\right)^2\mathbf{11}'\right\}.
\end{aligned}$$

where

$$\mathbf{S} \equiv (\mathbf{I} - \mathbf{L})^{-1}\mathbf{L},$$

with $\mathbf{I}$ *as the* T *dimensional identity matrix,* $\mathbf{L}$ *as the* T *dimensional lag matrix, and* $\mathbf{1}$ *as the* T *dimensional vector consisting of ones.*

Proof. See the appendix. ■

To find the minimum of $f(\gamma)$, we need to solve

$$0 = \frac{\partial}{\partial \gamma} f(\gamma) = T^{-1}(T+1)^{T^{-1}}\left(-2\gamma f_{2T} - 3\gamma^2 f_{2T} + 4\gamma^3 f_{4T}\right) + O\left(\gamma^4\right).$$

Disregarding the $O\left(\gamma^4\right)$ term, one solution is $\gamma = 0$ and the other two are given by

$$\begin{aligned}\gamma_{1,2}^* &= \frac{3f_{2T}}{8f_{4T}} \pm \sqrt{r_T}, \\ r_T &= \left(\frac{3f_{2T}}{8f_{4T}}\right)^2 + \frac{f_{2T}}{2f_{4T}}.\end{aligned}$$

The asymptotics of $\gamma_{1,2}^*$ may be analyzed by employing the following proposition. Without loss of generality, we assume that $\sigma^2 = 1$. The intuition is that if $\theta = 1$ and if we put $\varepsilon_0 = 0$ (which has no effect asymptotically), then $\mathbf{x} = (\varepsilon_1, ..., \varepsilon_T)'$, which means that $T^{-1/2}\mathbf{1}'\mathbf{x}$ converges to $W(1)$ where $W(t)$ is a standard Wiener process. Similarly, $T^{-2}\mathbf{x}'\mathbf{SS}'\mathbf{x}$ converges to $\int_0^1 W^*(t)^2\,dt$ where $W^*(t) = W(1) - W(t)$, etcetera. Moreover, observe that the same results should hold without the normality assumption. It should be enough that the ε_t form a martingale difference sequence, cf Chan and Wei (1987).

Proposition 2 *If $\theta = 1$ and $\sigma^2 = 1$, then as $T \to \infty$,*

$$\begin{aligned}T^{-1} f_{0T} &\xrightarrow{P} 1, \\ T^{-2} f_{2T} &\xrightarrow{d} f_2, \\ T^{-4} f_{4T} &\xrightarrow{d} f_4,\end{aligned}$$

where

$$\begin{aligned}f_2 &\equiv \frac{1}{3}W(1)^2 + \int_0^1 W^*(t)^2\,dt - \frac{1}{6} - 2W(1)\int_0^1 (1-t)\,W^*(t)\,dt, \\ f_4 &\equiv \frac{1}{45}W(1)^2 - \frac{1}{180} + \frac{2}{3}W(1)\int_0^1 (1-t)\,W^*(t)\,dt \\ &\quad + \int_{t=0}^1 \left(\int_{s=0}^t W^*(s)\,ds\right)^2 dt - \left\{\int_0^1 (1-t)\,W^*(t)\,dt\right\}^2 \\ &\quad - \frac{1}{3}W(1)\int_0^1 \left(2 - 3t^2 + t^3\right) W^*(t)\,dt,\end{aligned}$$

and where $\xrightarrow{P}$ and $\xrightarrow{d}$ denote convergence in probability and convergence in distribution respectively, $W^*(t) \equiv W(1) - W(t)$ and $W(t)$ is a standard Wiener process.

Proof. See the appendix. ■

Proposition 2 and the Slutsky theorem imply that

$$T^2 r_T = T^{-2}\left(\frac{3T^{-2}f_{2T}}{8T^{-4}f_{4T}}\right)^2 + \frac{T^{-2}f_{2T}}{2T^{-4}f_{4T}} \xrightarrow{d} \frac{f_2}{2f_4}$$

and so,

$$\begin{aligned} & T\gamma^*_{1,2} \\ = \; & T^{-1}\frac{3T^{-2}f_{2T}}{8T^{-4}f_{4T}} \pm \sqrt{T^2 r_T} \\ & \xrightarrow{d} \pm \sqrt{\left|\frac{f_2}{2f_4}\right|}\left(I_{\{f_2 f_4 \geq 0\}} + iI_{\{f_2 f_4 < 0\}}\right), \end{aligned}$$

where $I_{\{A\}}$ is the indicator function of the event A and i is the complex unit. Now, let γ^* be the solution that maximizes the approximate likelihood. Inserting into proposition 1,

$$f(\gamma^*) = T^{-1}(T+1)^{T^{-1}}(f_{0T} - A^*_T) + O\left(\gamma^{*3}\right),$$

where

$$A^*_T \equiv f_{2T}\gamma^{*2}.$$

Then, from (9), with $\theta^* = 1 - \gamma^*$,

$$l\left(\theta^*, \widehat{\sigma}^2_2\right) = c - \frac{T}{2}\log(f_{0T} - A^*_T) + O\left(\gamma^{*3}\right),$$

where

$$c \equiv -\frac{T}{2}\{\log(2\pi) + 1 - \log T\} - \frac{1}{2}\log(T+1).$$

Moreover, under H_0, the MLE of σ^2 is $\widehat{\sigma}^2_0 = T^{-1}f_{0T}$, and so,

$$f(0) = T^{-1}(T+1)^{T^{-1}} f_{0T},$$

implying that the maximum log likelihood is

$$l\left(1, \widehat{\sigma}^2\right) = c - \frac{T}{2}\log(f_{0T}).$$

Hence, defining the likelihood ratio test statistic as Q_T the ratio of the maximum likelihoods under H_0 and H_1 respectively, we have

$$\begin{aligned} & -2\log Q_T \\ = \; & T\log(f_{0T}) - T\log(f_{0T} - A^*_T) + O\left(\gamma^{*3}\right) \\ = \; & -T\log\left(1 - T^{-1}Z^*_T\right) + O\left(\gamma^{*2}\right), \end{aligned} \tag{11}$$

where

$$Z_T^* \equiv T\frac{A_T^*}{f_{0T}}.$$

Observe that regardless of which solution $\gamma_{1,2}^*$ that is chosen, as $T \to \infty$,

$$Z_T^* \xrightarrow{d} \frac{f_2}{2}\left|\frac{f_2}{f_4}\right|\left(I_{\{f_2 f_4 \geq 0\}} - I_{\{f_2 f_4 < 0\}}\right).$$

This motivates the statistic

$$Z_T = \frac{T f_{2T}}{2 f_{0T}}\left|\frac{f_{2T}}{f_{4T}}\right|.$$

(Simulations indicate that the power properties improve when the indicator terms are disregarded. Also, they are slightly better when incorporating f_{0T} in the statistic.) Observe that for large enough T, the approximation in (11) is equivalent to the statistic Z_T^*. It is also easily seen by Taylor expansion that $-2\log Q_T$ and Z_T^* have the same asymptotic distribution.

3 A simulation study on power

In this section, we compare the power properties of the approximative LR test statistic (Z_T) with the score test of Tanaka (1990) and the GLR test of Davis et al (1995). For $T = 50$, table 1 and figure 1 give the size adjusted local powers of these tests versus δ, where $\theta = 1 - \delta/T$, together with the power envelope. The latter is calculated under a known σ^2 and a known parameter θ under the alternative. The simulations were performed in Matlab 7.10, using 100 000 replications[1]. For $\gamma \leq 3$, the approximative LR test performs slightly worse than the score test, but for larger γ, its performance is clearly better. As expected, comparing to the GLR test, the latter is always about equally good or better than the approximative LR test, but up to about $\delta = 8$, the difference is very small. Moreover, note that overall, the power of the GLR test is very close to the power envelope. These numbers corroborate well with table 4.1 of Davis et al (1995). Also, except for the score test far from the null, the tests studied here perform better in terms of power than the ML estimator based tests of Vougas (2008), cf table 3 in the latter paper.

In table 2 and figure 2, we present corresponding results for $T = 200$. The conclusions are similar. As a by-product of these simulations, we got empirical critical values of the test statistics. These are given in table 3.

In order for the reader to be able to apply the approximative LR test for other sample sizes, we have performed response surface regressions for critical values at the common significance levels 0.1, 1 and 5 per cent. The regressions were based on 1 000 000 replications and $T \in \{25, 50, 100, 200, 400, 800\}$. The resulting regression equations are given in table 4.[2]

[1] We also simulated powers of the two least time consuming tests, i.e. the score test and the approximative LR test, for 10^6 replications. These results deviated very little (as a maximum one or two units in the third decimal point) from the ones presented here.

[2] In the first step, T^{-2} terms were included, but these were insignificant at the 1% level.

δ	Score	Approx. LR	GLR	Power envelope
0.5	5.3	5.3	5.3	5.5
1.0	6.2	6.1	6.1	6.2
1.5	7.8	7.6	7.5	7.8
2	10.1	9.7	9.6	10.0
3	16.2	16.0	15.9	16.4
4	24.0	24.2	24.0	24.6
5	32.0	32.7	32.9	33.3
6	39.5	41.2	41.8	42.1
7	46.4	49.1	50.1	50.3
8	52.3	56.0	57.4	57.6
10	61.8	67.0	69.4	70.0
12	69.3	75.2	78.3	79.3
15	77.2	83.7	87.3	88.8
20	85.6	91.5	94.9	96.4

Table 1: Local power in per cent for T=50, 100 000 replications.

δ	Score	Approx. LR	GLR	Power envelope
0.5	5.4	5.3	5.3	5.4
1.0	6.3	6.2	6.1	6.3
1.5	7.8	7.6	7.6	7.9
2	10.1	9.8	9.6	10.1
3	16.2	15.8	15.7	16.3
4	23.8	23.8	23.6	24.2
5	31.5	32.2	32.1	32.5
6	38.9	40.3	40.6	40.8
7	45.5	47.8	48.6	48.7
8	51.4	54.5	55.7	55.9
10	61.1	65.5	67.5	68.1
12	68.9	74.1	76.4	77.3
15	77.5	82.8	85.4	86.7
20	86.3	91.0	93.6	94.8

Table 2: Local power in per cent for T=200, 100 000 replications.

T	Score	Approx. LR	GLR
50	0.472	1.425	2.025
200	0.463	1.441	1.968

Table 3: Empirical critical values at the 5 per cent level. 1 000 000 replications where used for the score and approximative LR tests, and 100 000 replications were used for the GLR test.

Significance level	Regression equation
0.1%	$11.834 - 84.21T^{-1}$
1%	$3.187 - 14.80T^{-1}$
5%	$1.456 - 4.266T^{-1}$

Table 4: Response surface regressions from critical values of the approximative LR test. 1 000 000 replications were used, and the regressions were based on the sample sizes 25, 50, 100, 200, 400 and 800.

Figure 1 about here

Figure 2 about here

4 Extension

In this section, we will outline how our test may be generalized to the ARMA$(p, 1)$ model

$$\phi(L) y_t = \varepsilon_t - \theta\varepsilon_{t-1},$$

where L is the lag operator, θ and ε_t are as above and

$$\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p,$$

for some p. Moreover, we assume that $y_0 = y_{-1} = ... = y_{-p+1} = 0$. Then, defining the $T \times T$ matrix

$$\mathbf{\Phi} = \mathbf{I} - \sum_{i=1}^{p} \phi_i \mathbf{L}^i, \tag{12}$$

and we have via (2) that

$$\mathbf{\Phi y} = \mathbf{H}'\boldsymbol{\varepsilon}. \tag{13}$$

Consequently, the log likelihood is

$$l\left(\theta, \boldsymbol{\phi}, \sigma^2\right) = -\frac{T}{2}\log\left(2\pi\sigma^2\right) - \frac{\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y}}{2\sigma^2} - \frac{1}{2}\log\left(\det\mathbf{\Sigma}\right), \tag{14}$$

and assuming that $\mathbf{\Phi}$ is invertible,

$$\mathbf{\Sigma} = \mathbf{\Phi}^{-1}\mathbf{\Omega}\mathbf{\Phi}^{-1\prime},$$

with $\mathbf{\Omega}$ as before. Moreover, noting that $\det\mathbf{\Phi} = 1$, (14) simplifies into

$$l\left(\theta, \boldsymbol{\phi}, \sigma^2\right) = -\frac{T}{2}\log\left(2\pi\sigma^2\right) - \frac{\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y}}{2\sigma^2} - \frac{1}{2}\log\left(\det\mathbf{\Omega}\right).$$

For fixed θ and $\boldsymbol{\phi}$, it follows that the MLE of σ^2 is

$$\widetilde{\sigma}^2(\boldsymbol{\phi}) = T^{-1}\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y} = T^{-1}\mathbf{y}'\boldsymbol{\Phi}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Phi}\mathbf{y},$$

and we get

$$l\left(\theta, \boldsymbol{\phi}, \widetilde{\sigma}^2(\boldsymbol{\phi})\right) = -\frac{T}{2}\{\log(2\pi)+1\} - \frac{T}{2}\log\left\{\widetilde{\sigma}^2(\boldsymbol{\phi})\right\} - \frac{1}{2}\log(\det\boldsymbol{\Omega}), \quad (15)$$

where $\boldsymbol{\phi} \equiv (\phi_1, ..., \phi_p)'$. Here, via (12),

$$\begin{aligned}
& T\widetilde{\sigma}^2(\boldsymbol{\phi}) \\
= \; & \mathbf{y}'\left(\mathbf{I} - \sum_{i=1}^{p}\phi_i\mathbf{L}^i\right)'\boldsymbol{\Omega}^{-1}\left(\mathbf{I} - \sum_{i=1}^{p}\phi_i\mathbf{L}^i\right)\mathbf{y} \\
= \; & \mathbf{y}'\boldsymbol{\Omega}^{-1}\mathbf{y} - 2\sum_{i=1}^{p}\phi_i\mathbf{y}'\boldsymbol{\Omega}^{-1}\mathbf{L}^i\mathbf{y} + \sum_{i=1}^{p}\sum_{j=1}^{p}\phi_i\phi_j\mathbf{y}'\mathbf{L}'^i\boldsymbol{\Omega}^{-1}\mathbf{L}^j\mathbf{y} \\
= \; & m - 2\boldsymbol{\phi}'\mathbf{m} + \boldsymbol{\phi}'\mathbf{M}\boldsymbol{\phi},
\end{aligned}$$

where, putting $\mathbf{J}_i \equiv \boldsymbol{\Omega}^{-1}\mathbf{L}^i$, $\mathbf{J}_{ij} \equiv \mathbf{L}'^i\boldsymbol{\Omega}^{-1}\mathbf{L}^j$,

$$\begin{aligned}
m &\equiv \mathbf{y}'\boldsymbol{\Omega}^{-1}\mathbf{y}, \\
\mathbf{m} &\equiv (\mathbf{y}'\mathbf{J}_1\mathbf{y}, ..., \mathbf{y}'\mathbf{J}_p\mathbf{y})', \\
\mathbf{M} &\equiv \begin{pmatrix} \mathbf{y}'\mathbf{J}_{11}\mathbf{y} & \cdots & \mathbf{y}'\mathbf{J}_{1p}\mathbf{y} \\ \vdots & & \vdots \\ \mathbf{y}'\mathbf{J}_{p1}\mathbf{y} & \cdots & \mathbf{y}'\mathbf{J}_{pp}\mathbf{y} \end{pmatrix}.
\end{aligned}$$

Now, to minimize over $\boldsymbol{\phi}$, we find

$$T\frac{\partial}{\partial\boldsymbol{\phi}}\widetilde{\sigma}^2(\boldsymbol{\phi}) = -2\mathbf{m} + (\mathbf{M}+\mathbf{M}')\boldsymbol{\phi} = 2(-\mathbf{m} + \mathbf{M}\boldsymbol{\phi}),$$

implying that the MLE of $\boldsymbol{\phi}$ is $\widetilde{\boldsymbol{\phi}} = \mathbf{M}^{-1}\mathbf{m}$. This yields

$$T\widetilde{\sigma}^2 = T\widetilde{\sigma}^2\left(\widetilde{\boldsymbol{\phi}}\right) = m - \mathbf{m}'\mathbf{M}^{-1}\mathbf{m}. \quad (16)$$

Hence, with $\theta = 1-\gamma$, (15) implies

$$l\left(\theta, \widetilde{\boldsymbol{\phi}}, \widetilde{\sigma}^2(\boldsymbol{\phi})\right) = -\frac{T}{2}\{\log(2\pi)+1\} - \frac{T}{2}\log\{g(\gamma)\}. \quad (17)$$

where

$$g(\gamma) \equiv \widetilde{\sigma}^2\eta^{T^{-1}}. \quad (18)$$

with η as in (8). Then, as above, Taylor expansion and minimization yields the following result.

Proposition 3

$$g(\gamma) = T^{-1}(T+1)^{T^{-1}}\left(g_{0T} - \gamma g_{1T} - \gamma^2 g_{2T} - \gamma^3 g_{3T} + \gamma^4 g_{4T}\right) + O\left(\gamma^5\right),$$

where

$$\begin{aligned} g_{0T} &\equiv f_{0T} - r_{0T}, \\ g_{1T} &\equiv 2r_{0T}, \\ g_{2T} &\equiv f_{2T} + r_{2T}, \\ g_{3T} &\equiv f_{2T} + r_{3T}, \\ g_{4T} &\equiv f_{4T} - r_{4T}, \end{aligned}$$

with f_{0T}*,* f_{2T} *and* f_{4T} *as above and where* r_{0T}*,* r_{2T}*,* r_{3T} *and* r_{4T} *are defined in the appendix.*

Proof. See the appendix. ■

The following proposition gives the asymptotics of the g terms. We note that the order of convergence for g_{0T}, g_{2T} and g_{4T} is the same as for the corresponding f terms, but the limits are different.

