Smooth Particle Hydrodynamics Applied to Fracture Mechanics

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Abstract

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A numerical method commonly referred to as smooth particle hydrodynamics (SPH) is implemented in two dimensions for solid mechanics in general and fracture mechanics in particular. The implementation is tested against a few analytical cases: a vibrating plate, a bending plate, a mode I crack and a mode II crack. A conclusion of these tests is that a better way of treating a shortcoming of SPH called tensile instability is needed. A study is made on the best choice of a vital parameter called the smoothing radius, and it is found that a good choice of the smoothing radius is roughly 1.5 times the initial particle spacing.
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1 Introduction

In many industrial and scientific applications it is necessary to simulate fracture and crack propagation. This can be done by the commonly used finite element method. If finite elements shall be used it is, however, necessary to define the path along which the crack will propagate before the problem can be solved.

Smooth particle hydrodynamics (SPH) is a less commonly used numerical method for solving a set of coupled partial differential equations in continuum mechanics. SPH is a meshfree particle method which makes it interesting for dynamical problems in fracture mechanics. This is because there is no need to define the path of propagation in advance.

The aim of this master thesis is implementing SPH in two dimensions and to test this implementation against a few analytically known benchmarking cases. These cases are:

- A vibrating plate
- A bending plate
- A modus I/II crack

This in order to examine if SPH gives reliable results or not.

The theory of SPH contains a parameter called the smoothing radius. This is often mentioned in literature as an important parameter, but it is seldom discussed what a good value is. Because of this the above benchmarking cases are in this thesis tested with different values of smoothing radius. This is in order to find some optimal value.

The SPH formulation used in this thesis is very similar to what was done by Gray et al. [1], who also treated the case of the vibrating plate. A variation of the smoothing radius was however not made by Gray et al., who also used a different implementation of the boundary conditions.

Attempts at modelling fracture with SPH has also been made by Gray and Monaghan [2]. The way fracture is modelled in this thesis is however quite different from this and is more similar to what was done by Simkins and Li [3] (who worked with a different method called meshfree Galerkin). As far as known, this way of modelling fracture has not been used previously with SPH.
2 Theory

2.1 Problem

We are interested in the following set of coupled partial differential equations:

\[
\frac{dx_a}{dt} = v_a
\]  
(2.1)

\[
\frac{dv_a}{dt} = \frac{1}{\rho} \frac{\partial \sigma_{a\beta}}{\partial x_\beta}
\]  
(2.2)

\[
\frac{d\rho}{dt} = -\rho \frac{\partial v_\beta}{\partial x_\beta}
\]  
(2.3)

\[
\frac{d\sigma_{a\beta}}{dt} = E \left( \dot{\varepsilon}_{a\beta} + \frac{\nu}{1-2\nu} \dot{\varepsilon}_{\kappa\kappa} \delta_{a\beta} \right)
\]  
(2.4)

where

\[
\dot{\varepsilon}_{a\beta} = \frac{1}{2} \left( \frac{\partial v_a}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_a} \right),
\]  
(2.5)

and \(a, \beta, \kappa \in \{x, y\}\). This system represents how a solid elastic material behaves in two dimensions. The system represents plane strain but could as well represent plane stress simply by replacing Young’s modulus \(E\) and Poisson’s ratio \(\nu\) by \(E'\) and \(\nu'\) according to:

\[
\begin{align*}
E' &= E \frac{1+2\nu}{(1+\nu)^2}, \\
\nu' &= \frac{\nu}{1+\nu}.
\end{align*}
\]

In addition to solving this set one needs some way of modelling cracks.

2.2 Smooth Particle Hydrodynamics

As can be seen from equations 2.1 to 2.5 we have time derivatives expressed as spatial derivatives. What SPH does is providing a way of approximating spatial derivatives. This is done by starting with expressing the spatial derivative in terms of the Dirac function \(\delta(\bar{x})\):

\[
\frac{\partial f}{\partial x_\beta}(\bar{x}_i) = \int_\Omega \frac{\partial f}{\partial x_\beta}(\bar{x}) \delta(\bar{x} - \bar{x}_i) dA,
\]

and then replacing the Dirac function with the so called kernel function, \(W\), that is:

\[
\frac{\partial f}{\partial x_\beta}(\bar{x}_i) \approx \int_\Omega \frac{\partial f}{\partial x_\beta}(\bar{x}) W(\bar{x} - \bar{x}_i, h) dA.
\]  
(2.6)
The kernel $W$ has a similar shape to a gaussian as seen in figure 2.1 and is dependent on an additional parameter, called the smoothing radius, $h$. Equation 2.6 is motivated by the fact that the kernel is normalised:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\hat{x}, h) \, dx \, dy = 1$$

and approaches the Dirac function when this smoothing radius goes to zero:

$$\lim_{h \to 0} W(\hat{x}, h) = \delta(\hat{x}).$$

![Figure 2.1: The kernel compared to a normalised gaussian.](image)

There are several kernels used in SPH. Here the (perhaps most commonly used) cubic spline kernel is utilized. This is defined as

$$W(\bar{x}) = \begin{cases} 
\alpha_k \left( \frac{2}{3} r^2 + \frac{1}{6} r^3 \right) & r \in [0, 1] \\
\alpha_k \frac{6}{5} (2 - r)^3 & r \in (1, 2) \\
0 & r \in (2, \infty)
\end{cases}$$

(2.7)

where

$$r = \sqrt{\frac{x \cdot x}{h}}$$

and $\alpha_k$ has value

$$\alpha_k = \frac{15}{7\pi h^2},$$

in two dimensions. The cubic spline kernel is the one that is plotted (in 1D) in figure 2.1. As can be seen from equation 2.7 $W(\hat{x} - \hat{x}_i)$ is zero outside a distance $2h$ of $\hat{x}_i$. This is important since one can avoid differentiating the
function $f$ in equation 2.6 by moving the differentiation to the kernel by using Gauss’s theorem:

$$\frac{\partial f}{\partial x_\beta}(\bar{x}_i) \approx \int_\Omega f(\bar{x}) W(\bar{x} - \bar{x}_i) dA - \int_\Omega f(\bar{x}) \frac{\partial W(\bar{x} - \bar{x}_i)}{\partial x_\beta} dA = -\int_\Omega f(\bar{x}) \frac{\partial W(\bar{x} - \bar{x}_i)}{\partial x_\beta} dA$$

(2.8)

where the domain of integration $\Omega$ is the circle with radius $2h$, centred at $\bar{x}_i$:

$$\Omega(\bar{x}_i) = \{ \bar{x} : ||\bar{x} - \bar{x}_i|| < 2h \},$$

and the first term is zero because the kernel is zero on the boundary of $\Omega$.

In SPH the field variables of the system is represented by particles. A particle $j$ has properties such as:

- position, $\bar{x}_j$
- velocity, $\bar{v}_j$
- stress, $\sigma_{\alpha\beta}^j$
- density, $\rho_j$
- mass per thickness, $m_j$

Now, if $\bar{x}_i$ is the position of particle $i$ equation 2.8 is discretized using these particles. The integral goes to a sum over all neighbouring particles inside $\Omega$ and $dA$ goes to the area, $A_j$, of neighbour $j$:

$$\frac{\partial f}{\partial x_\beta}(\bar{x}_i) = -\sum_{j \in \Omega} f_j \frac{\partial W(\bar{x}_j - \bar{x}_i)}{\partial x_\beta} A_j,$$

That is, one collects all particles within a distance $2h$ of particle $i$ and weight their value of $f_j$, their area and the derivative of the kernel. It might be more helpful to think of this step in terms of equation 2.6, which is illustrated in figure 2.2. One can think of SPH as centring the kernel at the position of particle $i$ and weight $\frac{\partial f}{\partial x_\beta}$ of each neighbour with the kernel, even if this is not technically correct.

