Hyperbolic geometry: history, models, and axioms

Sverrir Thorgeirsson

Abstract

The aim of this paper is to give an overview of hyperbolic geometry, which is a geometry of constant negative curvature that satisfies Euclid's axioms with the exception of the parallel postulate. The history of the subject and a model-free perspective of the main geometric results are presented, after which the common models are introduced and analyzed.

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1 Introduction

In this paper, an emphasis has been placed on using primary sources when discussing historical results. Therefore it was of great help that papers by two very important authors, Beltrami and Lobachevsky, have recently been translated in English and can be found in [Stillwell, 1996] and [Lobachevsky, 2010]. The latter was especially important for finding and understanding identities in hyperbolic trigonometry in the third section, in which calculations are by necessity carried out in some detail.

Special thanks go to my advisor Vera Koponen for agreeing to supervise me on this topic and to my girlfriend Amanda Andén for illustrating the paper.

2 Historical overview

2.1 Discovery and early history

In the ancient world, geometry was used as a practical tool to solve problems in fields such as architecture and navigation. As fragmented knowledge grew, mathematicians felt the need to approach geometry in a more systematic fashion. This resulted in a breakthrough in Greece around 300 BC with the publication of Euclid's *Elements*, a mathematical treatise that was regarded as a paradigm of rigorous mathematical reasoning for the next two thousand years [Mueller, 1969]. In this work, Euclid wrote definitions, axioms and postulates which give the foundation of what we now call *Euclidean geometry*. The five postulates in *Elements* are interesting in particular, and can be paraphrased as follows (compare with [Euclid, 1908, page 154-155]):

1. There is one and only one line segment between any two given points.
2. Any line segment can be extended continuously to a line.
3. There is one and only one circle with any given centre and any given radius.
4. All right angles are congruent to one another.
5. If a line falling on two lines make the interior angles on the same side less than two right angles, then those two lines, if extended indefinitely, meet on the side on which the angles are less than two right angles.

The fifth postulate, which is seemingly the most complex one, is called *the parallel postulate*, as a pair of *parallel* lines is interpreted as two lines that do not intersect. Given the other four postulates, the postulate is equivalent to *Playfair's axiom*,[1] which has a simpler formulation:

- Given a line and a point not on the line, there is at most one line through the point that is parallel to the given line.

The parallel postulate was for long suspected of being superfluous in Euclid's axiomatic system and hence there were numerous attempts to decude it from the other four postulates. [Cannon et al., 1997] and other sources lists many mathematicians who attempted this, beginning as early as the fifth century.[2] By assuming that the postulate was false and looking for a contradiction, they discovered many interesting and counterintuitive results. The following is a brief discussion of the most well-known attempts (for more details see [Gray, 2007, pages 82-88]):

- The Italian GEROLAMO SACCHERI (1667-1733) showed in the year of his death that one of the following statements must be true:

  1. Every triangle has angle sum less than π.
  2. Every triangle has angle sum equal to π.
  3. Every triangle has angle sum greater than π.

  Saccheri proved that the third statement leads to a contradiction under Euclid's first four postulates. However, his proof of the falseness of the first statement (Theorem 3.4) was flawed. The second statement can be shown to be equivalent to the parallel postulate, so if Saccheri's proof had been correct, he would have succeeded in his task of proving the parallel postulate.

[1]Those axioms are not equivalent in general however, since Playfair's axiom posits uniqueness. Spherical geometry is a counterexample. See [Taimina and Henderson, 2005] for further discussion.

[2]One of the first documented attempt was by the Greek philosopher PROCLUS (412 - 485 AD), who wrote a commentary on Euclid's work in which he attempted to prove the parallel postulate using an assumption that ultimately could not be proven by the other four postulates. Note: [Adler, 1987] says that attempts to prove the postulate began with PTOLEMY (367 - 283 BC), but Adler apparently confuses Ptolemy I Soter, a Macedonian general, with Claudius Ptolemy, a Greek mathematician who was born much later but whose work is mentioned by Proclus, see the English translation of Proclus in [Proclus, 1970].

- The Swiss JOHANN LAMBERT (1728-1777) showed many interesting geometric results as a result of attempting to prove the parallel postulate, some of which are considered in the next section.[3] Unlike Saccheri, Lambert acknowledged that he could find no contradiction by assuming the negation of the parallel postulate, but instead he protested on somewhat philosophical grounds: if the angle sum of every triangle is indeed less than π, then there exists an absolute measure of length (like for angles, which is 2π). Evidently Lambert considered this a logical absurdity, quoting the Latin phrase *quantitas dari sed non per se intelligi potest* (there can not be a quantity known in itself) in his essay *Theorie der Parallellinen*,[4] published in 1786, but in the same essay Lambert admits that this belief must be amended.

- The French ADRIEN-MARIE LEGENDRE (1752-1833) made many contributions to mathematics, for example the Legendre transformation, but he was nevertheless one of many who constructed an erroneous proof of the parallel postulate by showing that the angle sum of a triangle equals π. Legendre's proof was published in his textbook *Éléments de géométrie* in 1794 which was used widely in France for many years.[5]

It was not until the 19th century when mathematicians abandoned these efforts for reasons which will now be explained. Consider an axiomatic system that includes Euclid's first four postulates but replaces the fifth one with the following:

Axiom 2.1 (The hyperbolic axiom)**.** Given a line and a point not on the line, there are *infinitely* many lines through the point that are parallel to the given line.

A consistent model of this axiomatic system implies that the parallel postulate is logically independent of the first four postulates. Deep and independent investigation by JÁNOS BOLYAI (1802-1860) from Hungary and

[3]Due to Lambert's great number of discoveries, [Penrose, 2004, see page 44] suggests that he should be given credit as the first person to construct a non-Euclidean geometry. This is inaccurate since at least one non-Euclidean geometry, spherical geometry, has been known since ancient times. See further discussion in [Taimina and Henderson, 2005].

[4]See an English translation of the selected passage in [Gray, 2007, pages 86-88].

[5]His proof can be found in a 19th century English translation of the work, see [Legendre, 1841, pages 13-15].

Nikolai Lobachevsky (1793-1856) from Russia led them conclude that this axiomatic system, which we today call *hyperbolic geometry*, was seemingly consistent, hence these two mathematicians have traditionally been given credit for showing the logical independence of the parallel postulate and for the discovery of hyperbolic geometry.[6]

Hyperbolic geometry is an imaginative challenge that lacks important features of Euclidean geometry such as a natural coordinate system. Its discovery had implications that went against then-current views in theology and philosophy, with philosophers such as Immanuel Kant (1724-1804) having expressed the widely-accepted view at the time that our minds will impose a Euclidean structure on things *a priori*,[7] meaning essentially that the existence of non-Euclidean geometry is impossible. Only with the work of later mathematicians, hyperbolic geometry found acceptance, which occurred after the death of both Bolyai and Lobachevsky.

2.2 Generalizations and consistency

So far we have only seen the synthetic basis of hyperbolic geometry. Later in the 19th century, mathematicians developed an analytic understanding of hyperbolic geometry with the study of curved surfaces. By generalizing the subject, mathematicians could prove that hyperbolic geometry was just as consistent as Euclidean geometry, which early 19th century mathematicians could not do since they lacked the proper tools. Now we will discuss this history.