Proposition 4 *If* $\theta = 1$*,* $\sigma^2 = 1$ *and if all roots of* $\phi(z) = 0$ *are outside the complex unit circle, then as* $T \to \infty$,

$$\begin{aligned} T^{-1} g_{0T} &\xrightarrow{P} g_0, \\ T^{-1} g_{1T} &\xrightarrow{P} g_1, \\ T^{-2} g_{2T} &\xrightarrow{d} g_2, \\ T^{-3} g_{3T} &\xrightarrow{P} 0, \\ T^{-4} g_{4T} &\xrightarrow{d} g_4, \end{aligned}$$

where

$$\begin{aligned} g_0 &\equiv c^{(0)*} - \mathbf{c}'\mathbf{C}^{-1}\mathbf{c}, \\ g_1 &\equiv 2\mathbf{c}'\mathbf{C}^{-1}\mathbf{c}, \\ g_2 &\equiv \left(1 - \mathbf{1}_p'\mathbf{C}^{-1}\mathbf{c}\right)^2 C^{(0)2}\left(f_2 + \frac{1}{6}\right) - \frac{1}{6}\left(c^{(0)*} + \mathbf{c}'\mathbf{C}^{-1}\mathbf{c}\right), \\ g_4 &\equiv \left(1 - \mathbf{1}_p'\mathbf{C}^{-1}\mathbf{c}\right)^2 C^{(0)2}\left(f_4 + \frac{1}{180}\right) - \frac{1}{180}\left(c^{(0)*} + \mathbf{c}'\mathbf{C}^{-1}\mathbf{c}\right), \end{aligned}$$

where $\mathbf{1}_p$ *is a* p *dimensional vector of ones, the* k*th entry of the* p *dimensional vector* $\mathbf{c}$ *is* $c^{(k)*}$ *and the* (k,l)*th entry of the* $p \times p$ *matrix* $\mathbf{C}$ *is* $c^{(|k-l|)*}$*, defining* $c^{(k)*}$ *as the limit of* $\sum_{i=1}^{T-k-1} c_i c_{i+k}$ *where* $c_0, c_1, ...$ *are the coefficients of the moving average representation of the AR(p) process* $\phi(L) z_t = \varepsilon_t$*. Moreover,* $C^{(0)}$ *is the limit of* $\sum_{i=0}^{T-1} c_i$*.*

Proof. See the supplement. ■

It is interesting to see how the limits of this proposition translate to the AR(1) case. We give these in the following corollary. In particular, we note the special case $\phi_1 = 0$, where as expected, the limits coincide with the corresponding limits in the $p = 0$ case.

Corollary 5 *If $p = 1$, under the assumptions of proposition 4, we have*

$$\begin{aligned} g_0 &= 1, \\ g_1 &= 2\widetilde{\phi}, \\ g_2 &= f_2 - \frac{\widetilde{\phi}}{3}, \\ g_4 &= f_4 - \frac{\widetilde{\phi}}{90}, \end{aligned}$$

where

$$\widetilde{\phi} \equiv \phi_1^2 \left(1 - \phi_1^2\right)^{-1}.$$

Proof. This follows since $c^{(0)*} = \left(1 - \phi_1^2\right)^{-1} = \mathbf{C}$, $c^{(1)*} = \phi_1 \left(1 - \phi_1^2\right)^{-1} = \mathbf{c}$ and $C^{(0)} = (1 - \phi_1)^{-1}$. ■

Next, we need to solve

$$0 = \frac{\partial}{\partial \gamma} g(\gamma) = (T+1)^{T^{-1}} \left(-g_{1T} - 2\gamma g_{2T} - 3\gamma^2 g_{3T} + 4\gamma^3 g_{4T}\right).$$

There are three solutions to this equation, which are given by

$$\begin{aligned} \widehat{\gamma}_1 &= \frac{g_{3T}}{4g_{4T}} + \frac{2^{2/3} s_1 s^{-1/3} + s^{1/3}}{3 * 2^{7/3} g_{4T}}, \\ \widehat{\gamma}_{2,3} &= \frac{g_{3T}}{4g_{4T}} - \frac{2^{2/3}\left(1 \pm i\sqrt{3}\right) s_1 s^{-1/3} + \left(1 \mp i\sqrt{3}\right) s^{1/3}}{3 * 2^{10/3} g_{4T}}, \end{aligned}$$

where

$$s = s_2 + \sqrt{-4s_1^3 + s_2^2} = 2is_1^{3/2} s_0,$$

with

$$\begin{aligned} s_0 &= \sqrt{1 - \frac{1}{4} s_2^2 s_1^{-3}} - \frac{i}{2} s_2 s_1^{-3/2}, \\ s_1 &= 9g_{3T}^2 + 24 g_{2T} g_{4T}, \\ s_2 &= 54 g_{3T}^3 + 216 g_{2T} g_{3T} g_{4T} + 432 g_{1T} g_{4T}^2. \end{aligned}$$

Moreover, since

$$i^{\pm 1/3} = \frac{\sqrt{3} \pm i}{2},$$

we have

$$\begin{aligned}
\widehat{\gamma}_1 &= \frac{g_{3T}}{4g_{4T}} + \frac{\left(\sqrt{3}-i\right) s_0^{-1/3} s_1^{1/2} + \left(\sqrt{3}+i\right) s_0^{1/3} s_1^{1/2}}{24 g_{4T}}, \\
\widehat{\gamma}_{2,3} &= \frac{g_{3T}}{4g_{4T}} \\
&\quad - \frac{\left(1 \pm i\sqrt{3}\right)\left(\sqrt{3}-i\right) s_0^{-1/3} s_1^{1/2} + \left(1 \mp i\sqrt{3}\right)\left(\sqrt{3}+i\right) s_0^{1/3} s_1^{1/2}}{48 g_{4T}}.
\end{aligned}$$

implying

$$\begin{aligned}
\widehat{\gamma}_2 &= \frac{g_{3T}}{4g_{4T}} - \frac{\left(\sqrt{3}+i\right) s_0^{-1/3} s_1^{1/2} + \left(\sqrt{3}-i\right) s_0^{1/3} s_1^{1/2}}{24 g_{4T}}, \\
\widehat{\gamma}_3 &= \frac{g_{3T}}{4g_{4T}} + i \frac{s_0^{-1/3} s_1^{1/2} - s_0^{1/3} s_1^{1/2}}{12 g_{4T}}.
\end{aligned}$$

Relating to the corresponding result for $p = 0$, we find that it is most reasonable to choose any of the solutions $\widehat{\gamma}_1$ or $\widehat{\gamma}_2$ which, asymptotically, have the same modulus but opposite signs. This is so, because as is seen from proposition 4, the g terms have the same asymptotic order of magnitude as the corresponding f terms. Indeed, we find $s_1 \sim 24 g_{2T} g_{4T} = O_p\left(T^6\right)$, $s_2 = o_p\left(T^9\right)$[3], implying $s_0 \sim 1$[4] and

$$T\widehat{\gamma}_{1,2} \xrightarrow{d} \pm\sqrt{\left|\frac{g_2}{2g_4}\right|}\left(I_{\{g_2 g_4 \geq 0\}} + i I_{\{g_2 g_4 < 0\}}\right),$$

and $T\widehat{\gamma}_3$ is of smaller order. Hence, along the same lines as for $p = 0$, we will choose the test statistic

$$\widetilde{Z}_T = \frac{T g_{2T}}{2 g_{0T}} \left|\frac{g_{2T}}{g_{4T}}\right|.$$

As in the $p = 0$ case, from proposition 4 we get the asymptotic result

$$\widetilde{Z}_T \xrightarrow{d} \frac{g_2}{2} \left|\frac{g_2}{g_4}\right|.$$

Unfortunately, as is seen from proposition 4 and corollary 5, the asymptotic distribution of $\widetilde{Z}_T$ is non similar, i.e. it is a function of the AR parameters. (Observe that the Tanaka test is also non similar, although a simple correction of it is, see further below.) To alleviate this problem, we suggest to derive the critical values of the test by response surface regression on the AR parameters. Then, in a practical situation, the critical value may be obtained from this regression equation by inserting the estimated AR parameters. In the simulation section below, we will investigate the performance of this procedure.

[3] $X_T = O_p\left(T^n\right)$ means that $T^{-n} X_T$ tends in distribution to a non degenerate random variable as $T \to \infty$. $X_T = o_p\left(T^n\right)$ similarly means that $T^{-n} X_T$ tends to zero in probability as $T \to \infty$.

[4] $X_T \sim Y_T$ means that X_T / Y_T tends to one in probability as $T \to \infty$.

The GLR test may be readily generalized to the ARMA$(p, 1)$ case, by numerically maximizing the likelihood with the MLE of $\boldsymbol{\phi}$ inserted, i.e. maximizing (17) over γ, and comparing to (17) with $\gamma = 0$ inserted.

As for the score test, theorem 3 of Tanaka (1990) yields that under the ARMA$(p, 1)$ framework, $\widehat{r}S_T$ has the same asymptotic distribution as that of S_T under $p = 0$, where

$$S_T = T\frac{\mathbf{y}'\left(2\mathbf{I}+\mathbf{L}+\mathbf{L}'\right)^{-2}\mathbf{y}}{\mathbf{y}'\left(2\mathbf{I}+\mathbf{L}+\mathbf{L}'\right)^{-1}\mathbf{y}},$$

and

$$\widehat{r} = \frac{\sum_{j=0}^{\infty}\widehat{\varphi}_j^2}{\left(\sum_{j=0}^{\infty}\widehat{\varphi}_j\right)^2},$$

where $\widehat{\varphi}_j$ are ML estimates of the coefficients of the moving average representation. In particular, if $p = 1$, the corresponding population quantity is

$$r = \frac{\sum_{j=0}^{\infty}\phi_1^{2j}}{\left(\sum_{j=0}^{\infty}\phi_1^{j}\right)^2} = \frac{\left(1-\phi_1^2\right)^{-1}}{\left(1-\phi_1\right)^{-2}} = \frac{1-\phi_1}{1+\phi_1},$$

where ϕ_1 is estimated by

$$\widehat{\phi}_1 = \frac{\mathbf{y}'\widehat{\boldsymbol{\Omega}}^{-1}\mathbf{L}\mathbf{y}}{\mathbf{y}'\mathbf{L}'\widehat{\boldsymbol{\Omega}}^{-1}\mathbf{L}\mathbf{y}},$$

with

$$\widehat{\boldsymbol{\Omega}} = \left(1-\mathbf{L}\right)\left(1-\mathbf{L}\right)' + \boldsymbol{\delta}_1\boldsymbol{\delta}_1'.$$

5 Simulation of small sample size

In this section, by means of simulation, we compare the size properties of the tests in the case $p = 1$. The size of the approximate LR test is given both without corrected critical values, as well as with critical values corrected via response surface regression, with the estimated parameter inserted. The response surface regression was performed with 1 000 000 replications on the sets $T \in \{25, 50, 100, 200, 400, 800\}$ and $\phi_1 \in \{-0.7, -0.6, ..., 0.7\}$. (Inclusion of ± 0.8 gave rather unstable results.) The results are given in table 5.[5]

For sample sizes $T = 50$ and 200, in tables 6 and 7 and figures 3 and 4, we give the empirical sizes for the tests at the nominal 5% level. Critical values for the score test and for the GLR test are taken from table 3. As for the uncorrected approximative LR test, we use the response surface regression of table 5 with $\phi_1 = 0$. It is seen that the score test is oversized for negative ϕ_1,

[5] The procedure to obtain the regressions was to initially include the terms ϕ_1, ϕ_1^2, T^{-1} $\phi_1 T^{-1}$ and $\phi_1^2 T^{-1}$ and then succesively delete terms with the highest p value when testing if their coefficient is zero. The procedure was stopped when there were no p values larger than 1% left. We also tried bringing in terms involving T^{-2}, resulting in no real improvements.

Significance level	Regression equation
0.1%	$9.136 - 81.4T^{-1} - 17.3\phi_1^2 + 289\phi_1 T^{-1} + 720\phi_1^2 T^{-1}$
1%	$3.110 - 14.6T^{-1} - 4.93\phi_1^2 + 21.2\phi_1 T^{-1} + 89.9\phi_1^2 T^{-1}$
5%	$1.142 - 4.42T^{-1} - 3.14\phi_1^2 + 28.8\phi_1^2 T^{-1}$

Table 5: Response surface regressions fro critical values of the approximative LR test in the ARMA(1,1) case. 1 000 000 replications where used, and the regression is based on the sample sizes 25, 50, 100, 200, 400 and 800 and AR parameter -0.7, -0.6, ..., 0.7.

and undersized for positive ϕ_1, especially for $T = 50$. Moreover, note that the score test is undersized at $\phi_1 = 0$, especially for $T = 50$. This might be due to the estimation of ϕ_1. Without correction of the critical values, in general the approximative LR test is undersized (conservative), but including the correction, it works very well apart from in extreme cases. As long as ϕ_1 is not too close to one, the GLR test has good size properties, in particular for $T = 200$.

Figure 3 about here

Figure 4 about here

6 Conclusion

In this paper, we have compared the GLR test of Davis et al and the score test of Tanaka for noinvertibility with a new approximative LR test. In terms of local power, the GLR test outperforms the score test, although they are very close in the neighborhood of the null. However, unlike GLR, the score test takes an explicit form. The same is true for the approximative LR test, but in terms of power, this test is seen to outperform the score test while it is only slightly worse than GLR.

The tests are generalized to the ARMA$(p, 1)$ model. Unfortunately, the distribution of the approximative LR test turns out to depend on the nuisance parameters even asymptotically. As is seen in a simulation study for the $p = 1$ case, this may materialize in conservativeness in terms of size. However, when incorporating a correction of the critical values in terms of the estimated AR parameter, the size properties are better and comparable to the score and GLR tests.

Further extensions of our results could be generalizations to MA processes of higher order (cf Davis and Song, 2011) or to models including deterministic terms.

ϕ_1	Score	Approx. LR	Approx. LR with corrected c.v.	GLR
-0.8	9.0	0.3	3.1	4.4
-0.7	6.8	0.7	4.0	4.4
-0.6	5.8	1.3	4.3	4.5
-0.5	5.3	2.0	4.5	4.6
-0.4	4.9	2.8	4.6	4.8
-0.3	4.6	3.5	4.8	4.8
-0.2	4.4	4.1	4.9	5.0
-0.1	4.2	4.5	4.9	5.2
0	4.0	4.7	5.0	5.4
0.1	3.7	4.7	5.1	5.5
0.2	3.5	4.5	5.2	5.8
0.3	3.2	4.1	5.4	6.2
0.4	2.8	3.6	5.4	6.7
0.5	2.3	3.0	5.1	7.5
0.6	1.7	2.1	4.1	8.9
0.7	1.0	1.1	2.5	11.0
0.8	0.3	0.3	1.1	16.8

Table 6: Empirical sizes in per cent in the ARMA(1,1) case, T=50. 1 000 000 replications where used except for the GLR test, where the number of replications was 40 000.

ϕ_1	Score	Approx. LR	Approx. LR with corrected c.v.	GLR
-0.8	5.7	0.1	3.0	4.7
-0.7	5.4	0.4	5.1	4.7
-0.6	5.2	1.0	4.5	4.7
-0.5	5.1	1.8	4.6	4.7
-0.4	5.0	2.7	4.8	4.8
-0.3	4.9	3.6	5.0	4.8
-0.2	4.9	4.4	5.0	4.8
-0.1	4.8	4.9	5.1	4.9
0	4.8	5.1	5.2	4.9
0.1	4.7	4.9	5.2	5.0
0.2	4.6	4.4	5.1	5.1
0.3	4.6	3.7	5.1	5.2
0.4	4.5	2.8	4.9	5.4
0.5	4.4	1.9	4.7	5.6
0.6	4.3	1.1	4.2	5.9
0.7	4.1	0.5	3.5	6.5
0.8	3.7	0.1	1.1	7.8

Table 7: Empirical sizes in per cent in the ARMA(1,1) case, T=200. 1 000 000 replications where used except for the GLR test, where the number of replications was 40 000.