The area of particle $j$ is (note that $m_j$ is mass per thickness):

$$A_j = \frac{m_j}{\rho_j}$$

which gives us

$$\frac{\partial f}{\partial x_\beta}(\bar{x}_i) = \sum_{j \in \Omega} f_j \frac{\partial W_{ij}}{\partial x_\beta} m_j \rho_j,$$

(2.9)

where we have used the short hand notation

$$W_{ij} = W(\bar{x}_i - \bar{x}_j).$$
Figure 2.2: Illustration of how equation 2.6 is discretized with particles.

Note that the change in sign in equation 2.9 comes from that $W$ is an even function and thus $\frac{\partial W}{\partial x_\beta}$ is an odd function. Thus

$$\frac{\partial W_{ij}}{\partial x_\beta} = -\frac{\partial W_{ji}}{\partial x_\beta}.$$  

(2.10)

To create an SPH-discretization of equation 2.2, equation 2.9 leads to:

$$\frac{d v^i}{dt}_{\text{of particle } i} = \frac{1}{\rho_i} \sum_{j \in \Omega} \sigma_{i\beta} \frac{\partial W_{ij}}{\partial x_\beta} m_j \rho_j,$$  

(2.11)

where Einstein summation convention is applied over the greek letters. One can symmetrize this equation by realizing that the derivative of a constant function, $1(\bar{x}) = 1$, is zero, i.e.

$$0 = \frac{\partial 1}{\partial x_\beta} = \sum_{j \in \Omega} \frac{\partial W_{ij}}{\partial x_\beta} \frac{m_j}{\rho_j}.$$  

(2.12)

according to equation 2.9. Multiplying this zero with $\sigma_{i\beta}/\rho_i$ gives

$$0 = \frac{\sigma_{i\beta}}{\rho_i} \sum_{j \in \Omega} \frac{\partial W_{ij}}{\partial x_\beta} m_j \rho_j.$$  

(2.13)
and adding this to equation 2.11 gives us

$$\frac{d v^i_a}{dt} = \sum_{j \in \Omega} m_j \left( \sigma_{a \beta}^i + \sigma_{a \beta}^j \right) \frac{\partial W_{ij}}{\partial x_\beta}.$$  

(2.14)

Since one can interpret the “force contribution”, $F_a^{ij}$, on particle $i$ from particle $j$ as

$$F_a^{ij} = m_i m_j \left( \sigma_{a \beta}^i + \sigma_{a \beta}^j \right) \frac{\partial W_{ij}}{\partial x_\beta},$$  

(2.15)

equation 2.14 is now in agreement with Newton’s third law since it is clear from equation 2.15 that interchanging $i$ and $j$ gives

$$F_a^{ij} = -F_a^{ji}$$

using equation 2.10. Equation 2.14 is the final SPH-equation representing equation 2.2.

In the same manner, an estimate of $\frac{\partial v^i_a}{\partial x_\beta}$ leads to

$$\frac{\partial v^i_a}{\partial x_\beta} = \sum_{j \in \Omega} \left( v^j_a - v^i_a \right) \frac{\partial W_{ij}}{\partial x_\beta} \frac{m_j}{\rho_j},$$

which gives the SPH estimation

$$\frac{\partial v^i_a}{\partial x_\beta} = \sum_{j \in \Omega} \left( v^j_a - v^i_a \right) \frac{\partial W_{ij}}{\partial x_\beta} \frac{m_j}{\rho_j}.$$  

(2.16)

To get an SPH-discretization of equation 2.3 one can reformulate it to

$$\frac{d \rho_i}{dt} = -\frac{\partial}{\partial x_\beta} \left( \rho v_\beta \right) + v_\beta \frac{\partial \rho}{\partial x_\beta},$$

which according to equation 2.9 can be approximated as:

$$\frac{d \rho_i}{dt} = -\sum_{j \in \Omega} \rho_j v_\beta \frac{m_j}{\rho_j} \frac{\partial W_{ij}}{\partial x_\beta} + v_\beta \sum_{j \in \Omega} \rho_j \frac{\partial W_{ij}}{\partial x_\beta} \frac{m_j}{\rho_j},$$

or equivalently:

$$\frac{d \rho_i}{dt} = \sum_{j \in \Omega} m_j (v_\beta^j - v_\beta^i) \frac{\partial W_{ij}}{\partial x_\beta}.$$  

(2.17)

The collected semi-discrete system becomes:
\[
\frac{dx_i}{dt} = v_i \\
\frac{dv_i}{dt} = \sum_{j \in \Omega} m_j \left( \frac{\sigma^i_{a\beta} + \sigma^j_{a\beta}}{\rho_i \rho_j} \right) \frac{\partial W_{ij}}{\partial x_\beta} \tag{2.18}
\]

\[
\frac{d\rho_i}{dt} = \sum_{j \in \Omega} m_j (v_i^\alpha - v_j^\alpha) \frac{\partial W_{ij}}{\partial x_\beta} \tag{2.14}
\]

\[
\frac{d\sigma^i_{a\beta}}{dt} = E_1 + \nu \left( \dot{\epsilon}^{i}_{a\beta} + \nu_1 - 2\nu \dot{\epsilon}^{i}_{\kappa\kappa} \delta_{a\beta} \right) \tag{2.17}
\]

where \( \dot{\epsilon}^{i}_{a\beta} \) is determined from

\[
\dot{\epsilon}^{i}_{a\beta} = \frac{1}{2} \left( \frac{\partial v_i^a}{\partial x_\beta} + \frac{\partial v_j^\alpha}{\partial x_\alpha} \right) \tag{2.19}
\]

\[
\frac{\partial v_i^a}{\partial x_\beta} = \sum_{j \in \Omega} (v_j^\alpha - v_i^\alpha) \frac{\partial W_{ij}}{\partial x_\beta} \frac{m_j}{\rho_j} \tag{2.16}
\]

It should be emphasised that there are several forms of SPH equations that can be developed and not all forms are compatible with each other. For a more detailed investigation of the subject see for example Bonet and Lok [4].

Since we will frequently use the quantity smoothing radius per mean particle distance we define:

\[
\tilde{h} = \frac{h}{\frac{1}{2}(\Delta x + \Delta y)}
\]

where \( \Delta x \) and \( \Delta y \) are the initial particle spacings in the \( x \)- and \( y \)-direction. This quantity will be called the relative smoothing radius. \( \tilde{h} \) will be held constant for a given simulation.