We begin by introducing the notion of *Gaussian curvature*, which informally implies how a surface "bends" in a point x, denoted $\kappa(x)$. Define geodesic distance as distance travelled on a particular surface. Then a geodesic circle of radius r and with centre at a point x is the collection of all points on a surface whose geodesic distance from x equals r. Denote its area as $A(r,x)$. The Gaussian curvature of a point x on a surface is then the limiting difference of $A(r,x)$ and the area of a circle tangent to the surface at x as the radii approach 0.

[6]Another person associated with the discovery of hyperbolic geometry is Carl Friedrich Gauss (1777-1855), having worked on his ideas in private, see longer discussion of this for example in [Milnor, 1982].

[7]See page 248 in [Trudeau, 2001].

Definition 2.2. The Gaussian curvature at a point x on a surface is

$$\kappa(x) = \lim_{r \to 0+} 12 \frac{\pi r^2 - A(r,x)}{\pi r^4}.$$

The above formula is known as the Bertrand–Diquet–Puiseux theorem.[8] We chose it for its simplicity and since it compares neatly to the definition of the curvature at a point of a curve: the deviation from its tangent. A fundamental theorem by Gauss, *Theorema Egregium*, published in 1827, says that Gaussian curvature is intrinsic which means that it does not depend on how the surface is embedded into Euclidean 3-space.[9]

Mathematician Bernhard Riemann (1826-1866), who was a professor at Göttingen university like Gauss, expanded upon Theorema Egregium in a famous lecture in 1854. Riemann showed that two surfaces have different geometries if they have different Gaussian curvature at any point, and by generalizing Gaussian curvature to higher dimensions, that this was also valid for n-dimensional manifolds in n+1-space. Hence there is an infinite number of different geometries; not only one for each different (non-isomorphic) manifold, but also one for every definition of distance on such a manifold. This means that Euclidean geometry is not particularily relevant anymore; the whole subject of geometry can be reduced to the analysis of n-dimensional *Riemannian manifolds* and their intrinisic properties.

It is now worth noting the important fact that the surface of constant negative Gaussian curvature, called *pseudosphere*, admits hyperbolic geometry. This has been known at least since the 1830s, when German mathematician Ferdinand Minding (1806-1885) analyzed the trigonometry on such a surface.[10] In his 1854 lecture, Riemann also discussed this by stating that the angle sum of triangles on such a surface is always less than π, a fact that we know from synthetic geometry. Outside of this, Riemann avoided discussing non-Euclidean geometry as such, perhaps because the mathematical community was still hesitant about the subject.[11]

[8] First described in 1848 by its eponymous authors in the French journal *Journal de mathématiques pures et appliquées* (which can be found online). Gaussian curvature is generally defined as the product of *principal curvatures*, the formula chosen is an alternate definition.

[9] See more for example in [Gray, 2007].

[10] An English translation of Minding's papers seems not to be available, but this is discussed for example in [Stillwell, 1996, page 2].

[11] More information about Riemann's theories and an excerpt of an English translation from his 1854 lecture can be found in [Gray, 2007].

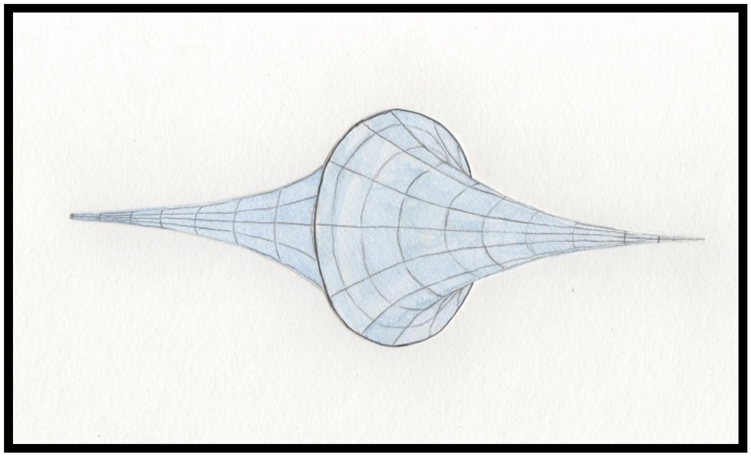

Figure 1: A *tractricoid*, which is an example of a pseudosphere. Note that as pseudosphere will necessarily contain singularities; no complete and regular surface of constant negative curvature exists in Euclidean 3-space by Hilbert's theorem (1901).

It was up to the Italian Eugenio Beltrami (1835-1899) to continue Riemann's work. In two papers published in 1868,[12] Beltrami considered models on the unit disk and the upper half-plane, to be analyzed in the fourth section of this paper, and showed their geometry fulfilled the axioms for hyperbolic geometry. Today we say that Beltrami's models establish that hyperbolic geometry is just as consistent as Euclidean geometry since Beltrami's models are defined entirely in Euclidean terms (note that at the time that Beltrami published his papers the consistency of Euclidean geometry was unquestioned). Beltrami's nevertheless did not set out to establish consistency of hyperbolic geometry, his modest aim was only to express Lobachvesky's ideas without "the necessity for a new order of entities and concepts" as he explains himself in his first paper [Stillwell, 1996, page 7].[13]

[12]These papers exist in English translation in [Stillwell, 1996], which we use here. There is also a very useful discussion of the papers in [Arcozzi, 2012].

[13]Some authors, such as Saul Stahl in *The Poincaré Half-plane: A Gateway to Modern Geometry*, state inaccurately that Beltrami used the surface of the pseudosphere to establish the consistency of hyperbolic geometry. Quoting Stahl, page 50: "In the first of the two papers published that year Beltrami pointed out that the trigonometry of the geodesics of the pseudosphere, a surface of Euclidean geometry that Minding had investigated as far back as 1840, was identical with the trigonometry of the hyperbolic plane. Consequently any self-contradiction that might arise in hyperbolic geometry would of necessity also

In Beltrami's second paper, he proves a remarkable result first noted by Lobachevsky: Euclidean geometry is contained within hyperbolic geometry by means of so-called *horospheres*, hence also making Euclidean geometry as consistent as hyperbolic geometry.[14] Horospheres are spheres with centre at infinity and Euclidean geometry can be realized on them if we interpret lines as horocycles (i.e. one-dimensional horospheres). This means that for inhabitants of hyperbolic space, Euclidean geometry would come as natural as spherical geometry comes for those who live in Euclidean space. As a matter of fact, Beltrami notes in the last sentence of his paper that elliptic (positive constant-curvature) geometry is also contained in hyperbolic space, making hyperbolic geometry the only geometry that contains all the constant-curvature geometries.[15]

We conclude this section by going back to Euclid and the re-evaluation of his work that happened two decades after Beltrami wrote his papers.[16] Euclid's *Elements* is not a rigorous work by the mathematical standards that were established in the 19th century. Definitions such as "a point is that which has no part" are not meaningful from a modern point of view and notions such as "betweenness" are left undefined by Euclid. In 1899, German David Hilbert (1862-1943) published *Grundlagen der Geometrie* in which he re-axiomatized Euclidean geometry.[17] With this work, Euclidean and non-Euclidean geometry is finally given some needed mathematical rigour, while avoiding the axiomatic theory of real numbers. According to [Greenberg, 2010, page 198], this is because the axiomatic theory of real numbers is somewhat controversial (for example due to the independence of the con-

constitute a self-contradiction of Euclidean geometry. In other words, Beltrami proved that that hyperbolic geometry was just as consistent as Euclidean geometry." However, embedding a part of the hyperbolic plane into Euclidean space is not enough to establish consistency of hyperbolic geometry, but Beltrami's model are.