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8 Appendix: Omitted proofs

8.1 Proof of proposition 1

Via (5) and (6),

$$\widehat{\sigma}^2 = T^{-1}\mathbf{y}'\mathbf{\Omega}^{-1}\mathbf{y}$$

where with $\eta = \det \mathbf{\Omega}$,

$$\mathbf{\Omega}^{-1} = \left(\mathbf{\Omega}_1 + \theta^2 \boldsymbol{\delta}_1 \boldsymbol{\delta}_1'\right)^{-1} = \mathbf{\Omega}_1^{-1} - \theta^2 \mathbf{\Omega}_1^{-1} \boldsymbol{\delta}_1 \boldsymbol{\delta}_1' \mathbf{\Omega}_1^{-1} \eta^{-1}.$$

Writing

$$\mathbf{A} \equiv (\mathbf{I} - \mathbf{L}')(\mathbf{I} - \theta\mathbf{L}')^{-1}(\mathbf{I} - \theta\mathbf{L})^{-1}(\mathbf{I} - \mathbf{L}),$$

we get

$$\boldsymbol{\Omega}_1^{-1} = \left(\mathbf{I} - \theta\mathbf{L}'\right)^{-1}\left(\mathbf{I} - \theta\mathbf{L}\right)^{-1} = \left(\mathbf{I} - \mathbf{L}'\right)^{-1}\mathbf{A}\left(\mathbf{I} - \mathbf{L}\right)^{-1},$$

implying

$$\boldsymbol{\Omega}^{-1} = \left(\mathbf{I} - \mathbf{L}'\right)^{-1}\mathbf{B}\left(\mathbf{I} - \mathbf{L}\right)^{-1}, \tag{19}$$

where because $\left(\mathbf{I} - \mathbf{L}\right)^{-1}\boldsymbol{\delta}_1 = \mathbf{1}$,

$$\mathbf{B} = \mathbf{A} - \theta^2\mathbf{A}\mathbf{1}\mathbf{1}'\mathbf{A}\eta^{-1}. \tag{20}$$

Hence, with $\mathbf{x} = \left(\mathbf{I} - \mathbf{L}\right)^{-1}\mathbf{y}$, we get

$$\widehat{\sigma}^2 = T^{-1}\mathbf{x}'\mathbf{B}\mathbf{x}. \tag{21}$$

Now, with $\theta = 1 - \gamma$ and $\mathbf{S} \equiv \left(\mathbf{I} - \mathbf{L}\right)^{-1}\mathbf{L}$, Taylor expansion yields

$$\begin{aligned} & \left\{\mathbf{I} - \left(1 - \gamma\right)\mathbf{L}\right\}^{-1}\left(\mathbf{I} - \mathbf{L}\right) \\ = \; & \left(\mathbf{I} + \gamma\mathbf{S}\right)^{-1} = \mathbf{I} - \gamma\mathbf{S} + \gamma^2\mathbf{S}^2 - \gamma^3\mathbf{S}^3 + \gamma^4\mathbf{S}^4 + O\left(\gamma^5\right), \end{aligned}$$

implying

$$\mathbf{A} = \mathbf{I} - \gamma\mathbf{A}_1 + \gamma^2\mathbf{A}_2 - \gamma^3\mathbf{A}_3 + \gamma^4\mathbf{A}_4 + O\left(\gamma^5\right), \tag{22}$$

where

$$\begin{aligned} \mathbf{A}_1 &\equiv \mathbf{S}' + \mathbf{S}, \\ \mathbf{A}_2 &\equiv \mathbf{S}'^2 + \mathbf{S}'\mathbf{S} + \mathbf{S}^2, \\ \mathbf{A}_3 &\equiv \mathbf{S}'^3 + \mathbf{S}'^2\mathbf{S} + \mathbf{S}'\mathbf{S}^2 + \mathbf{S}^3, \\ \mathbf{A}_4 &\equiv \mathbf{S}'^4 + \mathbf{S}'^3\mathbf{S} + \mathbf{S}'^2\mathbf{S}^2 + \mathbf{S}'\mathbf{S}^3 + \mathbf{S}^4, \end{aligned}$$

which may be simplified into

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{1}\mathbf{1}' - \mathbf{I}, \\ \mathbf{A}_2 &= \left(T - 2\right)\mathbf{1}\mathbf{1}' + \mathbf{I} - \mathbf{S}\mathbf{S}', \\ \mathbf{A}_3 &= \left(T^2 - 3T + 3\right)\mathbf{1}\mathbf{1}' - \mathbf{I} - \mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}' - \mathbf{S}\mathbf{S}'\mathbf{1}\mathbf{1}' + 2\mathbf{S}\mathbf{S}', \\ \mathbf{A}_4 &= \frac{1}{6}\left(T - 3\right)\left(4T^2 - 9T + 8\right)\mathbf{1}\mathbf{1}' + \mathbf{I} - \left(T - 3\right)\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}' \\ & \quad - \left(T - 3\right)\mathbf{S}\mathbf{S}'\mathbf{1}\mathbf{1}' - 3\mathbf{S}\mathbf{S}' + \left(\mathbf{S}\mathbf{S}'\right)^2, \end{aligned}$$

and moreover, it follows that

$$\begin{aligned} \mathbf{1}'\mathbf{A}_1 &= \left(T - 1\right)\mathbf{1}', \\ \mathbf{1}'\mathbf{A}_2 &= \left(T - 1\right)^2\mathbf{1}' - \mathbf{1}'\mathbf{S}\mathbf{S}', \\ \mathbf{1}'\mathbf{A}_3 &= \frac{1}{6}\left(T - 2\right)\left(T - 1\right)\left(4T - 3\right)\mathbf{1}' - \left(T - 2\right)\mathbf{1}'\mathbf{S}\mathbf{S}', \\ \mathbf{1}'\mathbf{A}_4 &= \frac{1}{6}\left(T - 2\right)\left(T - 1\right)\left(2T^2 - 6T + 3\right)\mathbf{1}' - \left(T^2 - 3T + 3\right)\mathbf{1}'\mathbf{S}\mathbf{S}' \\ & \quad + \mathbf{1}'\left(\mathbf{S}\mathbf{S}'\right)^2. \end{aligned}$$

Consequently, via (22),

$$\begin{aligned}
& \mathbf{A11'A} \\
= & \left(\mathbf{I}-\gamma\mathbf{A}_1+\gamma^2\mathbf{A}_2-\gamma^3\mathbf{A}_3+\gamma^4\mathbf{A}_4\right)\mathbf{11'} \\
& \left(\mathbf{I}-\gamma\mathbf{A}_1+\gamma^2\mathbf{A}_2-\gamma^3\mathbf{A}_3+\gamma^4\mathbf{A}_4\right)+O\left(\gamma^5\right) \\
= & \mathbf{C}_0-\gamma\mathbf{C}_1+\gamma^2\mathbf{C}_2-\gamma^3\mathbf{C}_3+\gamma^4\mathbf{C}_4+O\left(\gamma^5\right),
\end{aligned}$$

where via simplifications,

$$\mathbf{C}_0 \equiv \mathbf{11'},$$

$$\mathbf{C}_1 \equiv \mathbf{11'A}_1+\mathbf{A}_1\mathbf{11'} = 2\left(T-1\right)\mathbf{11'},$$

$$\begin{aligned}
\mathbf{C}_2 & \equiv \mathbf{11'A}_2+\mathbf{A}_1\mathbf{11'A}_1+\mathbf{A}_2\mathbf{11'} \\
& = 3\left(T-1\right)^2\mathbf{11'}-\mathbf{11'SS'}-\mathbf{SS'11'},
\end{aligned}$$

$$\begin{aligned}
& \mathbf{C}_3 \\
\equiv & \mathbf{11'A}_3+\mathbf{A}_1\mathbf{11'A}_2+\mathbf{A}_2\mathbf{11'A}_1+\mathbf{A}_3\mathbf{11'} \\
= & \frac{1}{3}\left(T-1\right)\left(2T-3\right)\left(5T-4\right)\mathbf{11'}-\left(2T-3\right)\left(\mathbf{11'SS'}+\mathbf{SS'11'}\right),
\end{aligned}$$

$$\begin{aligned}
& \mathbf{C}_4 \\
\equiv & \mathbf{11'A}_4+\mathbf{A}_1\mathbf{11'A}_3+\mathbf{A}_2\mathbf{11'A}_2+\mathbf{A}_3\mathbf{11'A}_1+\mathbf{A}_4\mathbf{11'} \\
= & \frac{1}{3}\left(T-1\right)\left(9T^3-34T^2+41T-15\right)\mathbf{11'} \\
& -\left(3T^2-8T+6\right)\left(\mathbf{11'SS'}+\mathbf{SS'11'}\right)+\mathbf{11'}\left(\mathbf{SS'}\right)^2+\left(\mathbf{SS'}\right)^2\mathbf{11'} \\
& +\mathbf{SS'11'SS'}
\end{aligned}$$

Hence, via (20), (21), (8) and (10), Taylor expansion yields

$$f\left(\gamma\right)=T^{-1}\left(T+1\right)^{T^{-1}}\mathbf{x}'\left(\mathbf{D}_{0T}-\gamma^2\mathbf{D}_{2T}-\gamma^3\mathbf{D}_{2T}+\gamma^4\mathbf{D}_{4T}\right)\mathbf{x}+O\left(\gamma^5\right),$$

where

$$\begin{aligned}
\mathbf{D}_{0T} & \equiv \mathbf{I}-\frac{1}{T+1}\mathbf{11'}, \\
\mathbf{D}_{2T} & \equiv \frac{T^2-2T+4}{3\left(T+1\right)}\mathbf{11'}+\mathbf{SS'}-\frac{T+2}{6}\mathbf{I}-\frac{1}{T+1}\left(\mathbf{11'SS'}+\mathbf{SS'11'}\right), \\
\mathbf{D}_{4T} & \equiv \frac{8T^4+14T^3-309T^2+529T-602}{360\left(T+1\right)}\mathbf{11'}-\frac{T+8}{6}\mathbf{SS'} \\
& -\frac{\left(T+2\right)\left(2T^2-T-61\right)}{360}\mathbf{I}+\frac{T^2-2T+7}{3\left(T+1\right)}\left(\mathbf{11'SS'}+\mathbf{SS'11'}\right) \\
& +\left(\mathbf{SS'}\right)^2-\frac{1}{T+1}\mathbf{SS'11'SS'}-\frac{1}{T+1}\left\{\mathbf{11'}\left(\mathbf{SS'}\right)^2+\left(\mathbf{SS'}\right)^2\mathbf{11'}\right\}.
\end{aligned}$$

8.2 Proof of proposition 2

Observe that if $\theta = 1$, we have from (2) that with $\widetilde{\boldsymbol{\varepsilon}} = (\varepsilon_1, ..., \varepsilon_T)'$

$$\mathbf{x} = (\mathbf{I} - \mathbf{L})^{-1} (-\boldsymbol{\delta}_1, \mathbf{I} - \mathbf{L}) \begin{pmatrix} \varepsilon_0 \\ \widetilde{\boldsymbol{\varepsilon}} \end{pmatrix} = \widetilde{\boldsymbol{\varepsilon}} - \mathbf{1}\varepsilon_0.$$

Now,

$$f_{0T} = \mathbf{x}'\mathbf{x} - \frac{1}{T+1} (\mathbf{1}'\mathbf{x})^2,$$

where

$$\mathbf{x}'\mathbf{x} = (\widetilde{\boldsymbol{\varepsilon}} - \mathbf{1}\varepsilon_0)' (\widetilde{\boldsymbol{\varepsilon}} - \mathbf{1}\varepsilon_0) = \widetilde{\boldsymbol{\varepsilon}}'\widetilde{\boldsymbol{\varepsilon}} - 2\varepsilon_0 \mathbf{1}'\widetilde{\boldsymbol{\varepsilon}} + T\varepsilon_0^2$$

and

$$\mathbf{1}'\mathbf{x} = \mathbf{1}'\widetilde{\boldsymbol{\varepsilon}} - T\varepsilon_0.$$

Hence, it follows that

$$f_{0T} = \widetilde{\boldsymbol{\varepsilon}}'\widetilde{\boldsymbol{\varepsilon}} - \frac{1}{T+1} (\mathbf{1}'\widetilde{\boldsymbol{\varepsilon}})^2 - \frac{2}{T+1}\varepsilon_0 \mathbf{1}'\widetilde{\boldsymbol{\varepsilon}} + \frac{T}{T+1}\varepsilon_0^2,$$

and since $T^{-1}\widetilde{\boldsymbol{\varepsilon}}'\widetilde{\boldsymbol{\varepsilon}} \xrightarrow{P} 1$ and $T^{-1/2}\mathbf{1}'\widetilde{\boldsymbol{\varepsilon}}$ is standard normal for all T, it follows that $T^{-1} f_{0T} \xrightarrow{P} 1$. Similarly,

$$\begin{aligned} & f_{2T} \\ = \; & \frac{T^2 - 2T + 4}{3(T+1)} (\mathbf{1}'\mathbf{x})^2 + \mathbf{x}'\mathbf{S}\mathbf{S}'\mathbf{x} - \frac{T+2}{6}\mathbf{x}'\mathbf{x} - \frac{2}{T+1}\mathbf{x}'\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}, \end{aligned}$$

where the above results and

$$\mathbf{x}'\mathbf{S}\mathbf{S}'\mathbf{x} = \widetilde{\boldsymbol{\varepsilon}}'\mathbf{S}\mathbf{S}'\widetilde{\boldsymbol{\varepsilon}} - 2\varepsilon_0 \mathbf{1}'\mathbf{S}\mathbf{S}'\widetilde{\boldsymbol{\varepsilon}} + \varepsilon_0^2 \mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{1},$$

$$\begin{aligned} & \mathbf{x}'\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x} \\ = \; & \widetilde{\boldsymbol{\varepsilon}}'\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}'\widetilde{\boldsymbol{\varepsilon}} - T\varepsilon_0 \mathbf{1}'\mathbf{S}\mathbf{S}'\widetilde{\boldsymbol{\varepsilon}} - \varepsilon_0 \widetilde{\boldsymbol{\varepsilon}}'\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{1} + T\varepsilon_0^2 \mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{1}, \end{aligned}$$

yield, after simplification,

$$f_{2T} = \widetilde{\boldsymbol{\varepsilon}}'\mathbf{D}_{2T}\widetilde{\boldsymbol{\varepsilon}} - \varepsilon_0 Y_{21} + \varepsilon_0^2 Y_{22},$$

where

$$\begin{aligned} Y_{21} \equiv \; & 2T\frac{T^2 - 2T + 4}{3(T+1)}\mathbf{1}'\widetilde{\boldsymbol{\varepsilon}} + 2\mathbf{1}'\mathbf{S}\mathbf{S}'\widetilde{\boldsymbol{\varepsilon}} - \frac{T+2}{3}\mathbf{1}'\widetilde{\boldsymbol{\varepsilon}} - \frac{2T}{T+1}\mathbf{1}'\mathbf{S}\mathbf{S}'\widetilde{\boldsymbol{\varepsilon}} \\ & - \frac{2}{T+1}\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{1}\mathbf{1}'\widetilde{\boldsymbol{\varepsilon}}, \\ Y_{22} \equiv \; & T^2\frac{T^2 - 2T + 4}{3(T+1)} + \mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{1} - T\frac{T+2}{6} - \frac{2T}{T+1}\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{1}. \end{aligned}$$

Now, observing that

$$\mathbf{1}'\mathbf{SS}'\mathbf{1} = \sum_{t=1}^{T-1} t^2 = \frac{1}{6}T\left(2T-1\right)\left(T-1\right),$$

we find that by simplification,

$$Y_{22} = \frac{1}{6}T\frac{T-1}{T+1},$$

which shows that T^{-2} times the ε_0^2 term tends to zero with T. Similarly,

$$Y_{21} = -\frac{2}{3}\frac{\left(T-1\right)^2}{T+1}\mathbf{1}'\widetilde{\varepsilon} + \frac{2}{T+1}\mathbf{1}'\mathbf{SS}'\widetilde{\varepsilon},$$

which, because $\mathbf{1}'\mathbf{SS}'\widetilde{\varepsilon}$ is of order $T^{5/2}$, is of order $T^{3/2}$, and so, $T^{-2}Y_{21}$ tends to zero with T. Then, we find the limit of f_{2T} since by standard methods and elementary matrix manipulations,

$$\begin{aligned}
T^{-1}\widetilde{\varepsilon}'\widetilde{\varepsilon} \quad & \xrightarrow{P} 1, \\
T^{-1/2}\mathbf{1}'\widetilde{\varepsilon} \quad & \xrightarrow{d} W\left(1\right), \\
T^{-2}\widetilde{\varepsilon}'\mathbf{SS}'\widetilde{\varepsilon} \quad & \xrightarrow{d} \int_0^1 W^*\left(t\right)^2 dt, \\
T^{-3}\widetilde{\varepsilon}'\mathbf{11}'\mathbf{SS}'\widetilde{\varepsilon} \quad & \xrightarrow{d} W\left(1\right)\int_0^1 \left(1-t\right)W^*\left(t\right)dt,
\end{aligned}$$

In the same fashion, we get

$$f_{4T} = \widetilde{\varepsilon}'\mathbf{D}_{4T}\widetilde{\varepsilon} - \varepsilon_0 Y_{41} + \varepsilon_0^2 Y_{42},$$

where

$$\begin{aligned}
& Y_{41} \\
\equiv \quad & \left\{T\frac{8T^4+14T^3-309T^2+529T-602}{180\left(T+1\right)}\right. \\
& -\frac{\left(T+2\right)\left(2T^2-T-61\right)}{180} + 2\frac{T^2-2T+7}{3\left(T+1\right)}\mathbf{1}'\mathbf{SS}'\mathbf{1} \\
& \left.-\frac{2}{T+1}\mathbf{1}'\left(\mathbf{SS}'\right)^2\mathbf{1}\right\}\mathbf{1}'\widetilde{\varepsilon} \\
& +\left\{-\frac{T+8}{3} + 2T\frac{T^2-2T+7}{3\left(T+1\right)} - \frac{2}{T+1}\mathbf{1}'\mathbf{SS}'\mathbf{1}\right\}\mathbf{1}'\mathbf{SS}'\widetilde{\varepsilon} \\
& +\left(2-\frac{2T}{T+1}\right)\mathbf{1}'\left(\mathbf{SS}'\right)^2\widetilde{\varepsilon}
\end{aligned}$$

and

$$
\begin{aligned}
& Y_{42} \\
\equiv \quad & T^2 \frac{8T^4+14T^3-309T^2+529T-602}{360\,(T+1)} - \frac{T+8}{6}\mathbf{1}'\mathbf{SS}'\mathbf{1} \\
& -T\frac{(T+2)\left(2T^2-T-61\right)}{360} + 2T\frac{T^2-2T+7}{3\,(T+1)}\mathbf{1}'\mathbf{SS}'\mathbf{1} \\
& +\mathbf{1}'\left(\mathbf{SS}'\right)^2\mathbf{1} - \frac{1}{T+1}\left(\mathbf{1}'\mathbf{SS}'\mathbf{1}\right)^2 - \frac{2T}{T+1}\mathbf{1}'\left(\mathbf{SS}'\right)^2\mathbf{1}.
\end{aligned}
$$