### 2.3 XSPH

According to Gray et al. [1] it is usually better to replace the velocities in the right hand side of our semi-discrete system by the smeared velocity field, \( \tilde{v}_i^a \), calculated as

\[
\tilde{v}_i^a = v_i^a + \mu \sum_{j \in \Omega} (v_j^a - v_i^a) W_{ij} \frac{m_j}{\frac{1}{2}(\rho_j + \rho_i)} \tag{2.20}
\]

where \( \mu \) is typically chosen as \( \mu = 0.5 \). The variable \( \tilde{v}_i^a \) now represents the velocity of particle \( i \) that is weighted by the velocities of the neighbouring particles. This is sometimes referred to as XSPH.
2.4 Viscosity

Typically one needs to add some form of artificial viscosity, probably the most commonly used is the so called Monaghan artificial viscosity\(^1\):

\[\Pi_{ij} = \frac{-\alpha(c_i + c_j)\phi_{ij} + 2\beta \phi_{ij}^2}{\rho_i + \rho_j}, \quad (2.21)\]

where \(\phi_{ij}\) is given by the expression:

\[\phi_{ij} = h \min\{0, (\bar{x}_i - \bar{x}_j) \cdot (\bar{v}_i - \bar{v}_j)\} \frac{(\bar{x}_i - \bar{x}_j)^2 + (h/10)^2}{(\bar{x}_i - \bar{x}_j)^2}.
\]

A good choice for the parameters \(\alpha\) and \(\beta\) are\(^2\)

\[\alpha = 1 \quad \text{and} \quad \beta = 2.
\]

The parameter \(c_i\) in equation 2.21 is the speed of sound at particle \(i\), that is

\[c_i = \sqrt{\frac{K}{\rho_i}}, \quad (2.22)\]

where \(K\) is the bulk modulus of the material. This damping is added to equation 2.14 which is now changed to

\[
\frac{d\mathbf{v}_a^i}{dt} = \sum_{j \in \Omega} m_j \left( \frac{\sigma_{a\beta}^i + \sigma_{a\beta}^j}{\rho_i \rho_j} + \Pi_{ij} \delta_{a\beta} \right) \frac{\partial W_{ij}}{\partial x^\beta} - \gamma \mathbf{v}_a^i. \quad (2.23)
\]

The viscosity in equation 2.21 is good for stabilising the SPH equations, but if one is interested in the solution at equilibrium as much damping as possible might be wanted. To get more damping the term \(-\gamma \mathbf{v}_a^i\) is included. This term provides damping in the same way as one would introduce damping in an harmonic oscillator. A typical value of \(\gamma\) would be in the range \([10^{-3} \, \text{s}^{-1}, 0.1 \, \text{s}^{-1}]\).

2.5 Artificial stress

SPH is subject of a problem called tensile instability (see for example Swegle et al.\(^7\)). This means that during a tensile state the movements of the particles becomes unstable. The result of this is that particles tend to clump together. A way to resolve this was proposed by Gray et al.\(^1\). In this procedure an extra term is added to equation 2.23, which now becomes

\[
\frac{d\mathbf{v}_a^i}{dt} = \sum_{j \in \Omega} m_j \left( \frac{\sigma_{a\beta}^i + \sigma_{a\beta}^j + (R_{a\beta}^i + R_{a\beta}^j) f_{ij}^n}{\rho_i \rho_j} + \Pi_{ij} \delta_{a\beta} \right) \frac{\partial W_{ij}}{\partial x^\beta} - \eta \mathbf{v}_a^i. \quad (2.24)
\]

\(^1\)See Liu and Liu\(^5\) page 126.
\(^2\)See Monaghan\(^6\) page 551.
Where $R^{\alpha\beta}_{i}$ is the so-called artificial stress\(^3\) and $f_{ij}$ is a scaling factor.\(^4\) The artificial stress for a particle $i$ is determined by first rotating our coordinate system to determine the principal stresses, $\tilde{\sigma}^{xx}_i$ and $\tilde{\sigma}^{yy}_i$. This can be done by the transformation

$$
\tilde{\sigma}^{i}_{xx} = c^2 \sigma^{i}_{xx} + 2sc \sigma^{i}_{xy} + s^2 \sigma^{i}_{yy},
$$

$$
\tilde{\sigma}^{i}_{yy} = s^2 \sigma^{i}_{xx} - 2sc \sigma^{i}_{xy} + c^2 \sigma^{i}_{yy},
$$

where

$$
\begin{align*}
s &= \sin \theta, \\
c &= \cos \theta, \\
\theta &= \frac{1}{2} \tan\left( \frac{2\sigma^{i}_{xy}}{\sigma^{i}_{xx} - \sigma^{i}_{yy}} \right).
\end{align*}
$$

The artificial stresses $\tilde{R}^{i}_{xx}$ and $\tilde{R}^{i}_{yy}$ in this rotated coordinate system are then determined by

$$
\begin{align*}
\tilde{R}^{i}_{xx} &= \begin{cases} 
-\zeta \tilde{\sigma}^{i}_{xx} & \tilde{\sigma}^{i}_{xx} \geq 0 \\
0 & \tilde{\sigma}^{i}_{xx} < 0
\end{cases} 
\text{(2.25)}
\end{align*}
$$

and correspondingly for $\tilde{R}^{i}_{yy}$. The parameter $\zeta$ in equation 2.25 typically has the value $\zeta = 0.3$. The artificial stresses in the original coordinate system are then obtained by rotating back, i.e. by the transformation:

$$
\begin{align*}
R^{i}_{xx} &= c^2 \tilde{R}^{i}_{xx} + s^2 \tilde{R}^{i}_{yy} \\
R^{i}_{yy} &= s^2 \tilde{R}^{i}_{xx} + c^2 \tilde{R}^{i}_{yy} \\
R^{i}_{xy} &= sc(\tilde{R}^{i}_{xx} - \tilde{R}^{i}_{yy}).
\end{align*}
$$

Note that since the artificial stress in equation 2.25 is always negative it will help keep particles apart and thus prevent that particles clump together. The term $f_{ij}$ is determined by scaling the distance between particle $i$ and $j$ according to the original particle spacing using the kernel:

$$
f_{ij} = \frac{W(r_{ij})}{W(\frac{\Delta x + \Delta y}{2h})}
$$

where $r_{ij}$ is the distance between particle $i$ and $j$ divided by $h$. This means that if two particles get closer to each other $r_{ij}$ will decrease and $f_{ij}$ will increase, which will make the added term in equation 2.24 larger. According to Gray et al. [1] a good choice for the parameter $n$ in equation 2.24 is $n = 4$.

\(^3\)This is slightly different than what Gray et al. [1] calls artificial stress.