[14] Note that the full proof of this was not shown until 1995 by Ramsey and Richtmyer, according to [Greenberg, 2010, page 213-214]. Beltrami's proof only applies to the Euclidean plane.

[15] By a thesis of David Brander called *Isometric Embeddings between Space Forms* (2003), see section 5.2, spherical spaces cannot contain hyperbolic or Euclidean spaces for topological reasons. The thesis is available online.

[16] In the meantime, Felix Klein (1849-1925) showed that hyperbolic geometry is equiconsistent with projective geometry and Henri Poincaré (1854-1912) showed that hyperbolic geometry had applications in for example complex analysis and number theory, see translations of their original papers in [Stillwell, 1996].

[17] See a translation in [Hilbert, 1950].

tinuum hypothesis) and because it is better to avoid using such complicated tools if they are not strictly necessary. This is possible because geometry, as envisioned by Hilbert, is in a sense simpler than the theory of the real numbers.

There are nevertheless some limitations to what we can do with those kinds of axiomatizations of Euclidean geometry. Again quoting Greenberg, if we consider the "elementary" theory of plane geometry to be the theory of the geometry of straightedge and compass constructions on the plane, then we have that the elementary theory of the Euclidean plane is *undecidable*. By that, we mean that there exists no algorithm to decide if an arbitrary assertion is provable. The proof comes from the fact that each Euclidean plane is isomorphic to a field and any finitely axiomatized first-order theory of fields with $\mathbb{R}$ as a model is undecidable.[18]

In 1926-1927, Polish mathematician Alfred Tarski (1901-1983) constructed an axiomatic system for Euclidean geometry which is entirely in first-order logic and has the following attributes: it is decidable and it is complete (every assertion can be proved or refuted) [Tarski and Givant, 1999]. On the other hand, Tarski's axiomatization carefully avoids speaking much about arithmetic in order to avoid the criteria for Gödel's incompleteness theorems[19] and as a consequence lacks some expressive power (according to Greenberg, Tarski referred to this as elementary Euclidean geometry). Today we consider both Hilbert's and Tarski's axiomatizations to have some merits of their own, but in the rest of the text we will mostly refer to Hilbert's axiomatization.

3 Observations on the hyperbolic plane

In the previous section, we mentioned some well-known results from the axiomatic treatment of hyperbolic geometry. In this section, we will derive those results and others without resorting to a specific model of hyperbolic geometry. For synthetic results, we will use Hilbert's axiom system that is constructed with the three primitive terms *point*, *line* and *plane* and the three primitive relations *betweenness*, *containment* (e.g. line containing a point) and *congruence* (denoted $\cong$, a relation between line segments and

[18]This is according to [Greenberg, 2010, page 214] who cites M. Ziegler.

[19]By Gödel's incompleteness theorems (1931), Tarski's axiomatic system could not be both complete and decidable if it also contained some notion of arithmetic.

a relation between angles), but assume the negation of Hilbert's *Euclid's postulate* (which is analogous to the parallel postulate). As this section only gives a rough overview, we will not list Hilbert's 21 axioms here of Euclidean geometry but they can be found in [Hilbert, 1950].

Before proceeding further, we explain three notations from Euclidean geometry that may not be familiar to all readers:

1. *Foot* f of some point x for a line l refers to the point f on a line l so that the two lines l, and the line determined by f and x, are perpendicular.

2. The notation $A - B - C$ means that the point C is *between* the points A and C. $A - B - C - D$ implies that B is between A and C, and C is between B and D.

3. $\angle$ABC means the angle determined by the vertex at B and the line segments AB and BC.

3.1 Triangles, polygons and circles

We begin with two definitions. The second definition makes assumptions that we will not prove here.

Definition 3.1. A **Saccheri quadrilateral** is a four-sided polygon $ABCD$ with two equal sides, AD and BC, perpendicular to its base AB. The angles at C and D are congruent and are called the **summit angles** of the quadrilateral.

Definition 3.2. On the hyperbolic plane, given a line l and a point p not contained by l, there are two parallel lines to l that contains p and move arbitrarily close to l in two directions which we call left and right. Those lines are called the **left** and **right limiting parallels** to l through p. We say that these limiting parallels meet l at infinity. Let a be the foot of p on l, and let p_L and p_R be the points at infinity on the left and right limiting parallels, respectively, where l and the limiting parallels meet. Then we call the triangle with vertices at p, a and p_L (p_R) the **left** (**right**) **limit triangle** for l and p.

To prove the next theorem, which was initially done by Saccheri, we use a few results from absolute geometry[20] whose proofs we will omit here. This

[20] Absolute geometry is an incomplete axiomatic system that is neutral with respect to the parallel postulate or Hilbert's version thereof.

includes the AAS theorem, which states that two triangles are congruent if they have two congruent angles and a corresponding congruent side that is not between the angles, and the exterior angle theorem, stating that that for any angle of a triangle, its exterior angle will be greater than the sum of its other two angles. Note that to speak of angles (or line segments) being *greater* than each other, we preferably need to notion of *measure* (denoted $m\angle ABC$ for the angle at $\angle ABC$). We do not expand on this here but instead refer to [Greenberg, 1993] for full development and proofs of the aforementioned theorems.

Theorem 3.3. *In hyperbolic geometry, the summit angles of a Saccheri quadrilateral are acute.*

Proof. Let the Saccheri quadrilateral $ABCD$ be given. Consider the unique line l that contains both the points A and B (Hilbert's first and second axioms). We construct two left limit triangles: one for D and l (with vertices at D, A and a point α at infinity) and one for C and l (with vertices at C, B and α). As $\angle B\alpha C \cong \angle A\alpha D$, $\angle CB\alpha \cong \angle DA\alpha$ and $BC \cong AD$, the AAS theorem from absolute geometry gives us that $\angle \alpha CB \cong \angle \alpha DA$, implying $m\angle \alpha CB = m\angle \alpha DA$. Next construct a triangle with vertices at C, D and α. Let E be some point so that D is between E and C (Hilbert's order axioms) and then by the exterior angle theorem, $m\angle ED\alpha > m\angle EC\alpha$, so $m\angle ED\alpha + m\angle \alpha DA > m\angle EC\alpha + m\angle \alpha CB$ which implies that $m\angle EDA > m\angle ECB = m\angle DCB$ and thus $m\angle EDA + m\angle CDA = \pi > m\angle DCB + m\angle CDA$. As $m\angle DCB = m\angle CDA$, we get that $m\angle DCB = m\angle CDA < \pi/2$. □

As a consequence of Theorem 3.3, no rectangles exist in hyperbolic geometry. This is because rectangles are Saccheri quadrilaterals with right summit angles. This also means that *Lambert quadrilaterals*, which are quadrilaterals with three right angles, must have a fourth acute angle.