Here, because

$$
\mathbf{1}'\left(\mathbf{SS}'\right)^2\mathbf{1} = \sum_{t=1}^{T-1}\left(t\frac{2T-t-1}{2}\right)^2 = \frac{1}{30}T\,(2T-1)\,(T-1)\left(2T^2-2T+1\right),
$$

it follows that

$$
Y_{42} = \frac{1}{120}T\,(2T+9)\,(T-1)\,\frac{T-2}{T+1},
$$

and so, $T^{-4}Y_{42}$ tends to zero with T. We also find that

$$
\begin{aligned}
& Y_{41} \\
= \quad & -\frac{1}{180}\,(T-1)\,\frac{8T^3+22T^2-167T+122}{T+1}\mathbf{1}'\widetilde{\boldsymbol{\varepsilon}} \\
& -\frac{2}{3}\frac{T^2-2T+4}{T+1}\mathbf{1}'\mathbf{SS}'\widetilde{\boldsymbol{\varepsilon}} + \frac{2}{T+1}\mathbf{1}'\left(\mathbf{SS}'\right)^2\widetilde{\boldsymbol{\varepsilon}},
\end{aligned}
$$

which because of previous arguments and the fact that $\mathbf{1}'\left(\mathbf{SS}'\right)^2\widetilde{\boldsymbol{\varepsilon}}$ is of order $T^{9/2}$ implies that $T^{-4}Y_{41}$ tends to zero with T. The limit of f_{4T} follows using the results above, proposition 1 and

$$
\begin{aligned}
T^{-4}\widetilde{\boldsymbol{\varepsilon}}'\left(\mathbf{SS}'\right)^2\widetilde{\boldsymbol{\varepsilon}} \quad & \xrightarrow{d} \int_{t=0}^{1}\left(\int_{s=0}^{1} W^*(s)\,ds\right)^2 dt, \\
T^{-5}\widetilde{\boldsymbol{\varepsilon}}'\mathbf{SS}'\mathbf{11}'\mathbf{SS}'\widetilde{\boldsymbol{\varepsilon}} \quad & \xrightarrow{d} \left\{\int_0^1 (1-t)\,W^*(t)\,dt\right\}^2, \\
T^{-5}\widetilde{\boldsymbol{\varepsilon}}'\mathbf{11}'\left(\mathbf{SS}'\right)^2\widetilde{\boldsymbol{\varepsilon}} \quad & \xrightarrow{d} \frac{1}{6}W(1)\int_0^1\left(2-3t^2+t^3\right)W^*(t)\,dt.
\end{aligned}
$$

8.3 Proof of proposition 3

Proposition 1 gives the expansion of $f(\gamma) = T^{-1} m \eta^{T^{-1}}$. As for $\mathbf{m}$, note that from (19) and (20), for all i, with $\mathbf{x} = (\mathbf{I} - \mathbf{L})^{-1} \mathbf{y}$,

$$
\begin{aligned}
& \mathbf{y}'\mathbf{J}_i\mathbf{y} \\
= \; & \mathbf{x}'\mathbf{B}\mathbf{L}^i\mathbf{x} = \mathbf{x}'\mathbf{A}\mathbf{L}_T^i\mathbf{x} - \theta^2\eta^{-\mathbf{1}}\mathbf{x}'\mathbf{A}\mathbf{1}\mathbf{1}'\mathbf{A}\mathbf{L}_T^i\mathbf{x} \\
= \; & \mathbf{x}'\left(\mathbf{I} - \gamma\mathbf{A}_1 + \gamma^2\mathbf{A}_2 - \gamma^3\mathbf{A}_3 + \gamma^4\mathbf{A}_4\right)\mathbf{L}_T^i\mathbf{x} \\
& - (1-\gamma)^2\eta^{-\mathbf{1}}\mathbf{x}'\left(\mathbf{C}_0 - \gamma\mathbf{C}_1 + \gamma^2\mathbf{C}_2 - \gamma^3\mathbf{C}_3 + \gamma^4\mathbf{C}_4\right)\mathbf{L}_T^i\mathbf{x} + O\left(\gamma^5\right) \\
= \; & \mathbf{x}'\left(\mathbf{E}_0 - \gamma\mathbf{E}_1 + \gamma^2\mathbf{E}_2 - \gamma^3\mathbf{E}_3 + \gamma^4\mathbf{E}_4\right)\mathbf{L}_T^i\mathbf{x} + O\left(\gamma^5\right)
\end{aligned}
$$

where via simplifications,

$$
\begin{aligned}
\mathbf{E}_0 &= \mathbf{I} - \frac{1}{T+1}\mathbf{1}\mathbf{1}', \\
\mathbf{E}_1 &= -\mathbf{I} + \frac{1}{T+1}\mathbf{1}\mathbf{1}' = -\mathbf{E}_0, \\
\mathbf{E}_2 &= \mathbf{I} - \frac{2T^2 - 5T + 12}{6(T+1)}\mathbf{1}\mathbf{1}' - \mathbf{S}\mathbf{S}' + \frac{1}{T+1}\left(\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}' + \mathbf{S}\mathbf{S}'\mathbf{1}\mathbf{1}'\right), \\
\mathbf{E}_3 &= -\mathbf{I} + \frac{2T^2 - 5T + 9}{3(T+1)}\mathbf{1}\mathbf{1}' + 2\mathbf{S}\mathbf{S}' - \frac{2}{T+1}\left(\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}' + \mathbf{S}\mathbf{S}'\mathbf{1}\mathbf{1}'\right) = \mathbf{E}_0 - 2\mathbf{E}_2, \\
\mathbf{E}_4 &= \mathbf{I} + \frac{4T^4 + 16T^3 - 281T^2 + 576T - 720}{180(T+1)}\mathbf{1}\mathbf{1}' - 3\mathbf{S}\mathbf{S}' \\
& \quad + \frac{2T^2 - 5T + 24}{6(T+1)}\left(\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}' + \mathbf{S}\mathbf{S}'\mathbf{1}\mathbf{1}'\right) + \left(\mathbf{S}\mathbf{S}'\right)^2 \\
& \quad - \frac{1}{T+1}\left\{\mathbf{1}\mathbf{1}'\left(\mathbf{S}\mathbf{S}'\right)^2 + \left(\mathbf{S}\mathbf{S}'\right)^2\mathbf{1}\mathbf{1}' + \mathbf{S}\mathbf{S}'\mathbf{1}\mathbf{1}'\mathbf{S}\mathbf{S}'\right\}.
\end{aligned}
$$

It follows that

$$\mathbf{m} = \mathbf{m}_0 - \gamma\mathbf{m}_1 + \gamma^2\mathbf{m}_2 - \gamma^3\mathbf{m}_3 + \gamma^4\mathbf{m}_4 + O\left(\gamma^5\right),$$

where for $j = 0, 1, 2, 3, 4$,

$$\mathbf{m}_j \equiv \left(\mathbf{x}'\mathbf{E}_j\mathbf{L}\mathbf{x}, ..., \mathbf{x}'\mathbf{E}_j\mathbf{L}^p\mathbf{x}\right)'.$$

In a similar fashion,

$$\mathbf{M} = \mathbf{M}_0 - \gamma\mathbf{M}_1 + \gamma^2\mathbf{M}_2 - \gamma^3\mathbf{M}_3 + \gamma^4\mathbf{M}_4 + O\left(\gamma^5\right),$$

where for $j = 0, 1, 2, 3, 4$,

$$\mathbf{M}_j \equiv \begin{pmatrix} \mathbf{x}'\mathbf{L}'\mathbf{E}_j\mathbf{L}\mathbf{x} & \cdots & \mathbf{x}'\mathbf{L}'\mathbf{E}_j\mathbf{L}^p\mathbf{x} \\ \vdots & & \vdots \\ \mathbf{x}'\mathbf{L}'^p\mathbf{E}_j\mathbf{L}\mathbf{x} & \cdots & \mathbf{x}'\mathbf{L}'^p\mathbf{E}_j\mathbf{L}^p\mathbf{x} \end{pmatrix}.$$

Moreover, because

$$\begin{aligned}
& \mathbf{M}^{-1} \\
= \; & \left(\mathbf{I}_T - \gamma\mathbf{M}_0^{-1}\mathbf{M}_1 + \gamma^2\mathbf{M}_0^{-1}\mathbf{M}_2 - \gamma^3\mathbf{M}_0^{-1}\mathbf{M}_3 + \gamma^4\mathbf{M}_0^{-1}\mathbf{M}_4\right)^{-1}\mathbf{M}_0^{-1} \\
= \; & \mathbf{P}_0 + \gamma\mathbf{P}_1 + \gamma^2\mathbf{P}_2 + \gamma^3\mathbf{P}_3 + \gamma^4\mathbf{P}_4 + O\left(\gamma^5\right),
\end{aligned}$$

where, defining $2_c(\mathbf{A},\mathbf{B}) = \mathbf{AB} + \mathbf{BA}$ etcetera,

$$\begin{aligned}
\mathbf{P}_0 &\equiv \mathbf{M}_0^{-1}, \\
\mathbf{P}_1 &\equiv \mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1}, \\
\mathbf{P}_2 &\equiv \left\{\left(\mathbf{M}_0^{-1}\mathbf{M}_1\right)^2 - \mathbf{M}_0^{-1}\mathbf{M}_2\right\}\mathbf{M}_0^{-1}, \\
\mathbf{P}_3 &\equiv \left\{\left(\mathbf{M}_0^{-1}\mathbf{M}_1\right)^3 - 2_c\left(\mathbf{M}_0^{-1}\mathbf{M}_1, \mathbf{M}_0^{-1}\mathbf{M}_2\right) + \mathbf{M}_0^{-1}\mathbf{M}_3\right\}\mathbf{M}_0^{-1}, \\
\mathbf{P}_4 &\equiv \left\{\left(\mathbf{M}_0^{-1}\mathbf{M}_1\right)^4 - 3_c\left(\mathbf{M}_0^{-1}\mathbf{M}_1, \mathbf{M}_0^{-1}\mathbf{M}_1, \mathbf{M}_0^{-1}\mathbf{M}_2\right) + \left(\mathbf{M}_0^{-1}\mathbf{M}_2\right)^2\right. \\
&\quad \left. + 2_c\left(\mathbf{M}_0^{-1}\mathbf{M}_1, \mathbf{M}_0^{-1}\mathbf{M}_3\right) - \mathbf{M}_0^{-1}\mathbf{M}_4\right\}\mathbf{M}_0^{-1},
\end{aligned}$$

we get

$$\mathbf{m}'\mathbf{M}^{-1}\mathbf{m} = q_{0T} + \gamma q_{1T} + \gamma^2 q_{2T} + \gamma^3 q_{3T} + \gamma^4 q_{4T} + O\left(\gamma^5\right),$$

where

$$\begin{aligned}
q_{0T} &\equiv \mathbf{m}_0'\mathbf{P}_0\mathbf{m}_0, \\
q_{1T} &\equiv -2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_1 + \mathbf{m}_0'\mathbf{P}_1\mathbf{m}_0, \\
q_{2T} &\equiv 2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_2 - 2\mathbf{m}_0'\mathbf{P}_1\mathbf{m}_1 + \mathbf{m}_0'\mathbf{P}_2\mathbf{m}_0 + \mathbf{m}_1'\mathbf{P}_0\mathbf{m}_1, \\
q_{3T} &\equiv -2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_3 + 2\mathbf{m}_0'\mathbf{P}_1\mathbf{m}_2 - 2\mathbf{m}_0'\mathbf{P}_2\mathbf{m}_1 \\
&\quad + \mathbf{m}_0'\mathbf{P}_3\mathbf{m}_0 - 2\mathbf{m}_1'\mathbf{P}_0\mathbf{m}_2 + \mathbf{m}_1'\mathbf{P}_1\mathbf{m}_1, \\
q_{4T} &\equiv 2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_4 - 2\mathbf{m}_0'\mathbf{P}_1\mathbf{m}_3 + 2\mathbf{m}_0'\mathbf{P}_2\mathbf{m}_2 - 2\mathbf{m}_0'\mathbf{P}_3\mathbf{m}_1 \\
&\quad + \mathbf{m}_0'\mathbf{P}_4\mathbf{m}_0 + 2\mathbf{m}_1'\mathbf{P}_0\mathbf{m}_3 - 2\mathbf{m}_1'\mathbf{P}_1\mathbf{m}_2 + \mathbf{m}_1'\mathbf{P}_2\mathbf{m}_1 \\
&\quad + \mathbf{m}_2'\mathbf{P}_0\mathbf{m}_2.
\end{aligned}$$

Observe that because $\mathbf{E}_1 = -\mathbf{E}_0$ and $\mathbf{E}_3 = \mathbf{E}_0 - 2\mathbf{E}_2$, corresponding equalities hold for the $\mathbf{m}_j$ and $\mathbf{M}_j$, implying

$$\begin{aligned}
\mathbf{P}_0 &= \mathbf{M}_0^{-1}, \\
\mathbf{P}_1 &= -\mathbf{M}_0^{-1} = -\mathbf{P}_0, \\
\mathbf{P}_2 &= \left(\mathbf{I} - \mathbf{M}_0^{-1}\mathbf{M}_2\right)\mathbf{M}_0^{-1}, \\
\mathbf{P}_3 &= \left\{-\mathbf{I} + 2\mathbf{M}_0^{-1}\mathbf{M}_2 + \mathbf{M}_0^{-1}\left(\mathbf{M}_0 - 2\mathbf{M}_2\right)\right\}\mathbf{M}_0^{-1} = 0, \\
\mathbf{P}_4 &= \left\{-\mathbf{I} + \mathbf{M}_0^{-1}\mathbf{M}_2 + \left(\mathbf{M}_0^{-1}\mathbf{M}_2\right)^2 - \mathbf{M}_0^{-1}\mathbf{M}_4\right\}\mathbf{M}_0^{-1},
\end{aligned}$$

and

$$
\begin{aligned}
q_{1T} &= 2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_0 - \mathbf{m}_0'\mathbf{P}_0\mathbf{m}_0 = \mathbf{m}_0'\mathbf{P}_0\mathbf{m}_0, \\
q_{2T} &= -\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_0 + 2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_2 + \mathbf{m}_0'\mathbf{P}_2\mathbf{m}_0, \\
q_{3T} &= -3\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_0 + 4\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_2 + 2\mathbf{m}_0'\mathbf{P}_2\mathbf{m}_0, \\
q_{4T} &= -2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_2 + 2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_4 + \mathbf{m}_0'\mathbf{P}_2\mathbf{m}_0 + \mathbf{m}_0'\mathbf{P}_4\mathbf{m}_0 \\
&\quad +2\mathbf{m}_0'\mathbf{P}_2\mathbf{m}_2 + \mathbf{m}_2'\mathbf{P}_0\mathbf{m}_2.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& \mathbf{m}'\mathbf{M}^{-1}\mathbf{m}\eta^{-T^{-1}} \\
= \; & (T+1)^{-T^{-1}} \left\{ r_{0T} + \gamma r_{1T} + \gamma^2 r_{2T} + \gamma^3 r_{3T} + \gamma^4 r_{4T} + O\left(\gamma^5\right) \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
r_{0T} &\equiv q_{0T}, \\
r_{1T} &\equiv q_{0T} + q_{1T} = 2q_{0T} = 2r_{0T}, \\
r_{2T} &\equiv -\frac{T-4}{6} q_{0T} + q_{1T} + q_{2T} = -\frac{T-10}{6} q_{0T} + q_{2T}, \\
r_{3T} &\equiv -\frac{T-1}{3} q_{0T} - \frac{T-4}{6} q_{1T} + q_{2T} + q_{3T} = -\frac{T-2}{2} q_{0T} + q_{2T} + q_{3T}, \\
r_{4T} &\equiv \frac{2T^3 + 13T^2 - 143T + 38}{360} q_{0T} - \frac{T-1}{3} q_{1T} - \frac{T-4}{6} q_{2T} + q_{3T} + q_{4T} \\
&= \frac{2T^3 + 13T^2 - 263T + 158}{360} q_{0T} - \frac{T-4}{6} q_{2T} + q_{3T} + q_{4T}.
\end{aligned}
$$

Hence, via proposition 1, (16) and (18),

$$
g(\gamma) = T^{-1}(T+1)^{T^{-1}} \left(g_{0T} - \gamma g_{1T} - \gamma^2 g_{2T} - \gamma^3 g_{3T} + \gamma^4 g_{4T} \right) + O\left(\gamma^5\right),
$$

where

$$
\begin{aligned}
g_{0T} &\equiv f_{0T} - r_{0T}, \\
g_{1T} &\equiv r_{1T} = 2r_{0T}, \\
g_{2T} &\equiv f_{2T} + r_{2T}, \\
g_{3T} &\equiv f_{2T} + r_{3T}, \\
g_{4T} &\equiv f_{4T} - r_{4T}.
\end{aligned}
$$

This completes the proof.

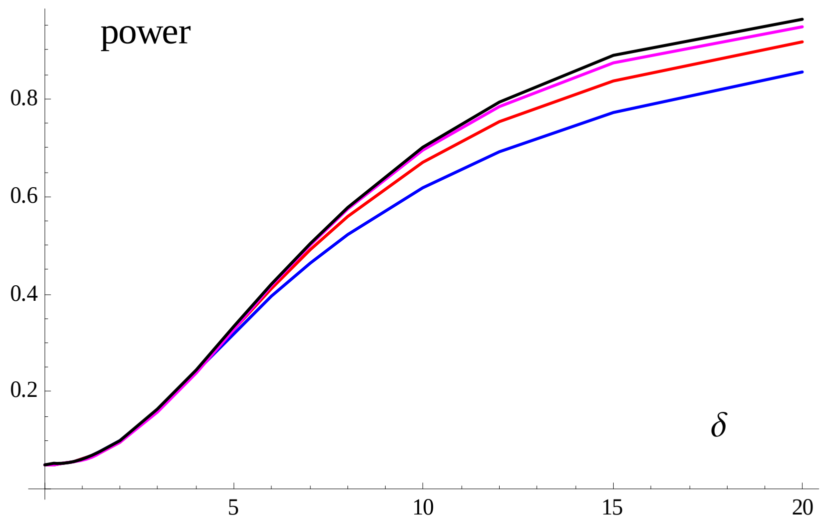

Figure 1: Local size adjusted power, $\theta = 1 - \delta/T$, $T = 50$, 100000 replications. Score test in blue, approximative LR test in red, GLR test in magenta and power envelope in black.