\(^4\)It should be emphasised that $n$ in the exponent of $f_{ij}^n$ now means $f_{ij}$ to the power $n$. 
2.6 Time Stepping

The semi-discrete system is time-stepped with Heun’s method, first the field variables are predicted with

\[
\bar{x}_{n+1}^p = \bar{x}_n + \Delta t \tilde{\bar{\nu}}^n
\]

\[
\bar{\nu}_{n+1}^p = \bar{\nu}_n + \Delta t \frac{d \bar{\nu}^n}{dt}
\]

\[
\bar{\rho}_{n+1}^p = \bar{\rho}_n + \Delta t \frac{d \bar{\rho}^n}{dt}
\]

\[
\bar{\sigma}_{n+1}^p = \bar{\sigma}_n + \Delta t \frac{d \bar{\sigma}^n}{dt}
\]

where the subscript \(p\) now designates predicted (not particle index), the predicted derivatives \((\frac{d \bar{\nu}^n}{dt})_p, (\frac{d \bar{\rho}^n}{dt})_p, (\frac{d \bar{\sigma}^n}{dt})_p\) are then calculated using these values and the final field variables are calculated as

\[
\bar{x}_{n+1} = \bar{x}_{n+1}^p + \frac{\Delta t}{2}(\tilde{\bar{\nu}}_{n+1}^p - \tilde{\bar{\nu}}^n)
\]

\[
\bar{\nu}_{n+1} = \bar{\nu}_{n+1}^p + \frac{\Delta t}{2}((\frac{d \bar{\nu}^n}{dt})_p - \frac{d \bar{\nu}^n}{dt})
\]

\[
\bar{\rho}_{n+1} = \bar{\rho}_{n+1}^p + \frac{\Delta t}{2}((\frac{d \bar{\rho}^n}{dt})_p - \frac{d \bar{\rho}^n}{dt})
\]

\[
\bar{\sigma}_{n+1} = \bar{\sigma}_{n+1}^p + \frac{\Delta t}{2}((\frac{d \bar{\sigma}^n}{dt})_p - \frac{d \bar{\sigma}^n}{dt}).
\]

Note that position is moved with the smeared velocity from equation 2.20.

In addition some extra smearing will be done in the beginning of each time-step according to

\[
\tilde{\bar{\nu}}^n \leftarrow (1 - \eta) \bar{\nu}^n + \eta \bar{\nu}^n. \quad (2.26)
\]

That is, the velocity is assigned a weighted value of itself and the smeared velocity from equation 2.20. The parameter \(\eta\) is in the range: \(\eta \in [0, 1]\). If some of the particles in our system are fixed this will introduce additional damping to the system. The time step is chosen according to: \(^5\)

\[
\Delta t = C \frac{\Delta x}{c_0}, \quad (2.27)
\]

where \(\Delta x\) is the initial particle spacing, \(c_0\) is the initial speed of sound in the material, and \(C\) is a dimensionless constant.

---

2.7 Cracks

A crack, describing a discontinuity, is represented as a curve consisting of straight line segments, as can be seen in figure 2.3a. If two particles are neighbours or not is determined by if they can “see” each other. A crack can be seen as a wall which breaks that line of sight. For example the “circle”-particle in figure 2.3b is neighbour with the “square”-particle but not with the “star”-particle. This condition is fairly easy to check. The “circle”- and “star”-particle are neighbours if the equation

\[ \bar{x}_{\text{circle}} + \lambda_1 (\bar{x}_{\text{star}} - \bar{x}_{\text{circle}}) = \bar{x}_1 + \lambda_2 (\bar{x}_2 - \bar{x}_1) \]

has solutions for \( \lambda_1 \) and \( \lambda_2 \) in the range

\[ \lambda_1, \lambda_2 \in [0, 1]. \]

This reduces checking the visibility condition to solving a two dimensional linear equation system.

It is desirable that the crack is a part of the actual material. If all particles in a body would be moving with a constant velocity you would want the crack to be moving with the body. To get this effect the start and end of each segment is “floating” half way between a pair of particles, as can be seen in figure 2.3a.

Figure 2.3: A crack represented as straight line segments, floating between pairs of particles.
3 Benchmarking cases

For the following benchmarking cases we wish to test the numerical implementation for four different values of the relative smoothing radius:

\[
\tilde{h} = 1.3, \quad \tilde{h} = 1.5, \quad \tilde{h} = 1.7 \& \tilde{h} = 1.9,
\]

and at the same time vary the initial number of particles.

3.1 Vibrating Plate

We have a rectangular plate with length \(L\) and width \(D\), as seen in figure 3.1. The left end at \(x = 0\) is fixed in a wall and the other end is freely movable. If \(L > 10D\) the dynamics of this system is well approximated by this one dimensional partial differential equation:

\[
\frac{E \cdot D^2}{12 \rho} \frac{\partial^4 u_y}{\partial x^4} = -\frac{\partial^4 u_y}{\partial t^4}
\]

\[
\left. u_y \right|_{x=0} = 0,
\]

\[
\left. \frac{\partial u_y}{\partial x} \right|_{x=0} = 0,
\]

\[
\left. \frac{\partial^2 u_y}{\partial x^2} \right|_{x=L} = 0,
\]

\[
\left. \frac{\partial^3 u_y}{\partial x^3} \right|_{x=L} = 0
\]

where \(u_y\) is the displacement in the \(y\)-direction. If we start with zero displacements at time \(t = 0\):

\[
\left. u_y \right|_{t=0} = 0,
\]

and solve our system by the method of separation of variables the solution for the fundamental mode becomes:

\[
u_y(x, t) = \frac{\cos a_0 + \cosh a_0}{\sinh a_0} \left( \sin \lambda_0 x - \sinh \lambda_0 x \right) - \frac{\sin a_0 + \sinh a_0}{\cosh a_0} \left( \cos \lambda_0 x - \cosh \lambda_0 x \right) \frac{H_0 \sin \omega_0 t}{(3.1)}
\]

where \(\lambda_0 = \frac{a_0}{L}, \quad \omega_0 = \frac{\lambda_0^2 D}{12 \rho}\) and \(a_0 = 1.8751\).
To test if SPH captures the dynamics correctly we give the system an initial impulse which corresponds to the fundamental mode. That is, starting from initial velocity:

\[
\frac{\partial u_y}{\partial t}\bigg|_{t=0} = \frac{(\cos \alpha_0 + \cosh \alpha_0)(\sin(\lambda_0 x - \sinh \lambda_0 x) - (\sin \alpha_0 + \sinh \alpha_0)(\cos \lambda_0 x - \cosh \lambda_0 x)}{(\cos \alpha_0 + \cosh \alpha_0)(\sin(\lambda_0 L - \sinh \lambda_0 L) - (\sin \alpha_0 + \sinh \alpha_0)(\cos \lambda_0 L - \cosh \lambda_0 L))} H_0 \omega_0,\]

(3.2)

which will make equation 3.1 the analytical solution. The parameter \(H_0\) is the maximum displacement in the plate at \(x = L\).

The plate is discretized with particles as seen in figure 3.2. Particles are placed equally spaced between

\[
x \in [0, L - \frac{\Delta x}{2}] \text{ and } y \in [-(D - \frac{\Delta y}{2}), (D - \frac{\Delta y}{2})].
\]

Particles with \(x \leq 0\) are constrained from moving and the others are free. It might seem strange that particles aren't placed all the way to \(x = L\) and \(y = \pm \frac{D}{2}\), but we motivate this by the fact that a particle actually represents a somewhat larger domain than just a point. In some sense a particle positioned at \((x, y)\) could be thought of as representing the domain

\[
(x, y) \in \left[\frac{x - \Delta x}{2}, \frac{x + \Delta x}{2}\right] \times \left[\frac{y - \Delta y}{2}, \frac{y + \Delta y}{2}\right].
\]

The problem is solved only with the damping provided by that in equation 2.21. That is, the additional damping parameters \(\gamma\) and \(\eta\) in equations 2.23 and 2.26 are both set to zero.