The next theorem characterizes hyperbolic geometry. Again we assume the AAS theorem in our proof, compare with [Hartshorne, 2000, page 310].

Theorem 3.4. *The angle sum of a hyperbolic triangle is less than* π.

Proof. Let the triangle ABC be given and let D and E be the midpoints of AB and AC, whose existence are guaranteed by Hilbert's axioms. Let l be the unique line that contains D and E. Let M, F and G be the of A, B, C, respectively, on l. We have that $m\angle BFD = m\angle AMD, m\angle BDF = m\angle ADM$ and $BD = AD$ so by the AAS theorem the triangles BFD and

AMD are congruent and thus $BF = AM$. We can show in the same way that the triangles CGE and AME are congruent, so $CG = AM$. Therefore we get that $BF = CG$, so $FGCB$ is a Saccheri quadrilateral with base FG. Now we consider two cases:

1. M is within ABC, meaning that a ray beginning at any vertex of the triangle and containing M will intersect the triangle. For the first summit angle in $FGCB$, we have $m\angle FBC = m\angle FBA + m\angle ABC = m\angle BAM + m\angle ABC$. Likewise for the second summit angle we get $m\angle GCB = m\angle GCA + m\angle ACB = m\angle CAM + m\angle ACB$. The sum is $m\angle FBC + m\angle GCB = m\angle BAM + m\angle ABC + m\angle CAM + m\angle ACB = m\angle BAC + m\angle ABC + m\angle ACB$.

2. M is not within of ABC. Assume that M is to the right of E without loss of generality. As before we have that $m\angle FBC = m\angle BAM + m\angle ABC$, we need to recalculate the next term $m\angle GCB = m\angle ACB - m\angle ECG = m\angle ACB - m\angle EAM$ but their sum is the same as before $m\angle FBC + m\angle GCB = m\angle BAM + m\angle ABC + m\angle ACB - m\angle EAM = m\angle ABC + m\angle ACB + m\angle BAC$.

In both cases we have that angle sum of ABC equals the measure of the summit angles of the Saccheri quadrilateral, which we showed in Theorem 3.3 to be less than π.

□

The difference between the angle sum of a triangle and π is hence non-zero and is called *the angular defect* of the triangle. We can generalize this notion to quadrilaterals and in fact to any polygons.

Definition 3.5. The **angular defect** of a polygon with n sides is the number $(n-2)\cdot\pi -$ (the angle sum of the polygon).

The angular defect of any hyperbolic convex polygon is positive; in other words, the angle sum of the n-gon is less than $(n-2)\cdot\pi$. We can show this inductively: in the base case, a convex quadrilateral (4-gon) can be divided in two triangles by either of its diagonals, each with angle sum less than π, so the angle sum of the quadrilateral will be less than 2π. In general, a convex n-gon can be divided into a $(n-1)$-gon and a triangle by drawing a line between the neighbours of any vertex, and as the triangle has angle sum less than π and the $(n-1)$-gon will have an angle sum less than $(n-3)\cdot\pi$, the n-gon will have an angle sum less than $(n-3)\cdot\pi+\pi=(n-2)\cdot\pi$.

We will now say a few things about the area of geometric objects on the hyperbolic plane. To do so, we need to introduce some differential geometry. Recall from the previous section that we can consider the hyperbolic plane to be a two dimensional Riemannian manifold with a constant negative Gaussian curvature (Definition 1.1). The following theorem is described by [Weisstein, 2005], its proof relies on Green's theorem and will not be given here. Note that Riemannian manifolds, which were discussed in the second section, are not defined rigorously in this paper. If the reader is unfamiliar with the term, one can think of the theorem as speaking of triangles on surfaces in 3-dimensional space.

Theorem 3.6 (Gauss-Bonnet theorem for triangles)**.** *If M is a two-dimensional Riemannian manifold with an embedded triangle T then*

$$\iint_T K\,dA = 2\pi - \int_{\delta T} \kappa_g\,ds$$

where K is Gaussian curvature, dA is the area measure for T, κ_g is the geodesic curvature of the boundary δT and ds the arc measure of T.

We apply the theorem to triangles of hyperbolic geometry.[21] The first thing we notice is that the area of the triangle will depend on what we choose for the Gaussian curvature, which can in our case be any negative number. The axioms for the hyperbolic plane are not enough to uniquely determine this curvature constant and hence a unique measure for area (or distance), therefore the axioms do not characterize the hyperbolic plane up to isomorphy.[22] We follow tradition and let the Gaussian curvature of hyperbolic space here and in the rest of the text equal -1. Then we get that $\iint_T K\,dA = -A$. Next we evaluate the line integral of the triangle. T is piecewise smooth and the line integral equals the integrals of the straight segments of T plus the sum of its exterior angles, call the angles α, β and γ. The integrals for the straight segments equal 0 so we will have that the line integral equals $\pi - m\angle\alpha + \pi - m\angle\beta + \pi - m\angle\gamma$. Thus by Theorem 3.6 $-A = 2\pi-(\pi-m\angle\alpha+\pi-m\angle\beta+\pi-m\angle\gamma)$ so $A = \pi-(m\angle\alpha+m\angle\beta+m\angle\gamma)$. We summarize this:

[21]Note that to understand fully how we can evaluate the line integral, some familiarity with integral calculus is required.

[22]The axioms will however characterize the hyperbolic plane only up to *homothety*, meaning the measure of distance will only differ by a constant.

Theorem 3.7 (Gauss-Bonnet formula)**.** *A hyperbolic triangle with angles* α*,* β *and* γ *has the area* $\pi - (m\angle\alpha + m\angle\beta + m\angle\gamma)$ *which is precisely its angular defect.*

That the area of a hyperbolic triangle is proportional to angle defect is a result first discovered by Lambert [Gray, 2007, page 84]. Just as we used the angular defect of a triangle to find the angular defect of a polygon, we can use the area of a hyperbolic triangle to find the area of a hyperbolic polygon.

Theorem 3.8. *The area of a convex polygon on the hyperbolic plane with* n *sides equals its angular defect.*

Proof. Choose any interior point of the polygon and draw lines from this point to the vertices of the polygon, dividing it into n triangles. The area of the polygon will equal the sum of the area of the triangles, which will in total be the difference between $n{\cdot}\pi$ and the angle sum of the polygon, minus the angles around the interior point, that is $n{\cdot}\pi -$(the angle sum of the polygon)$-$ $2\pi = (n-2)\cdot\pi -$ (the angle sum of the polygon), which is the polygon's defect. □

We will conclude this section by using the above theorem to obtain a formula for the area and circumference of a hyperbolic circle, but first we need to derive some hyperbolic trigonometry. We noted in the last section that hyperbolic geometry has an absolute measure of length. We will now clarify what we mean by this by following [Lobachevsky, 2010]. Take a limit triangle for a line l and a point p. The parallelism function Π returns the acute angle that the segment from p to its foot on l makes with the limiting parallel in the triangle. Line segments of the same length are congruent so Π only depends on the length L of the segment, not the segment itself, so the function has positive values as domain. We can extend it to non-postive values as well by adding the formula $\Pi(x) + \Pi(-x) = \pi$ to its definition. Lobachevsky showed by constructions involving horocircles and limit triangles that $(\tan\frac{1}{2}\Pi(x))^n = \tan\frac{1}{2}\Pi(nx)$ where n can be negative and fractional and x is the length of some segment. By choosing $\tan\frac{1}{2}\Pi(1) = e^{-1}$, which corresponds to -1 being our curvature constant, the fact that $(e^x)^n = e^{nx}$ helps us find the solution $\tan\frac{1}{2}\Pi(x) = e^{-x}$. This leads to the following equations