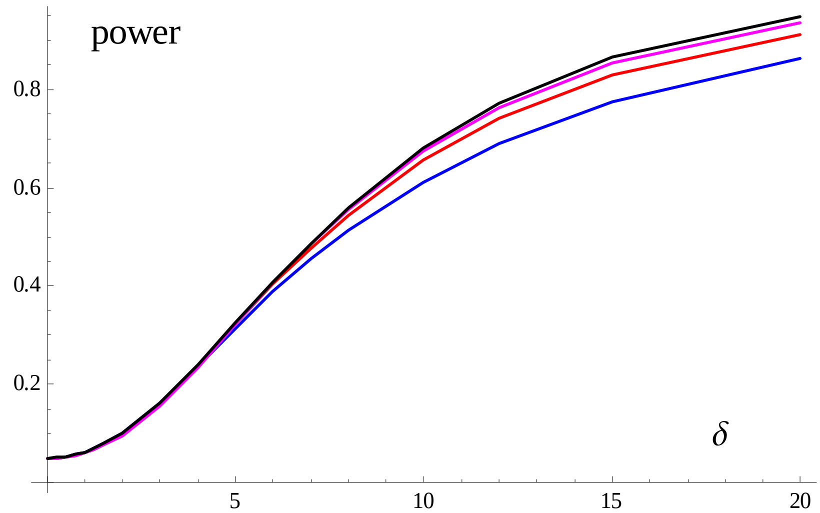

Figure 2: Local size adjusted power, $\theta = 1 - \delta/T$, $T = 200$, 100000 replications. Score test in blue, approximative LR test in red, GLR test in magenta and power envelope in black.

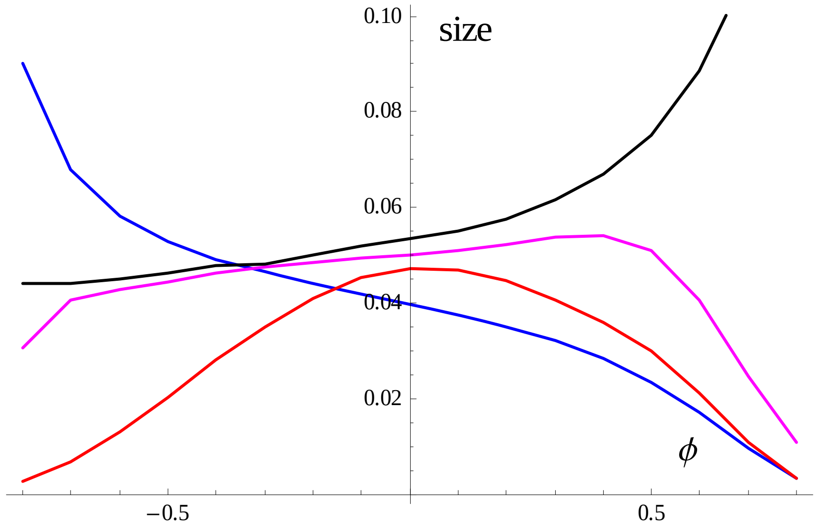

Figure 3: Size for $T = 50$, Score test in blue, approximative LR test in red, approximative LR test with corrected critical values in magenta and GLR test in black. 1 000 000 replications where used except for the GLR test, where the number of replications was 40 000.

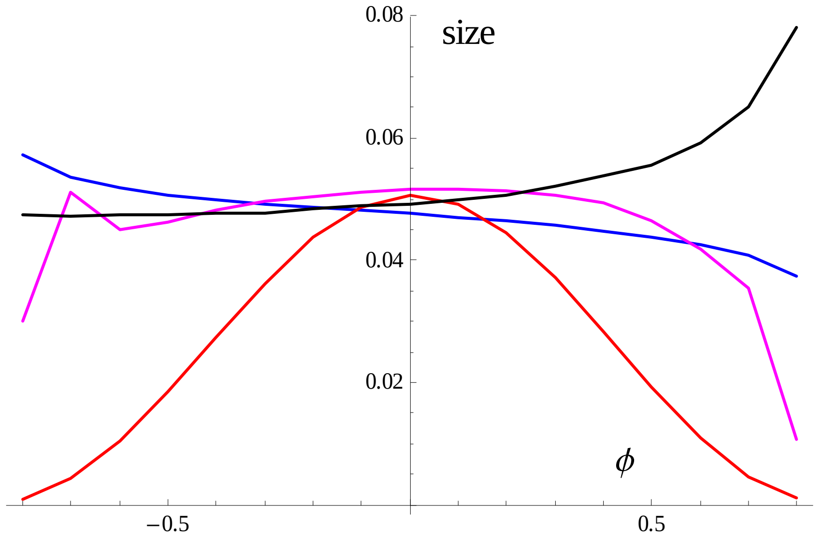

Figure 4: Size for $T = 200$, Score test in blue, approximative LR test in red, approximative LR test with corrected critical values in magenta and GLR test in black. 1 000 000 replications where used except for the GLR test, where the number of replications was 40 000.

A likelihood ratio type test for invertibility in moving average processes. Supplement: Proof of proposition 4

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June 27, 2013

Under H_0, as in the proof of proposition 2, we have

$$\mathbf{\Phi}\left(\mathbf{I}-\mathbf{L}\right)\mathbf{x}=\mathbf{\Phi}\mathbf{y}=\left(-\boldsymbol{\delta}_1,\mathbf{I}-\mathbf{L}\right)\begin{pmatrix}\varepsilon_0\\ \widetilde{\boldsymbol{\varepsilon}}\end{pmatrix},$$

and since $\mathbf{\Phi}$ and $\mathbf{I}-\mathbf{L}$ commute and $\mathbf{I}-\mathbf{L}$ is invertible,

$$\mathbf{\Phi}\mathbf{x}=-\mathbf{1}\varepsilon_0+\widetilde{\boldsymbol{\varepsilon}}.$$

Hence, writing $\mathbf{x}\equiv(x_1,...,x_T)$, we get

$$\phi\left(L\right)x_t=\varepsilon_t-\varepsilon_0,$$

i.e.

$$x_t=C\left(L\right)\varepsilon_t-C\left(1\right)\varepsilon_0,$$

where the inverse operator $C\left(L\right)$ fulfills $C\left(L\right)\phi\left(L\right)=1$.

Now, assume that all roots of $\phi\left(z\right)=0$ are outside the complex unit circle. In Larsson (1998), it is shown that under this assumption, certain functionals of $C\left(L\right)\varepsilon_t$ are asymptotically equivalent to the same functionals under the AR(1) framework. Indeed, with the assumption of zero initial values, a simplified version of lemma 2.2 in Larsson (1998) (which is in turn a slight extension of a univariate version of theorem 2.1. of Johansen, 1995) is given by

Lemma 1 *Let*

$$\begin{aligned}\phi\left(L\right)u_t &= \varepsilon_t,\\ \phi\left(L\right) &= 1-\phi_1L-...-\phi_pL^p,\end{aligned}$$

for $t=1,...,T$, *where all* ε_t *are independent standard normal and the starting values* $u_0=u_{-1}=...=u_{-p+1}=0$. *If all roots of* $\phi\left(z\right)=0$ *are outside the complex unit circle, then*

$$u_t=C_t^{(0)}\left(L\right)\varepsilon_t,$$

where $C_t^{(0)}(L) = \sum_{i=0}^{t-1} c_i^{(0)} L^i$ *with* $c_i^{(0)} = \sum_{i=1}^{\min(p-1,i)} c_{i-j}^{(0)} \phi_j$. *Moreover, for* $k = 1, 2$,

$$C_t^{(k-1)}(L) = C_t^{(k-1)}(1) + (1-L) C_t^{(k)}(L),$$

where $C_t^{(k)}(L) = \sum_{i=0}^{t-1} c_i^{(k)} L^i$ *with* $c_i^{(k)} = -\sum_{j=i+1}^{t-1} c_j^{(k-1)}$. *Furthermore, for some* $\delta > 0$, *the sums*

$$C_\infty^{(k)}(L) = \lim_{t\to\infty} C_t^{(k)}(L) = \sum_{i=1}^{\infty} c_i^{(k)} L^i, \;\; k = 0, 1, 2,$$

are absolutely convergent for $|L| < 1+\delta$, *and* $c_i^{(k)}$ *tend to zero exponentially fast as* $i \to \infty$.

At first, we study the asymptotics of $g_{0T} = f_{0T} - r_{0T}$. As for

$$T^{-1} f_{0T} = T^{-1} \mathbf{x}'\mathbf{x} - \frac{1}{T(T+1)} (\mathbf{1}'\mathbf{x})^2, \tag{1}$$

we have

$$\begin{aligned} & T^{-1}\mathbf{x}'\mathbf{x} \\ = \; & T^{-1}\sum_t x_t^2 = T^{-1}\sum_t \left\{ C_t^{(0)}(L)\varepsilon_t - C_t^{(0)}(1)\varepsilon_0 \right\}^2 \\ = \; & T^{-1}\sum_t \left\{ C_t^{(0)}(L)\varepsilon_t \right\}^2 - 2\varepsilon_0 T^{-1}\sum_t C_t^{(0)}(L)\varepsilon_t C_t^{(0)}(1) \\ & + \varepsilon_0^2 T^{-1}\sum_t \left\{ C_t^{(0)}(1) \right\}^2, \end{aligned} \tag{2}$$

where by lemma 1, the first term is

$$\begin{aligned} & T^{-1}\sum_t \left\{ C_t^{(0)}(L)\varepsilon_t \right\}^2 \\ = \; & T^{-1}\sum_t \sum_{i=0}^{t-1}\sum_{j=0}^{t-1} c_i^{(0)} c_j^{(0)} \varepsilon_{t-i}\varepsilon_{t-j} \\ = \; & \sum_{i=0}^{T-1}\sum_{j=0}^{T-1} c_i^{(0)} c_j^{(0)} T^{-1}\sum_t \varepsilon_{t-i}\varepsilon_{t-j} + R_1, \end{aligned} \tag{3}$$

where

$$R_1 \equiv -2T^{-1}\sum_t \sum_{i=0}^{t-1}\sum_{j=t}^{T-1} c_i^{(0)} c_j^{(0)} \varepsilon_{t-i}\varepsilon_{t-j} + T^{-1}\sum_t \sum_{i=t}^{T-1}\sum_{j=t}^{T-1} c_i^{(0)} c_j^{(0)} \varepsilon_{t-i}\varepsilon_{t-j}$$

tends to zero with T since the $c_i^{(0)}$ decay exponentially fast with i. The main term of (3) asymptotically behaves as its expectation, which is

$$\sum_{i=0}^{T-1}\sum_{j=0}^{T-1} c_i^{(0)} c_j^{(0)} T^{-1}\sum_t E(\varepsilon_{t-i}\varepsilon_{t-j}) = \sum_{i=0}^{T-1} c_i^{(0)2} T^{-1}\sum_t E\left(\varepsilon_{t-i}^2\right) = \sum_{i=0}^{T-1} c_i^{(0)2},$$

and because of the exponential decay of the $c_i^{(0)}$, this sum converges to a finite constant, $c^{(0)*}$ say. Hence, the probability limit of the first term of (2) is $c^{(0)*}$. It is similarly seen that

$$T^{-1}\sum_t C_t^{(0)}(L)\,\varepsilon_t C_t^{(0)}(1) \sim C^{(0)2}T^{-1}\sum_t \varepsilon_t,$$

where $C^{(0)}$ is the limit of $\sum_{i=0}^{T-1} c_i^{(0)}$, showing that the second term of (2) tends to zero with T. As for the last term, we find as above that

$$T^{-1}\sum_t \left\{C_t^{(0)}(1)\right\}^2 \sim \sum_{i=0}^{T-1}\sum_{j=0}^{T-1} c_i^{(0)}c_j^{(0)} \sim C^{(0)2},$$

and so, (2) implies

$$T^{-1}\mathbf{x}'\mathbf{x} \sim c^{(0)*} + C^{(0)2}\varepsilon_0^2. \tag{4}$$

As for the second term of (1), we find that

$$T^{-1/2}\mathbf{1}'\mathbf{x} = T^{-1/2}\sum_t x_t = T^{-1/2}\sum_t C_t^{(0)}(L)\,\varepsilon_t - T^{-1/2}\sum_t C_t^{(0)}(1)\,\varepsilon_0,$$

where as above, the first term satisfies

$$T^{-1/2}\sum_t C_t^{(0)}(L)\,\varepsilon_t \sim C^{(0)}T^{-1/2}\sum_t \varepsilon_t \sim C^{(0)}W(1),$$

and the second term tends to zero. Hence,

$$T^{-1/2}\mathbf{1}'\mathbf{x} \sim C^{(0)}W(1), \tag{5}$$

Moreover, note that

$$\begin{aligned} & T^{-2}\left(\mathbf{1}'\mathbf{x}\right)^2 \\ = \; & T^{-1}\left\{T^{-1/2}\sum_t C_t^{(0)}(L)\,\varepsilon_t\right\}^2 \\ & -2T^{-3/2}\varepsilon_0\left\{T^{-1/2}\sum_t C_t^{(0)}(L)\,\varepsilon_t\right\}\left\{\sum_t C_t^{(0)}(1)\right\} \\ & +\varepsilon_0^2T^{-2}\left\{\sum_t C_t^{(0)}(1)\right\}^2, \end{aligned} \tag{6}$$

where the first two terms are negligible but in fact, the third term gives a contribution because

$$T^{-2}\left\{\sum_t C_t^{(0)}(1)\right\}^2 = T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=0}^{t-1}\sum_{j=0}^{s-1} c_i^{(0)}c_j^{(0)} \sim \sum_{i=0}^{T-1}\sum_{j=0}^{T-1} c_i^{(0)}c_j^{(0)} \sim C^{(0)2}. \tag{7}$$

Hence,

$$T^{-2}\left(\mathbf{1}'\mathbf{x}\right)^2 \sim C^{(0)2}\varepsilon_0^2, \tag{8}$$

and together with (1) and (4),

$$T^{-1}f_{0T} \sim c^{(0)*}. \tag{9}$$

As for r_{0T}, we have from the proof of proposition 3 that

$$T^{-1}r_{0T} = \left(T^{-1}\mathbf{m}_0\right)'\left(T^{-1}\mathbf{M}_0\right)^{-1}\left(T^{-1}\mathbf{m}_0\right), \tag{10}$$

where $T^{-1}\mathbf{m}_0$ has components, for $k = 1, ..., p$,

$$T^{-1}\mathbf{x}'\mathbf{E}_0\mathbf{L}^k\mathbf{x} = T^{-1}\mathbf{x}'\mathbf{L}^k\mathbf{x} - \frac{1}{T(T+1)}\left(\mathbf{1}'\mathbf{x}\right)\left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right), \tag{11}$$

where as in (2),

$$\begin{aligned}
& T^{-1}\mathbf{x}'\mathbf{L}^k\mathbf{x} \\
= \; & T^{-1}\sum_t x_t x_{t-k} \\
= \; & T^{-1}\sum_t \left\{C_t^{(0)}(L)\varepsilon_t - C_t^{(0)}(1)\varepsilon_0\right\}\left\{C_{t-k}^{(0)}(L)\varepsilon_{t-k} - C_{t-k}^{(0)}(1)\varepsilon_0\right\} \\
= \; & T^{-1}\sum_t C_t^{(0)}(L)\varepsilon_t C_{t-k}^{(0)}(L)\varepsilon_{t-k} - \varepsilon_0 T^{-1}\sum_t C_t^{(0)}(L)\varepsilon_t C_{t-k}^{(0)}(1) \\
& -\varepsilon_0 T^{-1}\sum_t C_t^{(0)}(1)C_{t-k}^{(0)}(L)\varepsilon_t + \varepsilon_0^2 T^{-1}\sum_t C_t^{(0)}(1)C_{t-k}^{(0)}(1).
\end{aligned} \tag{12}$$

Here, as above,

$$T^{-1}\sum_t C_t^{(0)}(L)\varepsilon_t C_{t-k}^{(0)}(L)\varepsilon_{t-k} \sim T^{-1}\sum_t\sum_{i=1}^{t-1}\sum_{j=1}^{t-k-1} c_i^{(0)}c_j^{(0)}\varepsilon_{t-i}\varepsilon_{t-k-j},$$

which behaves as its expectation

$$\begin{aligned}
& T^{-1}\sum_t\sum_{i=1}^{t-1}\sum_{j=1}^{t-k-1} c_i^{(0)}c_j^{(0)}E\left(\varepsilon_{t-i}\varepsilon_{t-k-j}\right) \\
= \; & T^{-1}\sum_t\sum_{i=1}^{t-1}\sum_{j=k+1}^{t-1} c_i^{(0)}c_{j+k}^{(0)}E\left(\varepsilon_{t-i}\varepsilon_{t-j}\right) = T^{-1}\sum_t\sum_{i=1}^{t-k-1} c_i^{(0)}c_{i+k}^{(0)} \\
\sim \; & T^{-1}\sum_t\sum_{i=1}^{T-k-1} c_i^{(0)}c_{i+k}^{(0)} = \sum_{i=1}^{T-k-1} c_i^{(0)}c_{i+k}^{(0)},
\end{aligned}$$

which tends to a finite limit, $c^{(k)*}$ say. Moreover, in the usual manner, the second and third terms of (12) may be neglected, while as for the fourth term,

$$T^{-1}\sum_t C_t^{(0)}(1)C_{t-k}^{(0)}(1) = T^{-1}\sum_t\sum_{i=1}^{t-1}\sum_{j=1}^{t-k-1} c_i^{(0)}c_j^{(0)} \sim \sum_{i=1}^{T}\sum_{j=1}^{T} c_i^{(0)}c_j^{(0)} \sim C^{(0)2},$$

and so, (12) yields

$$T^{-1}\mathbf{x}'\mathbf{L}^{k}\mathbf{x} \sim c^{(k)*} + C^{(0)2}\varepsilon_0^2. \tag{13}$$

Further, analogous to (8),

$$T^{-2}\left(\mathbf{1}'\mathbf{x}\right)\left(\mathbf{1}'\mathbf{L}^{k}\mathbf{x}\right) \sim \varepsilon_0^2 T^{-2}\left\{\sum_t C_t^{(0)}(1)\right\}^2 \sim C^{(0)2}\varepsilon_0^2,$$

which together with (11) and (13) implies

$$T^{-1}\mathbf{x}'\mathbf{E}_0\mathbf{L}^{k}\mathbf{x} \sim c^{(k)*}. \tag{14}$$

It is similarly seen that the components of $T^{-1}\mathbf{M}_0$ satisfy, for $k,l = 1,...,p$,

$$T^{-1}\mathbf{x}'\mathbf{L}'^{k}\mathbf{E}_0\mathbf{L}^{l}\mathbf{x} = T^{-1}\mathbf{x}'\mathbf{L}'^{k}\mathbf{L}^{l}\mathbf{x} - \frac{1}{T(T+1)}\left(\mathbf{1}'\mathbf{L}'^{k}\mathbf{x}\right)\left(\mathbf{1}'\mathbf{L}^{l}\mathbf{x}\right) \sim c^{(|k-l|)*}.$$

Conclusively, from (10),

$$T^{-1}r_{0T} \sim \mathbf{c}'\mathbf{C}^{-1}\mathbf{c}, \tag{15}$$

where the kth entry in $\mathbf{c}$ is $c^{(k)*}$ and the (k,l)th entry of $\mathbf{C}$ is $c^{(|k-l|)*}$. Hence,

$$T^{-1}g_{0T} \sim c^{(0)*} - \mathbf{c}'\mathbf{C}^{-1}\mathbf{c}.$$

The limit result for g_{1T} follows immediately, since $g_{1T} = 2r_{0T}$.