To compare the numerical and analytical solution we solve the problem with SPH until an end time \(t_{end}\) corresponding to one analytical period, more precisely:

\[
t_{end} = \frac{2\pi}{\omega_0}.
\]

During this time the \(y\)-displacement of the end particle (marked in figure 3.2) is saved at equally spaced time intervals. This will give us the points seen in
Using these points we can determine the amplitude and frequency of the oscillations of the numerical solution by making a least square fit to the equation

$$u_y(t) = Ae^{-\kappa t} \sin(\omega t),$$

(3.3)

to determine the three parameters $A$, $\kappa$, and $\omega$. $A$ and $\omega$ will then be compared to the analytical values of the frequency and amplitude. The parameter $\kappa$ might seem superfluous, but should be included since the artificial viscosity in equation 2.21 introduces damping. This problem is very similar to the problem studied by Gray et al. [1] with a different implementation of the boundary conditions. Gray et al. used $h = 1.5$. 

Figure 3.2: Plate discretized with particles

Figure 3.3: $y$-displacement of the end particle as a function of time, together with the analytical curve from equation 3.1 and a least square fit to equation 3.3.
3.2 Bending Plate

For the same geometry as in figure 3.1 we start from equilibrium:

\[ u_y = 0 \text{ and } \frac{\partial u_y}{\partial t} = 0. \]

But now we prescribe the \( y \)-displacements of the particles at the end of the plate to be \( u_p^y \). This is illustrated in figure 3.4. The triangular particles are free to move in the \( x \)-direction but are during the time-stepping assigned a velocity in the \( y \)-direction according to:

\[ v_y(n) = \begin{cases} \frac{u_p^y}{N\Delta t} & n < N \\ 0 & n \geq N \end{cases}. \quad (3.4) \]

Where \( n \) is which time-step and \( N \) is the number of time-steps during which this loading takes place. The damping parameters \( \gamma \) and \( \eta \) were chosen to \( \gamma = 0.7 \text{s}^{-1} \) and \( \eta = 0.9 \) to get as much damping as possible. After loading the system and waiting for it to be damped sufficiently the following is checked:

1. The \( y \)-displacements of the centreline (particles originally located on \( y = 0 \)), which approximately should be\(^6\):

\[ u_y(x) = \frac{u_p^y}{2\beta^3}((\xi(x) - \alpha)^3 - 3\beta^2(\xi(x) - \alpha) + 2\beta^3) \]

where

\[ \xi(x) = -x \quad (3.4) \]

\[ \alpha = \frac{\Delta x}{2L} \quad \beta = 1 - \alpha, \]

provided that \( L > 10D \).

2. The angle at the end of the plate. Which analytically should be

\[ \theta_{\text{end}} = \frac{3u_p^y}{2\beta L} \]

according to KTH [8]. This angle is calculated as illustrated in figure 3.5. A linear least square fit is made to the \( x \)- and \( y \)-coordinates of the triangular particles in figure 3.4 to determine \( \frac{dy}{dx} \). The angle is then obtained by

\[ \theta_{\text{num}} = \frac{\pi}{2} - \tan^{-1}\left( \frac{dy}{dx} \right). \quad (3.5) \]

Figure 3.4: Plate with end prescribed.

Figure 3.5: End of the plate in figure 3.4 and how the angle $\theta$ was determined.
3.3 Modus I/II Crack

We have a square body of an elastic material in the domain $x, y \in [-\frac{L}{2}, \frac{L}{2}]$ as can be seen in figure 3.6. The body is considered infinitely thin and is therefore modelled as plain stress. In this body there is a straight crack going from $(x, y) = (-\frac{L}{2}, 0)$ to $(x, y) = (0, 0)$.

Two cases will be considered, a modus I and a modus II crack. During the time-stepping the frame of outermost particles in this body (marked in red) get prescribed velocities in both the $x$- and $y$-direction in the same way as in equation 3.4. The final displacements are shown in figure 3.7. As can be seen modus I is an opening crack and modus II is a sliding crack. Expressed in cylindrical coordinates these displacements are

$$u_x(r, \theta) = \frac{(1 + \nu)K_I}{4\pi E} \sqrt{2\pi r} \left((2\kappa - 1)\cos \frac{\theta}{2} - \cos \frac{3\theta}{2}\right) \quad (3.6)$$

$$u_y(r, \theta) = \frac{(1 + \nu)K_I}{4\pi E} \sqrt{2\pi r} \left((2\kappa + 1)\sin \frac{\theta}{2} - \sin \frac{3\theta}{2}\right) \quad (3.7)$$

in the modus I case and

$$u_x(r, \theta) = \frac{(1 + \nu)K_{II}}{4\pi E} \sqrt{2\pi r} \left((2\kappa + 3)\sin \frac{\theta}{2} + \sin \frac{3\theta}{2}\right) \quad (3.8)$$

$$u_y(r, \theta) = \frac{(1 + \nu)K_{II}}{4\pi E} \sqrt{2\pi r} \left((2\kappa - 3)\cos \frac{\theta}{2} + \cos \frac{3\theta}{2}\right) \quad (3.9)$$

in the modus II case\(^7\). The parameter $\kappa$ equals

$$\kappa = \frac{(3 - \nu)}{(1 + \nu)},$$

\(^7\)See KTH [8] page 238.
and $K_I$ and $K_{II}$ are constants called the stress intensity factors. This choice of

The analytical displacements will make the analytical displacements take

In addition, the stresses in the body will become

\begin{align}
\sigma_{xx} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \\
\sigma_{yy} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \\
\sigma_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}
\end{align}

for the modus I case and

\begin{align}
\sigma_{xx} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} (2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}) \\
\sigma_{yy} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \\
\sigma_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2})
\end{align}

for the modus II case, see Williams [9].

![Figure 3.7: Displacements applied to the frame (exaggerated).](image)

An important parameter when it comes to fracture mechanics is the so
called energy release rate, $G$, defined as:

$$G = -\frac{\partial \tilde{U}}{\partial a},$$

(3.16)
where $\tilde{U}$ is the elastic energy per thickness in the body, and $a$ is the length of the crack. This says how the potential energy stored in the body is changed when the crack grows. Theoretically the energy release rate should have the value
\[ G = \frac{K_I^2}{E} + \frac{K_{II}^2}{E}, \]  
that is
\[ G = \begin{cases} \frac{K_I^2}{E} & \text{in pure modus I} \\ \frac{K_{II}^2}{E} & \text{in pure modus II} \end{cases} \]  
The potential energy per thickness is given by
\[ \tilde{U} = \frac{1}{2} \int_V \sigma_{\alpha\beta} \epsilon_{\alpha\beta} dA. \]  
For our discrete system this will be calculated as
\[ \tilde{U} = \frac{1}{2} \sum_{j} \sigma_{\alpha\beta}^j \epsilon_{\alpha\beta}^j \frac{m_j}{\rho_j} \]  
where we calculate the strains using:
\[ \epsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) \]  
\[ \epsilon_{yy} = \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) \]  
\[ \epsilon_{xy} = \frac{1}{2G} \sigma_{xy} \]  
The issues that will be checked for this problem are:

1. The error-norms for stresses and displacements at equilibrium, determined as:
\[ e(\sigma_{\alpha\beta}) = \frac{1}{n} \sqrt{\sum_{j=1}^{n} (\sigma_{\alpha\beta}^j - \sigma_{\alpha\beta}(\tilde{x}_j))^2}, \]  
for the stresses and
\[ e(u_{\beta}) = \frac{1}{n} \sqrt{\sum_{j=1}^{n} (u_{\beta}^j - u_{\beta}(\tilde{x}_j))^2}, \]

\footnote{Note that the $m_j$ actually has dimension $kg/m$ in two dimensions and thus is the mass per thickness.}
\footnote{What is meant by equilibrium is that the norm of the velocities in the system:
\[ |v_{sys}| = \sqrt{\sum_j v_j^2} \]  
has decreased enough relative to its value at the time the system had been loaded.}
for the displacements. In equation 3.22 and 3.23 $\sigma_{\alpha\beta}(\bar{x}_j)$ and $u_{\beta}(\bar{x}_j)$ are the analytical values and the sum goes over all particles in the body. The factor $\frac{1}{n}$ must be included since the error otherwise will increase when the number of particles increases.