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{\tan\Pi(x)},$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{\sin \Pi(x)},$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \cos \Pi(x).$$

The functions sinh, cosh and tanh are called the *hyperbolic functions*. One of their many properties is that sinh and cosh parametrize the unit hyperbola. The hyperbolic functions satisfy various identities, proven by Lobachevsky, which resemble the trigonometric identities of Euclidean geometry. We state now the three main identities for hyperbolic right triangles without proof.[23]

Theorem 3.9. *For any right hyperbolic triangle with edges a, b and c, opposite angles A, B and the right angle C respectively, we have*

1. $\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}$ *(The hyperbolic law of sines)*

2. $\cos(A) = \frac{\tanh(b)}{\tanh(c)}$ *(Hyperbolic cosine law I)*

3. $\cos(B) = \frac{\tanh(a)}{\tanh(c)}$ *(Hyperbolic cosine law II)*

We are now in a position to prove the last theorem of this section.

Theorem 3.10. *A hyperbolic circle with with radius r has the circumference $2\pi \sinh(r)$ and the area $2\pi(cosh(r) - 1)$.*

Proof. Construct a regular polygon on the hyperbolic plane with n sides and a midpoint A so that the length of the line segment from A to any vertex equals r. We see that the polygon is composed of n congruent triangles that have the vertices x and two neighbouring vertices of the polygon. Take any such triangle and divide it into two congruent triangles by bisecting the angle at A. We take one of those triangles and call the vertex it has in common with the polygon B, the vertex at which it has a right angle C and the sides a, b and c opposite its respective angles A, B and C. Clearly the angle at A is π/n and $c = r$. By the hyperbolic law of sines, we have that $\sin(A)/\sinh(a) = \sin(C)/\sinh(c)$ so $\sinh(a) = \sin(\pi/n) \cdot \sinh(r)$. We now

[23] The identities are stated by [Lobachevsky, 2010] as this: 1. $\sin A \tan \Pi(a) = \sin B \; tan\Pi(b)$ (page 26) and 2. $\cos \Pi(c) \cos B = \cos \Pi(a)$ (page 25), which are equivalent formulations.

note that $a = \frac{p}{2n}$ where p is the perimeter of the polygon. As n goes to infinity, the perimeter of the polygon will approach that of a circle of radius r, so we subsitute a and calculate

$$\sinh(\tfrac{p}{2n}) = \sin(\pi/n) \cdot \sinh(r) \Rightarrow n \cdot \sinh(\tfrac{p}{2n}) = n \cdot \sin(\tfrac{\pi}{n}) \cdot \sinh(r)$$
$$\Rightarrow \lim_{n\to\infty} n \cdot \sinh(\tfrac{p}{2n}) = \lim_{n\to\infty} n \cdot \sin(\tfrac{\pi}{n}) \cdot \sinh(r).$$

To evaluate the limits $\lim_{n\to\infty} n \cdot \sin(\frac{\pi}{n})$ and $\lim_{n\to\infty} n \cdot \sinh(\frac{\pi}{n})$, we find the Taylor expansions of $\sin(\frac{\pi}{n})$ and $\sinh(\frac{p}{2n})$, which are $\frac{\pi}{n} - \frac{\pi^3}{n^3 \cdot 3!} + \frac{\pi^5}{n^5 \cdot 5!} - \cdots$ and $\frac{p}{2n} + \frac{p^3}{(2n)^3 \cdot 3!} + \frac{p^5}{(2n)^5 \cdot 5!} + \cdots$ respectively. As we multiply them both with n and take the limit as n approaches infinity, we see that the limits equal π and $\frac{p}{2}$ respectively. Thus our equation reduces to $\frac{p}{2} = \pi \cdot \sinh(r)$, that is $p = 2\pi \cdot \sinh(r)$, which is what we wanted to prove.

Now we calculate the area of the polygon. By Theorem 3.5, the area of the polygon equals its angle defect, and by considering the same triangle construction as before the angle defect equals $(n-2)\cdot\pi - n(2B)$. Thus we need to find the angle at B. We have that the the side $a = \frac{p}{2n} = \frac{2\pi \cdot \sinh(r)}{2n} = \frac{\pi \cdot \sinh(r)}{n}$. By hyperbolic cosine law II, $\cos(B) = \frac{\tanh(a)}{\tanh(c)} = \tanh\left(\frac{\pi \sinh(r)}{n}\right) \Big/ \tanh(r)$, so $B = \cos^{-1}\left(\tanh\left(\frac{\pi \sinh(r)}{n}\right) \Big/ \tanh(r)\right)$. When the number of sides of the polygon approaches infinity, its area will be that of a circle of radius r, so we calculate the limit

$$\lim_{n\to\infty} (n-2) \cdot \pi - n \left(2\cos^{-1}\left(\frac{\tanh\left(\frac{\pi \sinh(r)}{n}\right)}{\tanh(r)} \right) \right)$$
$$= \left(\lim_{n\to\infty} \left(n \left(\pi - 2\cos^{-1}\left(\frac{\tanh\left(\frac{\pi \sinh(r)}{n}\right)}{\tanh(r)} \right) \right) \right) \right) - 2\pi.$$

By using the substitution $n = 1/t$, this equals

$$\left(\lim_{t\to 0} \frac{\pi - 2\cos^{-1}\left(\frac{\tanh(\pi t \sinh(r))}{\tanh(r)}\right)}{t} \right) - 2\pi. \qquad (*)$$

The numerator approaches $\pi - 2\cos^{-1}(\tanh(0)/\tanh(r)) = \pi - 2\cos^{-1}(0) = \pi - 2(\pi/2) = 0$ and so does the denominator. We can use L'Hopital's rule as all the criteria we need is met. Set $u = \frac{\tanh(\pi t \sinh(r))}{\tanh r}$. Then the derivate of the numerator with respect to t equals $\frac{d}{dt}(\pi - 2\cos^{-1}(u))$ which by the chain rule equals $-2\frac{d(\cos^{-1}(u))}{du}\frac{du}{dt} = 2\frac{1}{\sqrt{1-u^2}}\frac{du}{dt}$. By using the chain rule again, we get that

$$\frac{d}{dt}\tanh(\pi\sinh(r)\cdot t) = \pi\sinh(r)\cdot\frac{d}{d(\pi\sinh(r)\cdot t)}\tanh(\pi\sinh(r)\cdot t)$$

$$= \frac{\pi\sinh(r)}{\cosh(\pi\sinh(r)\cdot t)}$$

and thus

$$2\frac{1}{\sqrt{1-u^2}}\frac{du}{dt} = 2\frac{1}{\sqrt{1-\left(\frac{\tanh(\pi t\sinh(r))}{\tanh r}\right)^2}}\cdot\frac{d}{dt}\frac{\tanh\left(\pi t\sinh(r)\right)}{\tanh r}$$

$$= 2\frac{\pi\sinh(r)/\cosh(\pi\sinh(r)\cdot t)}{\tanh(r)\sqrt{1-\left(\frac{\tanh(\pi t\sinh(r))}{\tanh r}\right)^2}} = \frac{2\pi\cosh(r)}{\cosh(\pi\sinh(r)\cdot t)\sqrt{1-\left(\frac{\tanh(\pi t\sinh(r))}{\tanh r}\right)^2}}.$$