Next, considering f_{2T}, we have

$$\begin{aligned} & T^{-2}f_{2T} \\ = & \frac{T^2-2T+4}{3T^2(T+1)}\left(\mathbf{1}'\mathbf{x}\right)^2 - \frac{T+2}{6T^2}\mathbf{x}'\mathbf{x} + T^{-2}\mathbf{x}'\mathbf{S}\mathbf{S}'\mathbf{x} \\ & -2\frac{1}{T^2(T+1)}\left(\mathbf{1}'\mathbf{x}\right)\left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}\right), \end{aligned} \tag{16}$$

where at first, we note that the tth component of $\mathbf{S}'\mathbf{x}$ is

$$\sum_{i=t+1}^{T} x_i = \sum_{i=t+1}^{T} C_i^{(0)}(L)\varepsilon_i - \sum_{i=t+1}^{T} C_i^{(0)}(1)\varepsilon_0,$$

and so,

$$\begin{aligned} & T^{-2}\mathbf{x}'\mathbf{S}\mathbf{S}'\mathbf{x} \\ = & T^{-2}\sum_t\left(\sum_{i=t+1}^{T} x_i\right)^2 \\ = & T^{-2}\sum_t\left\{\sum_{i=t+1}^{T} C_i^{(0)}(L)\varepsilon_i\right\}^2 - 2T^{-2}\varepsilon_0\sum_t\sum_{i=t+1}^{T} C_i^{(0)}(L)\varepsilon_i\sum_{i=t+1}^{T} C_i^{(0)}(1) \\ & +T^{-2}\varepsilon_0^2\sum_t\left\{\sum_{i=t+1}^{T} C_i^{(0)}(1)\right\}^2. \end{aligned} \tag{17}$$

Now, defining $s_t = \sum_{k=1}^{t} \varepsilon_k$, we have via lemma 1[1],

$$
\begin{aligned}
& \sum_{i=t+1}^{T} C_i^{(0)}(L)\,\varepsilon_i \\
= & \sum_{i=t+1}^{T} C_i^{(0)}(1)\,\varepsilon_i + \sum_{i=t+1}^{T} C_i^{(1)}(L)\,(1-L)\,\varepsilon_i \\
= & \sum_{i=t+1}^{T} \sum_{j=0}^{i-1} c_j^{(0)} \varepsilon_i + \sum_{i=t+1}^{T} \sum_{j=0}^{i-1} c_j^{(1)}\,(1-L)\,\varepsilon_{i-j} \\
= & \sum_{j=0}^{T-1} c_j^{(0)} \sum_{i=(j\vee t)+1}^{T} \varepsilon_i + \sum_{j=0}^{T-1} c_j^{(1)}\,(1-L) \sum_{i=(j\vee t)+1}^{T} \varepsilon_{i-j} \\
= & \sum_{j=0}^{T-1} c_j^{(0)}\,(s_T - s_{j\vee t}) + \sum_{j=0}^{T-1} c_j^{(1)} \varepsilon_{T-(j\vee t)}, \qquad (18)
\end{aligned}
$$

implying

$$
\begin{aligned}
& T^{-2} \sum_t \left\{ \sum_{i=t+1}^{T} C_i^{(0)}(L)\,\varepsilon_i \right\}^2 \\
= & T^{-2} \sum_t \left\{ \sum_{j=0}^{T-1} c_j^{(0)}\,(s_T - s_{j\vee t}) + \sum_{j=0}^{T-1} c_j^{(1)} \varepsilon_{T-(j\vee t)} \right\}^2 \\
= & T^{-2} \sum_t \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} c_j^{(0)} c_k^{(0)}\,(s_T - s_{j\vee t})\,(s_T - s_{k\vee t}) \\
& + 2T^{-2} \sum_t \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} c_j^{(0)} c_k^{(1)}\,(s_T - s_j)\,\varepsilon_{T-(k\vee t)} \\
& + T^{-2} \sum_t \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} c_j^{(1)} c_k^{(1)} \varepsilon_{T-(j\vee t)} \varepsilon_{T-(k\vee t)}. \qquad (19)
\end{aligned}
$$

[1] In the following, $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$.

As for the first term of (19), we find in the usual manner that

$$
\begin{aligned}
& T^{-2} \sum_t \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} c_j^{(0)} c_k^{(0)} \left(s_T - s_{j \vee t}\right) \left(s_T - s_{k \vee t}\right) \\
= & \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} c_j^{(0)} c_k^{(0)} T^{-2} \sum_t \left(s_T - s_{j \vee t}\right) \left(s_T - s_{k \vee t}\right) \\
= & \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} c_j^{(0)} c_k^{(0)} T^{-2} \sum_{t=j \vee k}^{T} \left(s_T - s_t\right)^2 \\
& +2 \sum_{j=0}^{T-1} c_j^{(0)} T^{-2} \sum_t \left(s_T - s_t\right) \sum_{k=t}^{T-1} c_k^{(0)} \left(s_T - s_k\right) \\
& +T^{-2} \sum_t \sum_{j=t}^{T-1} \sum_{k=t}^{T-1} c_j^{(0)} c_k^{(0)} \left(s_T - s_j\right) \left(s_T - s_k\right) \\
\sim & \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} c_j^{(0)} c_k^{(0)} T^{-2} \sum_t \left(s_T - s_t\right)^2 \\
\sim & C^{(0)2} \int_0^1 W^* (t)^2 \, dt, \qquad (20)
\end{aligned}
$$

and it is similarly seen that the second and third terms of (19) are asymptotically dominated by this term. Analogously, considering the second term of (17), we have

$$
\begin{aligned}
& T^{-2} \sum_t \sum_{i=t+1}^{T} C_i^{(0)} (L) \varepsilon_i \sum_{i=t+1}^{T} C_i^{(0)} (1) \\
= & T^{-2} \sum_t \left\{ \sum_{j=0}^{T-1} c_j^{(0)} \left(s_T - s_{j \vee t}\right) + \sum_{j=0}^{T-1} c_j^{(1)} \varepsilon_{T-(j \vee t)} \right\} \sum_{i=t+1}^{T} \sum_{k=1}^{i-1} c_k^{(0)} \\
\sim & T^{-1/2} \sum_{j=0}^{T-1} \sum_{i=1}^{T} c_j^{(0)} T^{-3/2} \sum_t \left(s_T - s_t\right) \sum_{k=1}^{i-1} c_k^{(0)},
\end{aligned}
$$

which tends to zero with T. However, the third term of (17) is not asymptotically negligible, but rather

$$\begin{aligned}
& T^{-2}\sum_{t}\left\{\sum_{i=t+1}^{T}C_i^{(0)}(1)\right\}^2 \\
= & T^{-2}\sum_{t=1}^{T}\sum_{i=t+1}^{T}\sum_{j=t+1}^{T}\sum_{k=0}^{i-1}\sum_{l=0}^{j-1}c_k^{(0)}c_l^{(0)} \\
= & T^{-2}\sum_{i=2}^{T}\sum_{j=2}^{T}(i\wedge j-1)\sum_{k=0}^{i-1}\sum_{l=0}^{j-1}c_k^{(0)}c_l^{(0)} \\
= & T^{-2}\sum_{i=2}^{T}\sum_{j=2}^{T}(i\wedge j-1)\sum_{k=0}^{T-1}\sum_{l=0}^{T-1}c_k^{(0)}c_l^{(0)} \\
& -2T^{-2}\sum_{i=2}^{T}\sum_{j=2}^{T}(i\wedge j-1)\sum_{k=0}^{T-1}\sum_{l=i}^{T-1}c_k^{(0)}c_l^{(0)} \\
& +T^{-2}\sum_{i=2}^{T}\sum_{j=2}^{T}(i\wedge j-1)\sum_{k=i}^{T-1}\sum_{l=i}^{T-1}c_k^{(0)}c_l^{(0)},
\end{aligned}$$

where the first term is

$$\begin{aligned}
& T^{-2}\sum_{i=2}^{T}\sum_{j=2}^{T}(i\wedge j-1)\sum_{k=0}^{T-1}\sum_{l=0}^{T-1}c_k^{(0)}c_l^{(0)} \\
= & C^{(0)2}T^{-2}\sum_{i=2}^{T}\sum_{j=2}^{T}(i\wedge j-1)+o(1) \\
= & C^{(0)2}T^{-2}\left\{2\sum_{i=2}^{T}\sum_{j=2}^{i-1}(j-1)+\sum_{i=2}^{T}(i-1)\right\}+o(1) \\
= & C^{(0)2}\left(\frac{1}{3}T-\frac{1}{2}\right)+o(1),
\end{aligned}$$

the second term is essentially

$$
\begin{aligned}
& T^{-2}\sum_{i=2}^{T}\sum_{j=2}^{T}(i\wedge j-1)\sum_{k=0}^{T-1}\sum_{l=i}^{T-1}c_k^{(0)}c_l^{(0)} \\
= \; & C^{(0)}T^{-2}\sum_{i=2}^{T}\sum_{j=2}^{T}(i\wedge j-1)\sum_{l=i}^{T-1}c_l^{(0)}+o\left(1\right) \\
= \; & C^{(0)}T^{-2}\sum_{l=2}^{T-1}c_l^{(0)}\sum_{i=2}^{l}\sum_{j=2}^{T}(i\wedge j-1)+o\left(1\right) \\
= \; & C^{(0)}T^{-2}\sum_{l=2}^{T-1}c_l^{(0)}\sum_{i=2}^{l}\left\{\sum_{j=2}^{i}(j-1)+\sum_{j=i+1}^{T}(i-1)\right\}+o\left(1\right) \\
= \; & o\left(1\right),
\end{aligned}
$$

and similarly, the third term is $o\left(1\right)$. Hence, it follows that

$$
T^{-2}\sum_{t}\left\{\sum_{i=t+1}^{T}C_i^{(0)}\left(1\right)\right\}^2=C^{(0)2}\left(\frac{1}{3}T-\frac{1}{2}\right)+o\left(1\right),
$$

implying via (17) and (20) that

$$
\begin{aligned}
& T^{-2}\mathbf{x}'\mathbf{S}\mathbf{S}'\mathbf{x} \\
= \; & C^{(0)2}\int_0^1 W^*\left(t\right)^2dt+\varepsilon_0^2C^{(0)2}\left(\frac{1}{3}T-\frac{1}{2}\right)+o\left(1\right).
\end{aligned} \tag{21}
$$

Similarly, via (18),

$$
\begin{aligned}
& T^{-5/2}\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x} \\
= \; & T^{-5/2}\sum_{t}\left(T-t\right)\sum_{i=t+1}^{T}x_i \\
= \; & T^{-5/2}\sum_{t}\left(T-t\right)\sum_{i=t+1}^{T}C_i^{(0)}\left(L\right)\varepsilon_i-T^{-5/2}\sum_{t}\left(T-t\right)\sum_{i=t+1}^{T}C_i^{(0)}\left(1\right)\varepsilon_0 \\
\sim \; & T^{-5/2}\sum_{t}\left(T-t\right)\sum_{j=0}^{T-1}c_j^{(0)}\left(s_T-s_{j\vee t}\right) \\
\sim \; & C^{(0)}\int_0^1\left(1-t\right)W^*\left(t\right)dt,
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
& T^{-3}\left(\mathbf{1}'\mathbf{x}\right)\left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}\right) \\
= \ & T^{-3}\sum_t x_t \sum_s (T-s)\sum_{i=s+1}^{T} x_i \\
= \ & T^{-3}\left\{\sum_t C_t^{(0)}(L)\,\varepsilon_t - \sum_t C_t^{(0)}(1)\,\varepsilon_0\right\} \\
& \left\{\sum_t (T-t)\sum_{i=t+1}^{T} C_i^{(0)}(L)\,\varepsilon_i - \sum_t (T-t)\sum_{i=t+1}^{T} C_i^{(0)}(1)\,\varepsilon_0\right\} \\
= \ & T^{-3}\sum_t C_t^{(0)}(L)\,\varepsilon_t \sum_t (T-t)\sum_{i=t+1}^{T} C_i^{(0)}(L)\,\varepsilon_i \\
& -\varepsilon_0 T^{-3}\sum_t C_t^{(0)}(L)\,\varepsilon_t \sum_t (T-t)\sum_{i=t+1}^{T} C_i^{(0)}(1) \\
& -\varepsilon_0 T^{-3}\sum_t C_t^{(0)}(1)\sum_t (T-t)\sum_{i=t+1}^{T} C_i^{(0)}(L)\,\varepsilon_i \\
& +\varepsilon_0^2 T^{-3}\sum_t C_t^{(0)}(1)\sum_t (T-t)\sum_{i=t+1}^{T} C_i^{(0)}(1), \qquad (23)
\end{aligned}
$$

where the first term converges to $C^{(0)2}W(1)\int_0^1(1-t)\,W^*(t)\,dt$, and the second and third terms vanish asymptotically. As for the fourth term, we have

$$
\begin{aligned}
& \sum_t C_t^{(0)}(1) \\
= \ & \sum_{t=1}^{T}\sum_{i=0}^{t-1} c_i^{(0)} = \sum_{i=0}^{T-1}(T-i)\,c_i^{(0)} = TC^{(0)} - C^{(1)} + o(1), \qquad (24)
\end{aligned}
$$

where $C^{(1)} \equiv \sum_{i=0}^{\infty} i c_i^{(0)}$, and

$$
\begin{aligned}
& T^{-3}\sum_t (T-t)\sum_{i=t+1}^{T} C_i^{(0)}(1) \\
= \ & T^{-3}\sum_t (T-t)\sum_{i=t+1}^{T}\sum_{j=0}^{i-1} c_j^{(0)} \\
= \ & T^{-3}\sum_t (T-t)\sum_{i=t+1}^{T}\sum_{j=0}^{T-1} c_j^{(0)} - T^{-3}\sum_t (T-t)\sum_{i=t+1}^{T}\sum_{j=i}^{T} c_j^{(0)},
\end{aligned}
$$

where

$$
\begin{aligned}
& T^{-3}\sum_{t}\left(T-t\right)\sum_{i=t+1}^{T}\sum_{j=0}^{T-1}c_{j}^{(0)} \\
= \ & C^{(0)}T^{-3}\sum_{t}\left(T-t\right)^{2}+o\left(T^{-1}\right)=C^{(0)}\left(\frac{1}{3}-\frac{1}{2}T^{-1}\right)+o\left(T^{-1}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& T^{-3}\sum_{t}\left(T-t\right)\sum_{i=t+1}^{T}\sum_{j=i}^{T}c_{j}^{(0)} \\
= \ & T^{-3}\sum_{t=1}^{T}\left(T-t\right)\sum_{j=t+1}^{T}\left(j-t\right)c_{j}^{(0)} \\
= \ & T^{-3}\sum_{j=2}^{T}jc_{j}^{(0)}\sum_{t=1}^{j-1}\left(T-t\right)-T^{-3}\sum_{j=2}^{T}c_{j}^{(0)}\sum_{t=1}^{j-1}\left(T-t\right)t=o\left(T^{-1}\right),
\end{aligned}
$$

and so,

$$
T^{-3}\sum_{t}\left(T-t\right)\sum_{i=t+1}^{T}C_{i}^{(0)}\left(1\right)=C^{(0)}\left(\frac{1}{3}-\frac{1}{2}T^{-1}\right)+o\left(T^{-1}\right). \tag{25}
$$

Hence, via (23),

$$
\begin{aligned}
& T^{-3}\left(\mathbf{1}'\mathbf{x}\right)\left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}\right) \\
= \ & C^{(0)2}W\left(1\right)\int_{0}^{1}\left(1-t\right)W^{*}\left(t\right)dt \\
& +\varepsilon_{0}^{2}TC^{(0)}\left\{C^{(0)}-T^{-1}C^{(1)}\right\}\left(\frac{1}{3}-\frac{1}{2}T^{-1}\right)+o_{p}\left(1\right) \\
= \ & C^{(0)2}W\left(1\right)\int_{0}^{1}\left(1-t\right)W^{*}\left(t\right)dt \\
& +\varepsilon_{0}^{2}\left\{\frac{1}{3}TC^{(0)2}-\frac{1}{2}C^{(0)2}-\frac{1}{3}C^{(0)}C^{(1)}\right\}+o_{p}\left(1\right).
\end{aligned} \tag{26}
$$