2. The numerical energy release rate calculated in two ways. Hypothesis for calculation of crack paths may be divided into those based on propagation and those based on projection. Propagation criteria study the crack tip state of a hypothetical extension of the crack whereas projection based criteria study the mechanical state at the tip. Both hypothesis are investigated here, namely:

(a) By the procedure illustrated in figure 3.9:
   i. At equilibrium determine the energy in the body when the crack has length $a$, $\bar{U}(a)$, using equation 3.18.
   ii. Forcing the crack open one step, waiting for equilibrium to occur and calculating the energy when the crack has length $a + \Delta x$, $\bar{U}(a + \Delta x)$.
   iii. Opening the crack one step further to determine $\bar{U}(a + 2\Delta x)$.
   iv. Numerically differentiating these values to determine $\frac{\partial \bar{U}}{\partial a}$ using an $O(h^2)$ estimate of the derivative:

$$\frac{\partial \bar{U}}{\partial a}(a) \approx \frac{-\frac{3}{2} \bar{U}(a) + 2 \bar{U}(a + \Delta x) - \frac{1}{2} \bar{U}(a + 2\Delta x)}{\Delta x},$$

and thus obtaining $G$ from equation 3.16. This value will be referred to as $G_1$ and is determined from propagation.

(b) At equilibrium taking the marked points in figure 3.8a and:
   i. Determine $K_I$ by a least square fit of their $y$-displacement to equation 3.7.
   ii. Determine $K_{II}$ by a least square fit of their $x$-displacement to equation 3.8.
   iii. Determine $G$ by using equation 3.17. This value will be referred to as $G_2$ and is determined from the state at the tip (projection).
Figure 3.8: In a) particles that $K_I$ and $K_{II}$ are determined from. In b) particles that the stresses are plotted for.

Figure 3.9: How the crack was forced to open to determine the energy release rate, using equation 3.24.
4 Results

4.1 Vibrating Plate

This problem was solved without the artificial stress from section 2.5. With artificial stress a disturbance can be seen in the bottom of the plate where the plate is attached to the wall, as seen in figure 4.1.

![Disturbance seen in the von Mises stress when artificial stress is included.](image)

Without artificial stress the relative error in frequency is shown in figure 4.2a for different values of smoothing radius and particle densities. The parameter \( n_y \) describes the number of particles in the \( y \)-direction across the height of the plate. Since all values in figure 4.2a are negative we see that SPH underestimates the frequency of the plate. That is, the plate oscillates slower than it should.

The relative error in amplitude is shown in figure 4.2b. Since all values are positive it is clear that SPH here overestimates the amplitude of the oscillations. In both figures the smallest error is about 3% and the relative error decreases when \( n_y \) is increased. The exception is for \( \tilde{h} = 1.3 \) where we see that the error increases with increasing \( n_y \). Overall \( \tilde{h} = 1.5 \) seems to be the best choice of relative smoothing radius for this problem. Note that figure 4.2a and 4.2b qualitatively looks the same except mirrored since the errors differ in sign.

That SPH has problems for \( \tilde{h} = 1.3 \) is also visible if one plots the von Mises stress as seen in figure 4.3. The figure is taken at a time close to one quarter of the period when the plate is at its highest position \((w > 0)\). It is possible to see some spikes on the lower side \((y < 0)\) close to where the plate is attached to the wall. This looks rather unphysical. Note that the top side \((y > 0)\) close to the wall in figure 4.3 looks a lot better than the bottom side.

This problem does not seem to exist for a larger smoothing radius. The corresponding plot for \((n_y, \tilde{h}) = (21, 1.5)\) is seen in figure 4.4. Here the von
Mises stress looks a lot more physical. Neither does the problem seem to exist for $(n_y, \tilde{h}) = (9, 1.3)$ as seen in figure 4.5. It is however present for $(n_y, \tilde{h}) = (15, 1.3)$.

![Figure 4.2: Relative error in amplitude and frequency for different values of $n_y$ and $\tilde{h}$.](image)

![Figure 4.3: Von Mises stress for $n_y=21$ and $\tilde{h}=1.3$ at a time $t = 1.5681s$.](image)
Figure 4.4: Von Mises stress for \( n_y = 21 \) and \( \tilde{h} = 1.5 \) at a time \( t = 1.5681 \) s.

Figure 4.5: Von Mises stress for \( n_y = 9 \) and \( \tilde{h} = 1.3 \) at a time \( t = 1.5648 \) s.

4.2 Bending Plate

This problem was also solved without artificial stress, since the same type of disturbance as that in figure 4.1 could be seen also here.

The centreline of the plate is seen in figure 4.6 for the case of \((n_y, \tilde{h}) = (21, 1.7)\) after running the simulation for 11 seconds. As can be seen the agreement with the analytical solution is very good. The error in end angle at this time is typically of order:

\[
|\theta_{\text{numerical}} - \theta_{\text{analytical}}| \sim 0.1^\circ,
\]

which should be considered very good. It is possible to see some boundary effects close to where the plate is attached to the wall (figure 4.7a) and at the end of the plate (figure 4.7b).

At \( t = 11 \) s the particles are positioned as can be seen in figure 4.8 which is fairly regular. However, if the simulation is continued for an extensive amount of time the particles position themself as seen in figure 4.9. The particles between \( x = 0 \) and \( x = 0.3 \) m have become irregularly spaced. In the bottom part of this region \((y < 0)\) particles have also formed lines. It thus seems
like when the system has been damped enough the particles start to clump together. You can also see a slight tendency of this in figure 4.8. For $x > 0.3m$ the particles in figure 4.9 look more regularly spaced. Since this problem occurs a study on different values of $n_y$ and $\tilde{h}$ is no longer interesting.

Figure 4.6: Centreline of plate for $(n_y, \tilde{h}) = (21, 1.7)$

(a) Centreline at beginning of plate.

(b) Centreline at end of plate.

Figure 4.7: Boundary effects seen at both ends of the plate.
4.3 Modus I Crack

Let $n_x$ be the number of particles across the width of the body. The $y$- displacements for the particles marked in figure 3.8a are shown in figure 4.10 for $(n_x, \tilde{h}) = (60, 1.7)$. We see that the numerical solution agrees well with the analytical solution, except very near the crack tip at $x_0 = 0$. Note that the four particles closest to the boundary at $x_0 = -0.1\, \text{m}$ in figure 4.10 actually are forced to agree with the analytical solution. This is from the way the boundary conditions are implemented.