Therefore (*) equals

$$\left(\lim_{t\to 0}\frac{2\pi\cosh(r)}{\cosh(\pi\sinh(r)\cdot t)\sqrt{1-\left(\frac{\tanh(\pi t\sinh(r))}{\tanh(r)}\right)^2}}\right) - 2\pi$$

$$= \frac{2\pi\cosh(r)}{\cosh(0)\cdot\sqrt{1-\frac{\tanh(0)}{\tanh(r)}}^2} - 2\pi = \frac{2\pi\cosh(r)}{1\cdot\sqrt{1-0}^2} - 2\pi = 2\pi\cosh(r) - 2\pi$$

$$= 2\pi(\cosh(r) - 1)$$

which is therefore the area of a hyperbolic circle of radius r.

□

4 Models of hyperbolic geometry

As mentioned in the second section, the fact that hyperbolic geometry is as consistent as Euclidean geometry was proved by considering models of the former within the latter. So far we have avoided a detailed discussion of those models; as seen in the last section it is possible to derive many interesting results only by working from the axioms. In this section however, we will introduce and discuss three of the common models. Two of them were discovered by Beltrami but are named after Klein and Poincaré, a convention that [Stillwell, 1996, page 35] calls "one of the injustices of nomenclature that are so common in mathematics".[24] In this paper, we will not ascribe these two models to any authors but instead call them *the projective disk model* and *the conformal disk model*. The third, *the hyperboloid model*, was discovered later and has special importance, being related to special relativity through Minkowski space as we will discuss at the end.

For each model, we will verify that it indeed describes Hilbert's axiomatization of hyperbolic geometry, using Hilbert's axioms for Euclidean geometry with the hyperbolic axiom in place of Hilbert's Axiom of Parallelism. As it is trivial to verify most of Hilbert's axioms on the th models, we will only consider the following:

- Hilbert's first two axioms, which taken together state that any two points determine a line.
- The hyperbolic axiom, see Axiom 2.1.

When appropriate, we will make use of our knowledge of Euclidean geometry to verify the above, as all of the models are defined in Euclidean terms. One technique to do so is to project from one model to another and here we assume that the reader is familiar with *orthogonal* and *stereographic* projections from linear algebra, but if not then Figure 2 and 4 may assist with visualisation.

We will also define a notion of distance on each model, if only so that the notion of two congruent line segments makes some sense.

[24]Stillwell goes further and suggests that the models in question should be called the Cayley-Beltrami and Riemann-Beltrami models. The third model that was discovered by Beltrami, commonly called the Poincaré half-plane model, Stillwell suggests that should be called the Liouville-Beltrami model. We will not adopt this suggestion here.

4.1 The projective disk model

We begin with the simplest and the earliest model, *the projective disk model*, also called the Beltrami-Klein model.[25] In n-dimensions, the model is the set of points within an n-dimensional unit ball

$$B^n = \{(x_1, x_2, \cdots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$$

with lines represented as straight chords, i.e. line segments with endpoints on the boundary sphere

$$\delta B^n = \{(x_1, x_2, \cdots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

To avoid confusion, we shall refer to lines and points within the model as hyperbolic lines and hyperbolic points. Lines and points in the usual understanding will be called Euclidean lines and points. Hyperbolic points on δB^2 are not in a strict sense a part of the model as they represent points at infinity, which we see better after introducing hyperbolic distance. We call them *limiting points* (compare to the notion from the previous section of limiting parallels and triangles). Clearly a hyperbolic line has two limiting points.

We shall for the most part restrict our analysis to the two-dimensional case B^2, when the model is set in the open unit disk. Here it is straightforward to verify Hilbert's first two axioms; we have that two points on the Euclidean plane determine a line l so if those two points are in B^n, they determine the hyperbolic line that that is equivalent to the Euclidean line segment determined by the limiting points of l.

In the same vein, we can easily verify the hyperbolic axiom on B^2: Take a hyperbolic line h and a hyperbolic point p not on h. Refer to one of h's limiting points as s. There exists a Euclidean line segment l that is parallel in the Euclidean sense to h and contains p. Let l have endpoints on δB^2, then we have that l is a hyperbolic line. Refer to the limiting point of l that is adjacent to s as t. Since s and t are not the same point, there are infinitely many points between s and t on δB^2 (one the arc that does not contain the other endpoint of l). Any of them determines together with p a hyperbolic line that is parallel to h in the hyperbolic sense since they cannot intersect. Thus there is an infinite amount of hyperbolic lines that contain p and do not intersect h, so the proof is complete.

[25] For this model and the next one, our main reference is [Hvidsten, 2012].

Next we need to define distance on B^n, or a *metric*, which is a binary function that defines distance. There are certain natural conditions which such a metric needs to fulfill. For all hyperbolic points x and y, we must have

1. $d(x, y) \geq 0$ with equality only when $x = y$.
2. $d(x, y) = d(y, x)$.
3. (The triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all hyperbolic points z.

The Euclidean metric is not sufficient in our model since it will fail the triangle inequality; this is a consequence of the Pythagorean theorem's failure to hold on the hyperbolic plane as it is equivalent to the parallel postulate. Therefore we need a different metric. We first have to introduce the term *cross-ratio*.

Definition 4.1. The cross-ratio of four collinear points a, b, x, y is

$$[a, b, x, y] = \frac{|a - x| \cdot |b - y|}{|a - y| \cdot |b - x|}$$

when $|s - t|$ is the Euclidean distance between two points s and t.

The cross-ratio is an important tool in projective geometry, from which this model takes its name and where the following metric was first used.

Definition 4.2. Given two hyperbolic points a and b on B^n, let the limiting points of the hyperbolic line which they determine be x and y so that $x - a - b - y$. Then the hyperbolic distance between a and b is given by the metric

$$d(a, b) = \frac{1}{2} |\log[b, a, x, y]|$$

when log signifies the natural logarithm.

The constant $\frac{1}{2}$ could be any positive real number but the one we chose happens to corresponds to -1 as the curvature of space. It is now clear to see what was meant by the boundary sphere being the set of hyperbolic points at infinity. We take an example on B^2. Let a and b be two hyperbolic points on the real number line. If b approaches the point $(1, 0)$ then the hyperbolic distance between those points is $\lim_{x\to 1} \frac{1}{2} \log \frac{s}{(1-x)}$ with s being some positive number as long as a is any point firmly within the disk. This limit approaches positive infinity.

It is trivial to show that our hyperbolic metric fulfills the first two metric conditions that were stated above. Proving the triangle inequality requires more work but a proof can be found in [McMullen, 2002, page 157].