Also, we have to refine (8). A modification of (6) is

$$
\begin{aligned}
& T^{-1}\left(\mathbf{1}'\mathbf{x}\right)^{2} \\
= \ & \left\{T^{-1/2}\sum_{t}C_{t}^{(0)}\left(L\right)\varepsilon_{t}\right\}^{2} \\
& -2T^{-1/2}\varepsilon_{0}\left\{T^{-1/2}\sum_{t}C_{t}^{(0)}\left(L\right)\varepsilon_{t}\right\}\left\{\sum_{t}C_{t}^{(0)}\left(1\right)\right\} \\
& +\varepsilon_{0}^{2}T^{-1}\left\{\sum_{t}C_{t}^{(0)}\left(1\right)\right\}^{2},
\end{aligned}
$$

where the first term converges to $C^{(0)2}W(1)^2$, the second term is negligible and as for the third term, we refine (7) into

$$\begin{aligned}
& T^{-1}\left\{\sum_t C_t^{(0)}(1)\right\}^2 \\
= \; & T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=0}^{t-1}\sum_{j=0}^{s-1} c_i^{(0)}c_j^{(0)} = T^{-1}\sum_{i=0}^{T-1}\sum_{j=0}^{T-1} c_i^{(0)}c_j^{(0)}(T-i)(T-j) \\
= \; & T\left(\sum_{i=0}^{T-1} c_i^{(0)}\right)^2 - 2\sum_{i=0}^{T-1} ic_i^{(0)}\sum_{j=0}^{T-1} c_j^{(0)} + o(1) \\
= \; & TC^{(0)2} - 2C^{(0)}C^{(1)} + o(1),
\end{aligned}$$

implying

$$T^{-1}(\mathbf{1}'\mathbf{x})^2 = C^{(0)2}W(1)^2 + \varepsilon_0^2\left(TC^{(0)2} - 2C^{(0)}C^{(1)}\right) + o(1). \tag{27}$$

Hence, inserting this together with (4), (21) and (26) into (16), we obtain

$$\begin{aligned}
& T^{-2}f_{2T} \\
= \; & \frac{T^2-2T+4}{3T(T+1)}\left\{C^{(0)2}W(1)^2 + \varepsilon_0^2\left(TC^{(0)2} - 2C^{(0)}C^{(1)}\right)\right\} \\
& -\frac{1}{6}\left(c^{(0)*} + C^{(0)2}\varepsilon_0^2\right) \\
& +C^{(0)2}\int_0^1 W^*(t)^2\,dt + \varepsilon_0^2 C^{(0)2}\left(\frac{1}{3}T - \frac{1}{2}\right) \\
& -2\frac{T}{T+1}\left[C^{(0)2}W(1)\int_0^1 (1-t)W^*(t)\,dt\right. \\
& \left.+\varepsilon_0^2\left\{\frac{1}{3}TC^{(0)2} - \frac{1}{2}C^{(0)2} - \frac{1}{3}C^{(0)}C^{(1)}\right\}\right] \\
& +o_p(1) \\
= \; & C^{(0)2}\left(f_2 + \frac{1}{6}\right) - \frac{1}{6}c^{(0)*} + o_p(1).
\end{aligned} \tag{28}$$

To tackle $g_{2T} = f_{2T} + r_{2T}$, we need

$$\begin{aligned}
& q_{2T} \\
= \; & -\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_0 + 2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_2 + \mathbf{m}_0'\mathbf{P}_2\mathbf{m}_0 \\
= \; & -\mathbf{m}_0'\mathbf{M}_0^{-1}\mathbf{m}_0 + 2\mathbf{m}_0'\mathbf{M}_0^{-1}\mathbf{m}_2 + \mathbf{m}_0'\left(\mathbf{I} - \mathbf{M}_0^{-1}\mathbf{M}_2\right)\mathbf{M}_0^{-1}\mathbf{m}_0 \\
= \; & 2\mathbf{m}_0'\mathbf{M}_0^{-1}\mathbf{m}_2 - \mathbf{m}_0'\mathbf{M}_0^{-1}\mathbf{M}_2\mathbf{M}_0^{-1}\mathbf{m}_0,
\end{aligned} \tag{29}$$

where $\mathbf{m}_0$ and $\mathbf{M}_0$ have been treated above. Moreover, in the usual manner we find that $\mathbf{m}_2$ has components

$$\begin{aligned}
& \mathbf{x}'\mathbf{E}_2\mathbf{L}^k\mathbf{x} \\
= \; & \mathbf{x}'\mathbf{L}^k\mathbf{x} - \frac{2T^2-5T+12}{6(T+1)}(\mathbf{1}'\mathbf{x})\left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right) - \mathbf{x}'\mathbf{S}\mathbf{S}'\mathbf{L}^k\mathbf{x} \\
& + \frac{1}{T+1}\left\{(\mathbf{1}'\mathbf{x})\left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{L}^k\mathbf{x}\right) + \left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}\right)\left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right)\right\}, \qquad (30)
\end{aligned}$$

where as in (27),

$$T^{-1}(\mathbf{1}'\mathbf{x})\left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right) = C^{(0)2}W(1)^2 + \varepsilon_0^2\left(TC^{(0)2} - 2C^{(0)}C^{(1)}\right) + o(1). \qquad (31)$$

Similarly, as in (21),

$$T^{-2}\mathbf{x}'\mathbf{S}\mathbf{S}'\mathbf{L}^k\mathbf{x} = C^{(0)2}\int_0^1 W^*(t)^2\,dt + \varepsilon_0^2 C^{(0)2}\left(\frac{1}{3}T - \frac{1}{2}\right) + o(1), \qquad (32)$$

and as in (26),

$$\begin{aligned}
& T^{-3}(\mathbf{1}'\mathbf{x})\left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{L}^k\mathbf{x}\right) \\
= \; & C^{(0)2}W(1)\int_0^1 (1-t)\,W^*(t)\,dt \\
& + \varepsilon_0^2\left\{\frac{1}{3}TC^{(0)2} - \frac{1}{2}C^{(0)2} - \frac{1}{3}C^{(0)}C^{(1)}\right\} + o_p(1) \\
= \; & T^{-3}\left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}\right)\left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right). \qquad (33)
\end{aligned}$$

Hence, inserting into (30) together with (13), we find

$$\begin{aligned}
& T^{-2}\mathbf{x}'\mathbf{E}_2\mathbf{L}^k\mathbf{x} \\
= \; & -\frac{2T^2-5T+12}{6T(T+1)}\left\{C^{(0)2}W(1)^2 + \varepsilon_0^2\left(TC^{(0)2} - 2C^{(0)}C^{(1)}\right)\right\} \\
& - C^{(0)2}\int_0^1 W^*(t)^2\,dt - \varepsilon_0^2 C^{(0)2}\left\{\frac{1}{3}T - \frac{1}{2}\right\} \\
& + \frac{2T}{T+1}\left[C^{(0)2}W(1)\int_0^1 (1-t)\,W^*(t)\,dt\right. \\
& \left. + \varepsilon_0^2\left\{\frac{1}{3}TC^{(0)2} - \frac{1}{2}C^{(0)2} - \frac{1}{3}C^{(0)}C^{(1)}\right\}\right] + o_p(1) \\
= \; & -C^{(0)2}\left(f_2 + \frac{1}{6}\right) + o_p(1),
\end{aligned}$$

and it is easily seen that the components of $\mathbf{M}_2$ have the same behavior. Hence, it follows from (29) that

$$T^{-2}q_{2T} \sim -\left\{2\left(\mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right) - \left(\mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)^2\right\}C^{(0)2}\left(f_2 + \frac{1}{6}\right),$$

and since $q_{0T} = r_{0T}$ satisfies (15), we find

$$T^{-2} r_{2T} \sim -\frac{1}{6} \mathbf{c}' \mathbf{C}^{-1} \mathbf{c} - \left\{ 2 \left(\mathbf{1}' \mathbf{C}^{-1} \mathbf{c} \right) - \left(\mathbf{1}' \mathbf{C}^{-1} \mathbf{c} \right)^2 \right\} C^{(0)2} \left(f_2 + \frac{1}{6} \right).$$

Thus, together with (28),

$$T^{-2} g_{2T} \sim \left(1 - \mathbf{1}' \mathbf{C}^{-1} \mathbf{c} \right)^2 C^{(0)2} \left(f_2 + \frac{1}{6} \right) - \frac{1}{6} \left(c^{(0)*} + \mathbf{c}' \mathbf{C}^{-1} \mathbf{c} \right).$$

The limit result for $g_{3T} = f_{2T} + r_{3T}$ follows because as we have seen, f_{2T} is $O_p\left(T^2\right)$ and since it may be seen along similar lines as above that r_{3T} is $O_p\left(T^2\right)$ as well.

To tackle f_{4T}, we have

$$\begin{aligned} & T^{-4} f_{4T} \\ \equiv \; & \frac{8T^4 + 14T^3 - 309T^2 + 529T - 602}{360T^4 (T+1)} \left(\mathbf{1}' \mathbf{x} \right)^2 - \frac{T+8}{6T^4} \mathbf{x}' \mathbf{S} \mathbf{S}' \mathbf{x} \\ & - \frac{(T+2)\left(2T^2 - T - 61\right)}{360T^4} \mathbf{x}' \mathbf{x} + \frac{2\left(T^2 - 2T + 7\right)}{3T^4 (T+1)} \left(\mathbf{1}' \mathbf{x} \right) \left(\mathbf{1}' \mathbf{S} \mathbf{S}' \mathbf{x} \right) \\ & + T^{-4} \mathbf{x}' \left(\mathbf{S} \mathbf{S}' \right)^2 \mathbf{x} - \frac{1}{T^4 (T+1)} \left(\mathbf{1}' \mathbf{S} \mathbf{S}' \mathbf{x} \right)^2 \\ & - \frac{2}{T^4 (T+1)} \left(\mathbf{1}' \mathbf{x} \right) \left\{ \mathbf{1}' \left(\mathbf{S} \mathbf{S}' \right)^2 \mathbf{x} \right\}. \end{aligned} \tag{34}$$

The first new term to study is

$$\begin{aligned} & T^{-4} \mathbf{x}' \left(\mathbf{S} \mathbf{S}' \right)^2 \mathbf{x} \\ = \; & T^{-4} \sum_{t=2}^{T} \left(\sum_{i=2}^{t} \sum_{j=i}^{T} x_j \right)^2 \\ = \; & T^{-4} \sum_{t=2}^{T} \left\{ \sum_{i=2}^{t} \sum_{j=i}^{T} C_j^{(0)} (L) \varepsilon_i \right\}^2 \\ & -2T^{-4} \varepsilon_0 \sum_{t=2}^{T} \left\{ \sum_{i=2}^{t} \sum_{j=i}^{T} C_j^{(0)} (L) \varepsilon_i \right\} \left\{ \sum_{i=2}^{t} \sum_{j=i}^{T} C_j^{(0)} (1) \right\} \\ & + T^{-4} \varepsilon_0^2 \sum_{t=2}^{T} \left\{ \sum_{i=2}^{t} \sum_{j=i}^{T} C_j^{(0)} (1) \right\}^2, \end{aligned} \tag{35}$$

where as in (18),

$$
\begin{aligned}
& T^{-4} \sum_{t=2}^{T} \left\{ \sum_{i=2}^{t} \sum_{j=i}^{T} C_j^{(0)} (L) \varepsilon_i \right\}^2 \\
\sim \quad & T^{-4} \sum_{t=2}^{T} \left(\sum_{i=2}^{t} \sum_{j=i}^{T} \sum_{k=0}^{j-1} c_k^{(0)} \varepsilon_i \right)^2 \sim T^{-4} \left(\sum_{k=0}^{T-1} c_k^{(0)} \right)^2 \sum_{t=2}^{T} \left\{ \sum_{i=2}^{t} (s_T - s_{i-1}) \right\}^2 \\
\sim \quad & C^{(0)2} \int_{t=0}^{1} \left\{ \int_{s=0}^{t} W^* (s) \, ds \right\}^2 dt.
\end{aligned}
$$

The second term of (35) vanishes asymptotically, while as for the third term, we have

$$
\begin{aligned}
& T^{-4} \sum_{t=2}^{T} \left\{ \sum_{i=2}^{t} \sum_{j=i}^{T} C_j^{(0)} (1) \right\}^2 \\
= \quad & T^{-4} \sum_{t=2}^{T} \sum_{i_1=2}^{t} \sum_{i_2=2}^{t} \sum_{j_1=i_1}^{T} \sum_{j_2=i_2}^{T} \sum_{k_1=0}^{j_1-1} \sum_{k_2=0}^{j_2-1} c_{k_1}^{(0)} c_{k_2}^{(0)} \\
= \quad & T^{-4} \sum_{i_1=2}^{T} \sum_{i_2=2}^{T} (T + 1 - i_1 \vee i_2) \sum_{j_1=i_1}^{T} \sum_{j_2=i_2}^{T} \sum_{k_1=0}^{j_1-1} \sum_{k_2=0}^{j_2-1} c_{k_1}^{(0)} c_{k_2}^{(0)} \\
= \quad & T^{-4} \sum_{i_1=2}^{T} \sum_{i_2=2}^{T} (T + 1 - i_1 \vee i_2) \sum_{j_1=i_1}^{T} \sum_{j_2=i_2}^{T} \sum_{k_1=0}^{T} \sum_{k_2=0}^{T} c_{k_1}^{(0)} c_{k_2}^{(0)} \\
& -2T^{-4} \sum_{i_1=2}^{T} \sum_{i_2=2}^{T} (T + 1 - i_1 \vee i_2) \sum_{j_1=i_1}^{T} \sum_{j_2=i_2}^{T} \sum_{k_1=0}^{T} \sum_{k_2=j_2}^{T} c_{k_1}^{(0)} c_{k_2}^{(0)} \\
& +T^{-4} \sum_{i_1=2}^{T} \sum_{i_2=2}^{T} (T + 1 - i_1 \vee i_2) \sum_{j_1=i_1}^{T} \sum_{j_2=i_2}^{T} \sum_{k_1=j_1}^{T} \sum_{k_2=j_2}^{T} c_{k_1}^{(0)} c_{k_2}^{(0)}, \qquad (36)
\end{aligned}
$$

where in the usual manner, the first term is

$$
\begin{aligned}
& T^{-4}\sum_{i_1=2}^{T}\sum_{i_2=2}^{T}\left(T+1-i_1\vee i_2\right)\sum_{j_1=i_1}^{T}\sum_{j_2=i_2}^{T}\sum_{k_1=0}^{T}\sum_{k_2=0}^{T}c_{k_1}^{(0)}c_{k_2}^{(0)}\\
= \; & C^{(0)2}T^{-4}\sum_{i_1=2}^{T}\sum_{i_2=2}^{T}\left(T+1-i_1\vee i_2\right)\left(T+1-i_1\right)\left(T+1-i_2\right)+o\left(1\right)\\
= \; & 2C^{(0)2}T^{-4}\sum_{i_1=2}^{T}\sum_{i_2=2}^{i_1-1}\left(T+1-i_1\right)^2\left(T+1-i_2\right)\\
& +C^{(0)2}T^{-4}\sum_{i_1=2}^{T}\left(T+1-i_1\right)^3+o\left(1\right)\\
= \; & C^{(0)2}\left(\frac{2}{15}T-\frac{1}{3}\right)+o\left(1\right),
\end{aligned}
$$

and moreover,

$$
\begin{aligned}
& T^{-4}\sum_{i_1=2}^{T}\sum_{i_2=2}^{T}\left(T+1-i_1\vee i_2\right)\sum_{j_1=i_1}^{T}\sum_{j_2=i_2}^{T}\sum_{k_1=0}^{T}\sum_{k_2=j_2}^{T}c_{k_1}^{(0)}c_{k_2}^{(0)}\\
= \; & C^{(0)}T^{-4}\sum_{i_1=2}^{T}\sum_{i_2=2}^{T}\left(T+1-i_1\vee i_2\right)\left(T+1-i_1\right)\sum_{j_2=i_2}^{T}\sum_{k_2=j_2}^{T}c_{k_2}^{(0)}+o\left(1\right)\\
= \; & C^{(0)}T^{-4}\sum_{i_1=2}^{T}\sum_{i_2=2}^{T}\left(T+1-i_1\vee i_2\right)\left(T+1-i_1\right)\sum_{k_2=i_2}^{T}\left(k_2+1-i_2\right)c_{k_2}^{(0)}\\
& +o\left(1\right)\\
= \; & o\left(1\right).
\end{aligned}
$$

It is similarly seen that the third term of (36) is $o\left(1\right)$, and consequently, (36) implies

$$
T^{-4}\sum_{t=2}^{T}\left\{\sum_{i=2}^{t}\sum_{j=i}^{T}C_j^{(0)}\left(1\right)\right\}^2=C^{(0)2}\left(\frac{2}{15}T-\frac{1}{3}\right)+o\left(1\right).
$$

Hence, (35) yields

$$
\begin{aligned}
& T^{-4}\mathbf{x}'\left(\mathbf{SS}'\right)^2\mathbf{x}\\
= \; & C^{(0)2}\int_{t=0}^{1}\left\{\int_{s=0}^{t}W^*\left(s\right)ds\right\}^2dt+C^{(0)2}\varepsilon_0^2\left(\frac{2}{15}T-\frac{1}{3}\right)+o\left(1\right). \quad (37)
\end{aligned}
$$