The stress components for the particles marked in figure 3.8b are shown in figures 4.11a to 4.11c together with the analytical solutions. It can be seen that the $\sigma_{yy}$-component in figure 4.11b agrees very well with the analytical solution. For the $\sigma_{xx}$- and $\sigma_{xy}$-components the results agree very well except very close to the tip of the crack. Both in figure 4.11a and figure 4.11b we also see that there is some problem close to the boundary of the body (close to $x_0 = 0.1\, \text{m}$). Even though figures 4.11a to 4.11c are the results for $(n_x, \tilde{h}) = (60, 1.7)$ the qualitative look is the same as for other values of $n_x$ and $\tilde{h}$.

The relative error in energy release rate for the method when the crack is forced to open (propagation) is shown in figure 4.12a. For all values the error is very large, around 20%-30%, and since all values are negative the energy release rate is always underestimated. As can be seen in the figure 4.12a the error decreases when $n_x$ increases. The exception is the value of $(n_x, \tilde{h}) = (40, 1.5)$ which is much lower than the general trend. From figure 4.12a it is hard to see
that any particular value of $\hat{h}$ would be better.

The corresponding error when the energy release rate is calculated by projection is shown in figure 4.12b. As can be seen these errors are typically a factor 100 smaller compared to the values in figure 4.12a. Note that about half of the values in figure 4.12b overestimate and about half underestimate the energy release rate. For each value of $n_x$ the smallest error occur for $\hat{h} = 1.5$. It should however be emphasized that the relative error in $G_2$ only changes about 1% when $\hat{h}$ is varied between 1.3 and 1.9, as can be seen in figure 4.12b.

The error-norms for the $x$- and $y$-displacements (see equation 3.23) are shown in figure 4.13a for different values of $n_x$ and $\hat{h}$. Each $u_x$-error-norm, $e(u_x, \hat{h}, n_x)$, is normalised to the maximum error-norm that occurs for the tested values of $n_x$ and $\hat{h}$:

$$\frac{e(u_x, \hat{h}, n_x)}{\max_{n_x, \hat{h}} \{e(u_x, \hat{h}, n_x)\}}$$

and correspondingly for the $u_y$-error-norm. This is the reason why the largest value in figure 4.13a is 1 for both $u_x$ and $u_y$. Because of this normalisation, figure 4.13a says nothing about how large the error-norm in equation 3.23 actually is but rather how it changes when varying $n_x$ and $\hat{h}$. One can not even compare the error-norm in $u_x$ and in $u_y$ in figure 4.13a, except by shape of the curves. From figure 4.13a it is clear that the error decreases with increasing $n_x$ and that the displacements favour the lower values of $\hat{h}$. It is on the other hand worth noting that even if the error is larger for greater values of $\hat{h}$ the error also decreases more rapidly.

The corresponding normalised error-norms (equation 3.22) for the stresses are shown in figure 4.13b. Also here is it clear that the error decreases with increasing $n_x$. For $\sigma_{xx}$ and $\sigma_{yy}$ it is clear that $\hat{h} = 1.3$ and $\hat{h} = 1.5$ is favoured. For the $\sigma_{xy}$-component the error is however larger for $\hat{h} = 1.3$.

Overall figure 4.13 and 4.12 together indicates that $\hat{h} = 1.5$ seems to be the best choice of relative smoothing radius also for this problem.
Figure 4.10: $y$-displacements for the particles marked in figure 3.8a against their original $x$-position. In modus I with $(n_x, \hat{h}) = (60, 1.7)$.

Figure 4.11: Stress components of the particles marked in figure 3.8b against their original $x$-position. For $n_x = 60$ and $\hat{h} = 1.7$. In modus I.
Figure 4.12: Relative error for $G_1$ and $G_2$ for different values of $n_x$ and $\tilde{h}$. In modus I.

Figure 4.13: Normalised error-norms for the displacements and stresses for different values of $n_x$ and $\tilde{h}$.
4.4 Modus II Crack

The $x$-displacements of the particles marked in figure 3.8a are shown in figure 4.14. As can be seen the agreement with the analytical solution is very good, although the error clearly gets bigger closer to the crack tip. The stresses for the particles marked in figure 3.8b are shown in figure 4.15. We essentially have the same case as in modus I. The stresses follow the general trend of the analytical solution but clearly have some problems close to the edge of the body at $x_0 = 0.1$ and close to the tip of the crack at $x_0 = 0$. The $\sigma_{xy}$-component in figure 4.15c looks particularly good. It should be noted that $\sigma_{xx}$ and $\sigma_{yy}$ in figure 4.15a and 4.15b analytically should be antisymmetric around the $x$-axis but clearly isn’t close to the crack tip.

The relative error in energy release rate is shown in figure 4.16a for the case when the crack is forced to open (propagation). As can be seen the errors are also here of order 20%-30% and decreases slowly for increasing $n_x$. The exceptions are $\tilde{h} = 1.5$ (where the error only changes slightly), and $(n_x, \tilde{h}) = (40, 1.3)$ (which is exceptionally low and also differs in sign from the other values). Note that the lowest errors occur for $\tilde{h} = 1.3$ for each value of $n_x$.

The same error for when the energy release rate is determined from projection is shown in figure 4.16b. These values are typically a factor 10 lower than in figure 4.16a and a factor 10 larger than the corresponding modus I case in figure 4.12b. As can be seen the errors in figure 4.16b decrease with increasing $n_x$ and the error is smallest for $\tilde{h} = 1.9$ for all values of $n_x$. The variation with $\tilde{h}$ is however very slight. As can be seen the relative error in $G_2$ varies only about 0.5% for a given value of $n_x$.

The normalised error-norms (same as in section 4.3) for the displacements are shown in figure 4.17a and the normalised error-norms for the stresses are shown in figure 4.17b. As can be seen the errors decreases with increasing $n_x$ and are lowest for $\tilde{h} = 1.3$ for all values of $n_x$. It is also clear that for each $n_x$, the error in general increase with increasing $\tilde{h}$, even if the dependence of $\tilde{h}$ becomes less important for larger values of $n_x$. Because of this and since the variation in figure 4.16b is so slight $\tilde{h} = 1.3$ seems the most appropriate choice for this problem.
Figure 4.14: $x$-displacements for the particles marked in figure 3.8a against their original $x$-position. In modus II. For $(n_x, \tilde{h}) = (60, 1.7)$.

Figure 4.15: Stress components of the particles marked in figure 3.8b against their original $x$-position. For $(n_x, \tilde{h}) = (60, 1.7)$. In modus II.
Figure 4.16: Relative error for $G_1$ and $G_2$ for different values of $n_x$ and $\tilde{h}$. In modus II.

Figure 4.17: Normalised error-norms for the displacements and stresses for different values of $n_x$ and $\tilde{h}$.
5 Discussion

5.1 Vibrating Plate

The smallest error that SPH provided was about 3%, both for frequency and amplitude, which should be considered a quite good result. That the frequency is underestimated is not unexpected. Most numerical methods underestimate the wave-speed, which should lead to a lower frequency. It would have been more worrying if the plate would have vibrated faster than the analytical solution. It should also be noted that the analytical solution in equation 3.1 is only approximative to start with. Thus one can not expect a perfect agreement.