We conclude the discussion about the projective disk model by saying a few things about angles. As lines in this model are straight, the Euclidean notion of an angle is not preserved; otherwise we would have that the angle sum of a triangle is π which goes against what we know about hyperbolic geometry (Theorem 3.4). Thus it is not possible to measure angles on the model with a protractor and the angle formula is in fact very complicated. We will not give this formula here but the following definition illustrates how distorted the notion of an angle is on the model.

Definition 4.3. On B^n, two hyperbolic lines l and m are perpendicular if and only if they fulfill either of the following

1. l is a diameter of the disk and m is perpendicular to l in the Euclidean sense.

2. l is not a diameter of the disk. Then let s be the Euclidean point of intersection of the Euclidean tangents to the circle at the limiting points of l. If the Euclidean extension of the hyperbolic line m meets s, then m and l are perpendicular.

Some algebra which we skip here will show us that this is a symmetric relation. As there are four right angles at the point of intersection of a pair of perpendicular lines in both Euclidean and hyperbolic geometry, we see by the above definition that a right hyperbolic angle can be obtuse or acute in the Euclidean sense on this model.

The complications with angle measurement on the projective disk model is one of the reasons that the next model is often used instead.

4.2 The conformal disk model

The conformal disk model C^n, more commonly called the Poincaré disk model, is similar to the projective disk model as it is the set of points strictly within the unit n-dimensional ball. Lines, angles and distances are however defined differently as we now see.

Definition 4.4. On C^n, a hyperbolic line is either i) a Euclidean diameter of the unit sphere ii) a Euclidean circular arc that meets the endpoints on the boundary sphere at Euclidean right angles.

As in the previous model, the endpoints on the sphere are considered model-wise to be points at infinity and do not belong to the model. Hence all hperbolic lines extend to infinity, which is a property we know they should have. We know from Euclidean geometry the following fact; given two distinct points within a circle c, there is a unique circle d that contains those two points and intersect c at right angles. The circular arc of d within c is the hyperbolic line between the two points in this model, thuse there is a unique hyperbolic line that contains any two distinct points so Hilbert's first two axioms are fulfilled. Therefore we can borrow the same definitions of limiting points from the last model, which enables us to define a very similar metric as before:

Definition 4.5. Given two hyperbolic points a and b on C^n, let the limiting points of the hyperbolic line which they determine be x and y so that $x-a-b-y$. Then the hyperbolic distance between a and b is given by the metric

$$d(a,b) = \left|\log[b,a,x,y]\right|.$$

We will now show that there exists an isomorphism between the projective and the conformal disk models, that is a map that preserves points, lines, angles and the distance function. As usual, we consider the case when $n=2$ so we consider the map from B^2 to C^2. Let the unit sphere be given. Then let the unit disk within the the sphere contain the projective disk. Project orthogonally any hyperbolic point P on the disk from the north pole of the sphere onto the bottom hemisphere, yielding point Q, and then project Q stereographically from the north pole back onto the unit disk, yielding point P' on the conformal disk. We see that P' is a unique point given the point P and we let F be the function that takes the point P and returns P'. As orthogonal and stereographic projections are bijective, so is F, and thus the inverse of F, denoted F', is a well-defined function that maps the conformal disk to the projective disk. If the Cartesian coordinates of P is (x,y), then we have that

$$F(x,y) = \left(\frac{x}{1+\sqrt{1-x^2-y^2}}, \frac{y}{1+\sqrt{1-x^2-y^2}}\right)$$

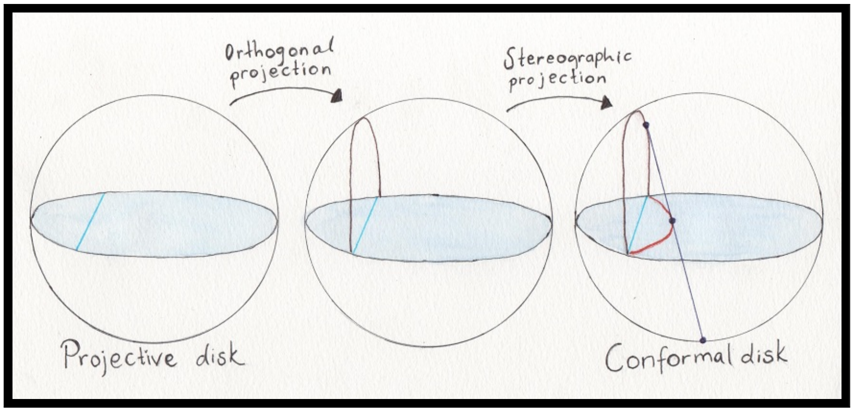

Figure 2

and

$$F'(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2} \right)$$

Clearly F preserves the notions of points between the models. Using the properties of stereographic and orthogonal projections, we will also see that the F preserves the lines and angles, for details see [Hvidsten, 2012]. We will sketch a method that shows the distance function is preserved. Let d_B and d_C be the distance metrics on the projective and conformal disk model respectively. Then we want to show that for two hyperbolic points P and Q on the projective disk, $d_B(P,Q) = d_C(F(P), F(Q))$. Assume without loss of generality that $P = (x_1, 0)$ and $Q = (x_2, 0)$ are on the real number line and that $x_1 < x_2$. Then

$$d_B(P,Q) = \frac{1}{2} \log \frac{|x_2 - (-1)| \cdot |x_1 - 1|}{|x_2 - 1| \cdot |x_1 - (-1)|} = \frac{1}{2} \log \frac{(x_2+1) \cdot (x_1 - 1)}{(x_2 - 1) \cdot (x_1 + 1)}$$

We have that that

$$F(P) = \left(\frac{x_1}{1+\sqrt{1-x_1^2}}, 0 \right), F(Q) = \left(\frac{x_2}{1+\sqrt{1-x_2^2}}, 0 \right)$$

so

$$d_C(F(P),F(Q)) = \log \left| \frac{\left(\frac{x_2}{1+\sqrt{1-x_2^2}}+1\right)\cdot\left(\frac{x_1}{1+\sqrt{1-x_1^2}}-1\right)}{\left(\frac{x_2}{1+\sqrt{1-x_2^2}}-1\right)\cdot\left(\frac{x_1}{1+\sqrt{1-x_1^2}}+1\right)} \right|$$

which after some laborious algebra simplifies to $\frac{1}{2}\log\frac{(x_2+1)\cdot(x_1-1)}{(x_2-1)\cdot(x_1+1)} = d_B(P,Q)$.

Thus we have that the isomorphism between the two models exists, so there is hyperbolic geometry on the conformal disk model. It follows that the hyperbolic axiom is valid here, but we nevertheless sketch a proof (only for C^2). Let a hyperbolic line h and a hyperbolic point p not on h be given. Call the limiting points of h x and y. Then there are two distinct hyperbolic lines determined by x and p, and y and p, that contain p and do not intersect h at any point. Furthermore, there are infinite such lines; let l be the line determined by p and x and let z be its limiting point that is not x. Then any point between z and x on the bounding circle, which are infinitely many, determines with p a line that contains p and does not intersect h.

As the name of the model suggests, the angle measure of the hyperbolic plane is preserved. To find the angle at the point of intersection of two hyperbolic lines, we find the Euclidean tangents to the corresponding circular arcs at the point and calculate the angle determined by the tangents in the usual Euclidean fashion. We now see why we skipped giving an angle formula in the previous section; the easiest way to calculates angles on the projective disk model is to project first to the conformal disk and then use the above procedure.