Moreover, as in (22),

$$\begin{aligned}
& T^{-5}\left(\mathbf{1}'\mathbf{SS}'\mathbf{x}\right)^2 \\
= & \left\{T^{-5/2}\sum_t (T-t)\sum_{i=t+1}^T x_i\right\}^2 \\
= & T^{-5}\left\{\sum_t (T-t)\sum_{i=t+1}^T C_i^{(0)}(L)\,\varepsilon_i\right\}^2 \\
& +2T^{-5}\varepsilon_0\left\{\sum_t (T-t)\sum_{i=t+1}^T C_i^{(0)}(L)\,\varepsilon_i\right\}\left\{\sum_t (T-t)\sum_{i=t+1}^T C_i^{(0)}(1)\right\} \\
& +T^{-5}\varepsilon_0^2\left\{\sum_t (T-t)\sum_{i=t+1}^T C_i^{(0)}(1)\right\}^2, \qquad (38)
\end{aligned}$$

where the first term satisfies

$$T^{-5}\left\{\sum_t (T-t)\sum_{i=t+1}^T C_i^{(0)}(L)\,\varepsilon_i\right\}^2 \sim C^{(0)2}\left\{\int_0^1 (1-t)\,W^*(t)\,dt\right\}^2,$$

the second term vanishes and (25) implies

$$T^{-5}\left\{\sum_t (T-t)\sum_{i=t+1}^T C_i^{(0)}(1)\right\}^2 = C^{(0)2}\left(\frac{1}{9}T-\frac{1}{3}\right)+o(1),$$

and so, via (38),

$$\begin{aligned}
& T^{-5}\left(\mathbf{1}'\mathbf{SS}'\mathbf{x}\right)^2 \\
= & C^{(0)2}\left\{\int_0^1 (1-t)\,W^*(t)\,dt\right\}^2 + C^{(0)2}\varepsilon_0^2\left(\frac{1}{9}T-\frac{1}{3}\right)+o_p(1). \qquad (39)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
& \mathbf{1}'\left(\mathbf{SS}'\right)^2\mathbf{x} \\
= & \frac{1}{2}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T} x_j \\
= & \frac{1}{2}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T}\left\{C_j^{(0)}(L)\,\varepsilon_i + C_j^{(0)}(1)\,\varepsilon_0\right\},
\end{aligned}$$

implying

$$
\begin{aligned}
& T^{-5}\left(\mathbf{1}'\mathbf{x}\right)\left\{\mathbf{1}'\left(\mathbf{SS}'\right)^{2}\mathbf{x}\right\} \\
= \; & \frac{1}{2}T^{-5}\left\{\sum_{t} C_t^{(0)}(L)\,\varepsilon_t\right\}\left\{\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T} C_j^{(0)}(L)\,\varepsilon_j\right\} \\
& +\frac{1}{2}T^{-5}\varepsilon_0\left\{\sum_{t} C_t^{(0)}(L)\,\varepsilon_t\right\}\left\{\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T} C_j^{(0)}(1)\right\} \\
& +\frac{1}{2}T^{-5}\varepsilon_0\left\{\sum_{t} C_t^{(0)}(1)\right\}\left\{\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T} C_j^{(0)}(L)\,\varepsilon_j\right\} \\
& +\frac{1}{2}T^{-5}\varepsilon_0^2\left\{\sum_{t} C_t^{(0)}(1)\right\}\left\{\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T} C_j^{(0)}(1)\right\}, \quad (40)
\end{aligned}
$$

where to handle the first term, we have

$$
\begin{aligned}
& T^{-9/2}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T} C_j^{(0)}(L)\,\varepsilon_j \\
\sim \; & C^{(0)}T^{-9/2}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\left(s_T-s_{i-1}\right) \\
\sim \; & C^{(0)}\int_{t=0}^{1}(2-t)\,t\int_{s=0}^{t}W^*(s)\,dsdt=\frac{1}{3}C^{(0)}\int_0^1\left(2-3t^2+t^3\right)W^*(t)\,dt,
\end{aligned}
$$

which yields

$$
\begin{aligned}
& \frac{1}{2}T^{-5}\left\{\sum_{t} C_t^{(0)}(L)\,\varepsilon_t\right\}\left\{\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T} C_j^{(0)}(L)\,\varepsilon_j\right\} \\
\sim \; & \frac{1}{6}C^{(0)2}W(1)\int_0^1\left(2-3t^2+t^3\right)W^*(t)\,dt. \quad (41)
\end{aligned}
$$

The second and third terms of (40) tend to zero with T, while

$$\begin{aligned}
& T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T}C_j^{(0)}(1) \\
= \;& T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T}\sum_{k=0}^{j-1}c_k^{(0)} \\
= \;& C^{(0)}T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}(T+1-i) \\
& -T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T}\sum_{k=j}^{T}c_k^{(0)}+o(1)\,,
\end{aligned}$$

where the first term is $C^{(0)}\left(\frac{4}{15}T-\frac{2}{3}\right)+o(1)$ and the second term is

$$\begin{aligned}
& T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T}\sum_{k=j}^{T}c_k^{(0)} \\
= \;& T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{k=i}^{T}(k+1-i)\,c_k^{(0)} \\
= \;& T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t \\
& \left\{\sum_{k=2}^{t+1}c_k^{(0)}\sum_{i=2}^{k}(k+1-i)+\sum_{k=t+2}^{T}c_k^{(0)}\sum_{i=2}^{t+1}(k+1-i)\right\} \\
= \;& \frac{1}{2}T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{k=2}^{t+1}c_k^{(0)}k\,(k-1) \\
& +\frac{1}{2}T^{-4}\sum_{t=1}^{T-1}(2T-t-1)\,t^2\sum_{k=t+2}^{T}c_k^{(0)}\,(2k-t-1) \\
= \;& \frac{1}{2}T^{-4}\sum_{k=2}^{T}c_k^{(0)}k\,(k-1)\sum_{t=k-1}^{T-1}(2T-t-1)\,t \\
& +\frac{1}{2}T^{-4}\sum_{k=3}^{T}c_k^{(0)}\sum_{t=1}^{k-2}(2T-t-1)\,t^2\,(2k-t-1) \\
= \;& o(1)
\end{aligned}$$

we get via (24) that

$$\begin{aligned}
& T^{-5}\left\{\sum_t C_t^{(0)}(1)\right\}\left\{\sum_{t=1}^{T-1}(2T-t-1)\,t\sum_{i=2}^{t+1}\sum_{j=i}^{T}C_i^{(0)}(1)\right\} \\
= \; & \left(C^{(0)}-T^{-1}C^{(1)}\right)C^{(0)}\left(\frac{4}{15}T-\frac{2}{3}\right)+o(1) \\
= \; & \frac{4}{15}C^{(0)2}T-\frac{4}{15}C^{(0)}C^{(1)}-\frac{2}{3}C^{(0)2}+o(1).
\end{aligned}$$

Hence, from (40) and (41),

$$\begin{aligned}
& T^{-5}(\mathbf{1}'\mathbf{x})\left\{\mathbf{1}'\left(\mathbf{SS}'\right)^2\mathbf{x}\right\} \\
= \; & \frac{1}{6}C^{(0)2}W(1)\int_0^1\left(2-3t^2+t^3\right)W^*(t)\,dt \\
& +\frac{1}{2}\varepsilon_0^2\left(\frac{4}{15}C^{(0)2}T-\frac{4}{15}C^{(0)}C^{(1)}-\frac{2}{3}C^{(0)2}\right)+o(1). \qquad (42)
\end{aligned}$$

Now, insertion of (27), (21), (4), (26), (37), (39) and (42) into (34) gives

$$\begin{aligned}
& T^{-4}f_{4T} \\
\equiv \; & \frac{8T^4+14T^3-309T^2+529T-602}{360T^3(T+1)} \\
& \left\{C^{(0)2}W(1)^2+\varepsilon_0^2\left(TC^{(0)2}-2C^{(0)}C^{(1)}\right)\right\} \\
& -\frac{T+8}{18T}\varepsilon_0^2C^{(0)2}-\frac{(T+2)\left(2T^2-T-61\right)}{360T^3}\left(c^{(0)*}+C^{(0)2}\varepsilon_0^2\right) \\
& +\frac{2\left(T^2-2T+7\right)}{3T(T+1)}\left[C^{(0)2}W(1)\int_0^1(1-t)\,W^*(t)\,dt\right. \\
& \left.+\varepsilon_0^2\left\{\frac{1}{3}TC^{(0)2}-\frac{1}{2}C^{(0)2}-\frac{1}{3}C^{(0)}C^{(1)}\right\}\right] \\
& +C^{(0)2}\int_{t=0}^1\left\{\int_{s=0}^t W^*(s)\,ds\right\}^2dt+\varepsilon_0^2C^{(0)2}\left(\frac{2}{15}T-\frac{1}{3}\right) \\
& -\frac{T}{T+1}\left[C^{(0)2}\left\{\int_0^1(1-t)\,W^*(t)\,dt\right\}^2+C^{(0)2}\varepsilon_0^2\left(\frac{1}{9}T-\frac{1}{3}\right)\right] \\
& -\frac{2T}{T+1}\left\{\frac{1}{6}C^{(0)2}W(1)\int_0^1\left(2-3t^2+t^3\right)W^*(t)\,dt\right. \\
& \left.+\frac{1}{2}\varepsilon_0^2\left(\frac{4}{15}C^{(0)2}T-\frac{4}{15}C^{(0)}C^{(1)}-\frac{2}{3}C^{(0)2}\right)\right\}+o_p(1) \\
= \; & C^{(0)2}\left(f_4+\frac{1}{180}\right)-\frac{1}{180}c^{(0)*}+o_p(1). \qquad (43)
\end{aligned}$$

Finally, we focus on $g_{4T} = f_{4T} - r_{4T}$, where

$$T^{-4} r_{4T} \sim \frac{1}{180} T^{-1} q_{0T} + T^{-4} q_{3T} + T^{-4} q_{4T}. \tag{44}$$

where it is easily seen that $T^{-4} q_{3T}$ vanishes asymptotically. To handle q_{4T}, we need $\mathbf{m}_4$ and $\mathbf{M}_4$. The components of $T^{-4}\mathbf{m}_4$ are

$$\begin{aligned}
& T^{-4}\mathbf{x}'\mathbf{E}_4\mathbf{L}^k\mathbf{x} \\
= \; & T^{-4}\mathbf{x}'\mathbf{L}^k\mathbf{x} \\
& + \frac{4T^4 + 16T^3 - 281T^2 + 576T - 720}{180T^4 (T+1)} (\mathbf{1}'\mathbf{x}) \left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right) \\
& - 3T^{-4}\mathbf{x}'\mathbf{S}\mathbf{S}'\mathbf{L}^k\mathbf{x} \\
& + \frac{2T^2 - 5T + 24}{6T^4 (T+1)} \left\{ (\mathbf{1}'\mathbf{x}) \left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{L}^k\mathbf{x}\right) + \left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right) \left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}\right) \right\} \\
& + T^{-4}\mathbf{x}' \left(\mathbf{S}\mathbf{S}'\right)^2 \mathbf{L}^k\mathbf{x} \\
& - \frac{1}{T^4 (T+1)} \left\{ (\mathbf{1}'\mathbf{x}) \mathbf{1}' \left(\mathbf{S}\mathbf{S}'\right)^2 \mathbf{L}^k\mathbf{x} + \left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right) \mathbf{1}' \left(\mathbf{S}\mathbf{S}'\right)^2 \mathbf{x} \right. \\
& \left. + \left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}\right) \left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{L}^k\mathbf{x}\right) \right\},
\end{aligned} \tag{45}$$

where as above, the first term vanishes while the second term behaves as in (31). Via (32), the third term vanishes, while the fourth term is as in (33). As for the remaining terms, we have as in (37) that

$$\begin{aligned}
& T^{-4}\mathbf{x}' \left(\mathbf{S}\mathbf{S}'\right)^2 \mathbf{L}^k\mathbf{x} \\
= \; & C^{(0)2} \int_{t=0}^{1} \left\{ \int_{s=0}^{t} W^*(s)\, ds \right\}^2 dt + C^{(0)2} \varepsilon_0^2 \left(\frac{2}{15} T - \frac{1}{3} \right) + o(1),
\end{aligned}$$

as in (42),

$$\begin{aligned}
& T^{-5} (\mathbf{1}'\mathbf{x}) \left\{ \mathbf{1}' \left(\mathbf{S}\mathbf{S}'\right)^2 \mathbf{L}^k\mathbf{x} \right\} \\
= \; & \frac{1}{6} C^{(0)2} W(1) \int_0^1 \left(2 - 3t^2 + t^3\right) W^*(t)\, dt \\
& + \frac{1}{2} \varepsilon_0^2 \left(\frac{4}{15} C^{(0)2} T - \frac{4}{15} C^{(0)} C^{(1)} - \frac{2}{3} C^{(0)2} \right) + o(1) \\
= \; & T^{-5} \left(\mathbf{1}'\mathbf{L}^k\mathbf{x}\right) \left\{ \mathbf{1}' \left(\mathbf{S}\mathbf{S}'\right)^2 \mathbf{x} \right\},
\end{aligned}$$

while as in (39),

$$\begin{aligned}
& T^{-5} \left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{x}\right) \left(\mathbf{1}'\mathbf{S}\mathbf{S}'\mathbf{L}^k\mathbf{x}\right) \\
= \; & C^{(0)2} \left\{ \int_0^1 (1-t) W^*(t)\, dt \right\}^2 + C^{(0)2} \varepsilon_0^2 \left(\frac{1}{9} T - \frac{1}{3} \right) + o_p(1).
\end{aligned}$$

Hence, (45) yields

$$
\begin{aligned}
& T^{-4}\mathbf{x}'\mathbf{E}_4\mathbf{L}^k\mathbf{x} \\
= \; & \frac{4T^4+16T^3-281T^2+576T-720}{180T^3\left(T+1\right)} \\
& \left\{C^{(0)2}W\left(1\right)^2+\varepsilon_0^2\left(TC^{(0)2}-2C^{(0)}C^{(1)}\right)\right\} \\
& +\frac{2T^2-5T+24}{3T\left(T+1\right)}\left\{C^{(0)2}W\left(1\right)\int_0^1\left(1-t\right)W^*\left(t\right)dt\right. \\
& \left.+\varepsilon_0^2\left\{\frac{1}{3}TC^{(0)2}-\frac{1}{2}C^{(0)2}-\frac{1}{3}C^{(0)}C^{(1)}\right\}\right\} \\
& +C^{(0)2}\int_{t=0}^1\left\{\int_{s=0}^t W^*\left(s\right)ds\right\}^2dt+C^{(0)2}\varepsilon_0^2\left(\frac{2}{15}T-\frac{1}{3}\right) \\
& -\frac{2T}{T+1}\left\{\frac{1}{6}C^{(0)2}W\left(1\right)\int_0^1\left(2-3t^2+t^3\right)W^*\left(t\right)dt\right. \\
& \left.+\frac{1}{2}\varepsilon_0^2\left(\frac{4}{15}C^{(0)2}T-\frac{4}{15}C^{(0)}C^{(1)}-\frac{2}{3}C^{(0)2}\right)\right\} \\
& -\frac{T}{T+1}\left\{C^{(0)2}\left\{\int_0^1\left(1-t\right)W^*\left(t\right)dt\right\}^2+C^{(0)2}\varepsilon_0^2\left(\frac{1}{9}T-\frac{1}{3}\right)\right\} \\
& +o_p\left(1\right) \\
= \; & C^{(0)2}\left(f_4+\frac{1}{180}\right)+o_p\left(1\right).
\end{aligned}
$$

Since, as is readily seen, the components of $T^{-4}\mathbf{M}_4$ have the same behavior, we have since

$$
\begin{aligned}
& q_{4T} \\
= \; & -2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_2+2\mathbf{m}_0'\mathbf{P}_0\mathbf{m}_4+\mathbf{m}_0'\mathbf{P}_2\mathbf{m}_0+\mathbf{m}_0'\mathbf{P}_4\mathbf{m}_0 \\
& +2\mathbf{m}_0'\mathbf{P}_2\mathbf{m}_2+\mathbf{m}_2'\mathbf{P}_0\mathbf{m}_2 \\
= \; & -2\mathbf{m}_0'\mathbf{M}_0^{-1}\mathbf{m}_2+2\mathbf{m}_0'\mathbf{M}_0^{-1}\mathbf{m}_4+\mathbf{m}_0'\left(\mathbf{I}-\mathbf{M}_0^{-1}\mathbf{M}_2\right)\mathbf{M}_0^{-1}\mathbf{m}_0 \\
& +\mathbf{m}_0'\left\{\mathbf{I}-3\mathbf{M}_0^{-1}\mathbf{M}_2+\left(\mathbf{M}_0^{-1}\mathbf{M}_2\right)^2-2\mathbf{M}_0^{-1}\left(\mathbf{M}_0-2\mathbf{M}_2\right)-\mathbf{M}_0^{-1}\mathbf{M}_4\right\} \\
& \mathbf{M}_0^{-1}\mathbf{m}_0+2\mathbf{m}_0'\left(\mathbf{I}-\mathbf{M}_0^{-1}\mathbf{M}_2\right)\mathbf{M}_0^{-1}\mathbf{m}_2+\mathbf{m}_2'\mathbf{M}_0^{-1}\mathbf{m}_2 \\
\sim \; & 2\mathbf{m}_0'\mathbf{M}_0^{-1}\mathbf{m}_4-\mathbf{m}_0'\mathbf{M}_0^{-1}\mathbf{M}_4\mathbf{M}_0^{-1}\mathbf{m}_0
\end{aligned}
$$

that

$$
\begin{aligned}
& T^{-4}q_{4T} \\
\sim \; & 2\left(\mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)C^{(0)2}\left(f_4+\frac{1}{180}\right)-\mathbf{c}'\mathbf{C}^{-1}\mathbf{1}\mathbf{1}'\mathbf{C}^{-1}\mathbf{c}C^{(0)2}\left(f_4+\frac{1}{180}\right) \\
= \; & \left(2-\mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)\left(\mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)C^{(0)2}\left(f_4+\frac{1}{180}\right),
\end{aligned}
$$

and so, (15) and (44) imply

$$T^{-4}r_{4T} \sim \frac{1}{180}\mathbf{c}'\mathbf{C}^{-1}\mathbf{c} + \left(2 - \mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)\left(\mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)C^{(0)2}\left(f_4 + \frac{1}{180}\right).$$

Consequently, together with (43),

$$\begin{aligned} & T^{-4}g_{4T} \\ \sim\ & C^{(0)2}\left(f_4 + \frac{1}{180}\right) - \frac{1}{180}c^{(0)*} - \frac{1}{180}\mathbf{c}'\mathbf{C}^{-1}\mathbf{c} \\ & - \left(2 - \mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)\left(\mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)C^{(0)2}\left(f_4 + \frac{1}{180}\right) \\ =\ & \left(1 - \mathbf{1}'\mathbf{C}^{-1}\mathbf{c}\right)^2 C^{(0)2}\left(f_4 + \frac{1}{180}\right) - \frac{1}{180}\left(c^{(0)*} + \mathbf{c}'\mathbf{C}^{-1}\mathbf{c}\right). \end{aligned}$$

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