It is interesting that the problems for $\tilde{h} = 1.3$ occurs on the bottom side and not on the top side of the plate in figure 4.3. The bottom side should be in a tensile state while the top side should be in a compressed state. Thus one is lead to suspect that this problem is due to tensile instability. It is slightly troubling that this problem is seen for $(n_y, \tilde{h}) = (21, 1.3)$ and not for $(n_y, \tilde{h}) = (9, 1.3)$. This leads to the question if the same problem would occur for $\tilde{h} = 1.5$ if $n_y$ was increased beyond $n_y = 21$. It would be far more satisfying if one could find a way of treating tensile instability that doesn't result in the disturbance seen in figure 4.1 on page 25.

5.2 Bending plate

It is somewhat unexpected that particles clump together on the bottom side ($y < 0$) of the plate. This is the side that should be in a compressed state and if this would be due to tensile instability these problems should start to occur on the top side ($y > 0$). Nevertheless, that particles clump together is a typical sign of tensile instability. Thus these problems are suspected to be due to this.

Even though this problem clearly doesn't result in any convincing results figure 4.6 on page 28 still points to that it would be possible to achieve satisfying results if one would find a better way to treat tensile instability.

5.3 Modus I/II Crack

I isn't unexpected that we have problems close to the crack tip. Analytically the stresses and strains approach infinity at the tip. Due to the present singularity, the numerical problem is difficult. Also, when the number of particles in the body is increased particles will be initially located close and closer to the tip. So upon increasing the number of particles the peaks of the blue curves in figures 4.11a to 4.11c and figure 4.15a to 4.15c will increase. Thus, it isn't obvious that increasing the number of particles in the body will make the stresses of the particles closest to the crack tip better, since the problem is self similar.

The estimate of the energy release rate coming from projection clearly gave very good results compared to the estimate coming from propagation, which
was close to useless. This can be explained by the fact that the energy calculated using equation 3.18 probably isn't a very accurate estimate. Since the strains in equation 3.18 are determined from the stresses through equations 3.19 to 3.21 the energy is proportional to the stresses squared:

\[ \tilde{U} \propto \sigma^2. \]

Because of this, a small error in the stresses will have a big impact on the energy. Since it is clear from the figures in section 4.3 and 4.4 that we have problems both near the crack tip and the boundary it isn't unexpected that the estimated energy isn't very good. Since the highest stresses also occur close to the crack tip these errors will have the biggest effect on the energy estimate.

Another error source is that the differentiation in equation 3.24 is for simplicity taken with respect to the initial particle spacing, \( \Delta x \). But since the particles around the crack tip moves slightly in the \( x \)-direction during the loading of the structure, the crack might not grown exactly a distance \( \Delta x \) each time it is forced open.

It is worth noting that in the current formulation the location of the crack tip is a bit fuzzy. Since each node of the crack is located between a pair of particles, the end of the crack-segment closest to origo is initially located at

\[ (x, y) = \left( -\frac{\Delta x}{2}, 0 \right), \]

even though we assume that the crack tip is located in origo. If we would want to set up a problem where the crack tip is moved by some amount smaller than \( \Delta x/2 \) this wouldn't make any difference to the numerical formulation. Thus, it is obvious that the tip is located somewhere in the region

\[ (x, y) \in \left\{ -\frac{\Delta x}{2} < x < \frac{\Delta x}{2}, -\frac{\Delta y}{2} < y < \frac{\Delta y}{2} \right\}, \]

but we cannot say exactly where.

The non-existence of antisymmetry that can be see in figures 4.15a and 4.15b on page 34 could be explained by tensile instability. If we have an antisymmetric case where one side of a symmetry line is in a tensile state and the other side is in a compressed state this symmetry can be broken through the numerics in two ways:

1. The tensile instability should cause the largest problems on the tensile side.

2. The way we tackle tensile instability through the method in section 2.5 on page 11 breaks antisymmetry since we add an artificial term only on the tensile side.
5.4 General

Looking on all the benchmarking cases studied in this report a value of $\hat{h} = 1.5$ seems like a good choice.

It isn't unexpected that we have weird effects close to the boundaries in several of the benchmarking problems. Removing the first term in equation 2.8 actually relies on the fact that the domain $\Omega$ isn't truncated (see Liu and Liu [5] page 40). Thus some boundary effects are expected.

5.5 Suggestions for Further Investigations

There are several kinds of correction terms that have been suggested to obtain higher accuracy in SPH, see for example Bonet and Lok [4] or Oger et al. [10]. It would be interesting to see if these corrections would give better results for our benchmarking cases.

There are also several proposed ways to treat tensile instability that have not been tested in this study. Perhaps the most promising seems to be the use of additional “stress points” as suggested in Randles and Libersky [11]. It would definitely be interesting to see if this would solve the problems experienced in sections 4.1 and 4.2.

6 Conclusions

- The results of the benchmarking cases show that SPH is useful for problems in solid and fracture mechanics. It is however necessary to find a better way to treat tensile instability than by the method of artificial stress that has been implemented here.

- A study of the modus I/II crack shows that SPH has problems capturing the singularities that occur at the tip of the crack, where the stresses and strains approach infinity.

- It's recommended to use a smoothing radius equal to 1.5 times the initial particle spacing.

- The most appropriate way to determine the energy release rate is probably from projection, i.e. from the mechanical state at the tip of the crack.

- In SPH the position of the crack tip is slightly undefined. The tip is located within some region of width and height proportional to the particle spacing, but it is not possible to say where within this region.

- Even if a problem is analytically antisymmetric on different sides of a line the numerical solution might not be antisymmetric. This could be due to tensile instability or artificial stress.
References


A  Used Parameters

The parameters that were used for the different benchmarking cases are shown in table A.1 to A.3.

Table A.1: Parameters used for the vibrating plate.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$1, MPa$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$0.3$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$7750, kg/m^3$</td>
</tr>
<tr>
<td>$L$</td>
<td>$1, m$</td>
</tr>
<tr>
<td>$D$</td>
<td>$L/12$</td>
</tr>
<tr>
<td>$H_0$</td>
<td>$0.02L$ (eq 3.1)</td>
</tr>
<tr>
<td>$C$</td>
<td>$0.8$ (eq 2.27)</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$0$ (eq 2.26)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$0$ (eq 2.23)</td>
</tr>
</tbody>
</table>

Table A.2: Parameters used for the bending plate.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$1, MPa$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$0.3$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$7750, kg/m^3$</td>
</tr>
<tr>
<td>$L$</td>
<td>$1, m$</td>
</tr>
<tr>
<td>$D$</td>
<td>$L/12$</td>
</tr>
<tr>
<td>$u''_p$</td>
<td>$-0.04L$ (eq 3.4)</td>
</tr>
<tr>
<td>$N$</td>
<td>$4840$ (eq 3.4)</td>
</tr>
<tr>
<td>$C$</td>
<td>$0.9$ (eq 2.27)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$0.7$ (eq 2.23)</td>
</tr>
</tbody>
</table>

Table A.3: Parameters used for the modus 1/modus 2 crack.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$6, MPa$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$0.3$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$7750, kg/m^3$</td>
</tr>
<tr>
<td>$L$</td>
<td>$0.2, m$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$0.05$ (eq 2.23)</td>
</tr>
<tr>
<td>$N$</td>
<td>$8467$ (eq 3.4)</td>
</tr>
<tr>
<td>$C$</td>
<td>$0.09$ (eq 2.27)</td>
</tr>
<tr>
<td>$K_I = K_{II}$</td>
<td>$35.8, KPa$ (eq 3.6 to 3.9)</td>
</tr>
</tbody>
</table>