4.3 The hyperboloid model

As we have mentioned, there were complications in the previous two models; either angles or lines are distorted. Trying to solve this resembles the problem of depicting a globe on a flat map, which is not possible without some distortion. Like in the case of the globe, we can avoid the complications we encountered by considering the geometry on a two-dimensional surface in three-dimensional space, or to generalize, a n-dimensional surface embedded in $n+1$-dimensional space, when the surface has constant negative curvature. But as mentioned in the second section, no such surface can possibly exist in Euclidean 3-space, so in order to find a suitable model for general n-dimensional hyperbolic geometry we need to work in a different inner product space: *Lorentzian n-space*, which we denote as $\mathbb{L}^n$, a real vector space with properties that will now be defined.[26] If x and y are vectors in $\mathbb{L}^n$, their inner product is defined as the real number

$$x \circ y = -x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

.

From there we define the norm of a vector x to be the complex number

$$||x|| = (x \circ x)^{\frac{1}{2}}$$

which, being the square root of a real number, is either real and positive, zero or strictly imaginary. As a sphere with a positive real radius has constant positive curvature on its surface, it should make some sense that a sphere with imaginary radius has constant negative curvature on its surface. This kind of an object exists in Lorentzian space and therefore, for our model of n-dimensional hyperbolic geometry, we would like to take the surface of a $n+1$-dimensional sphere of unit imaginary radius, which is

$$P^n = \{x \in \mathbb{L}^{n+1} : ||x||^2 = -1\}$$

.

which we see that is the set of all the vectors (i.e. points) $x = (x_1, x_2, \cdots, x_{n+1})$ that fulfill the equation $x_1^2 - x_2^2 - \cdots - x_{n+1}^2 = 1$. This is a disconnected set since the equation describes a two-sheeted hyperboloid, so for our model we only take the positive sheet of P^n, that is we require $x_1 > 0$. We denote the model as H^n.

The next step is to define the hyperbolic lines of H^n:

Definition 4.6. A hyperbolic line in H^n is the intersection of H^n with the Euclidean plane spanned by two points on H^n and the origin.

A hyperbolic line is in fact a branch of a hyperbola. Unlike the other models, there is no space at infinity in H^2 so visually, hyperbolic lines go indefinitely far up on the hyperboloid.

We can now verify that Hilbert's first two axioms of hyperbolic geometry are true in this model.

Theorem 4.7. *Any two distinct points on H^2 determine a line.*

[26] The following definitions of the inner product and norm in Lorentzian space and the definition, distance and geodesics of the hyperboloid model follow approximately the development given by [Ratcliffe, 1994].

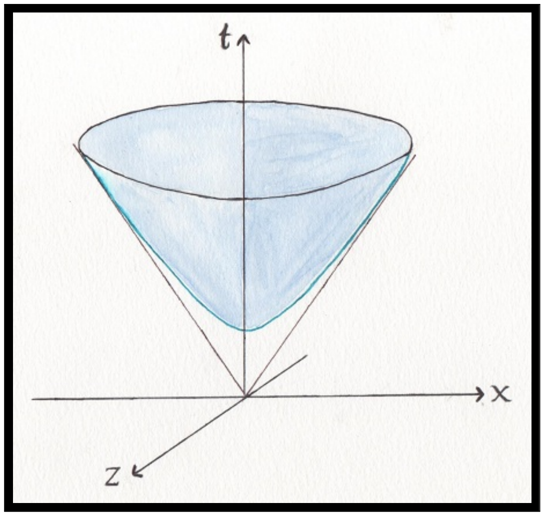

Figure 3: H^2 is the points on this hyperboloid. The hyperboloid is asymptotically bounded by the cone C^2, which is the set of vectors with $x_1 > 0$ and the norm zero.

Proof. We begin by showing that the distinct points x, y on H^2 and the origin are noncollinear (in the Euclidean sense). Assume that they are collinear. As x and y are on H^2, we have that $x = (\sqrt{x_2^2 + x_3^2 + 1}, x_2, x_3)$ and $y = (\sqrt{y_2^2 + y_3^2 + 1}, y_2, y_3)$. Since they are collinear with the origin, x and y are a scalar multiple of each other, so there must exist some real number k so that $k(\sqrt{x_2^2 + x_3^2 + 1}, x_2, x_3) = (\sqrt{y_2^2 + y_3^2 + 1}, y_2, y_3)$ so $k \cdot \sqrt{x_2^2 + x_3^2 + 1} = \sqrt{y_2^2 + y_3^2 + 1}$, $k \cdot x_2 = y_2$ and $k \cdot x_3 = y_3$. By making substitutions we get

$$k \cdot \sqrt{x_2^2 + x_3^2 + 1} = \sqrt{(k \cdot x_2)^2 + (k \cdot x_3)^2 + 1} = \sqrt{k^2 \cdot x_2^2 + k^2 \cdot x_3^2 + 1}$$

which implies that

$$k^2 \cdot (x_2^2 + x_3^2 + 1) = k^2 \cdot x_2^2 + k^2 \cdot x_3^2 + 1$$

$$\iff k^2 \cdot x_2^2 + k^2 \cdot x_3^2 + k^2 = k^2 \cdot x_2^2 + k^2 \cdot x_3^2 + 1 \iff k^2 = 1$$

and as k cannot be negative (since $(x_1 > 0$ per definition), we have that $k = 1$ and hence $x = y$, which is a contradiction. Hence x, y and the origin are

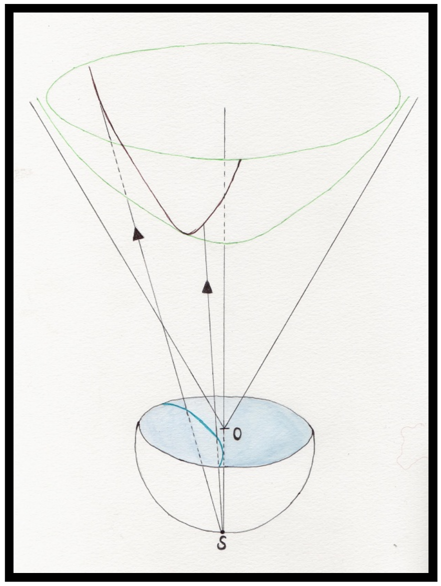

Figure 4

noncollinear and by our knowledge from absolute geometry, this means that those three points determine a plane. Thus this plane's intersection with H^2 gives a unique hyperbolic line determined by the plane which is determined by x and y.

□

On the other hand, it is not easy to verify the hyperbolic axiom of H^2 by using algebra. The simplest way to verify the axiom is by stereographically projecting the points and lines of the hyperboloid onto the conformal disk: Consider the south pole S of the unit sphere and for each point P on the hyperbolid, let P' be the unique point of intersection of the unit disk and the Euclidean line determined by S and P (see Figure 4). This a bijective mapping and it takes the hyperbolic lines of H^n and returns the hyperbolic lines of C^n, and vice versa, and as the hyperbolic axiom is valid on the conformal disk, it must be valid on the hyperboloid as a consequence. The mapping shows that there is an isomorphism between the hyperboloid model and the conformal disk model (and hence the projective disk model) but in order to show that, we also need to verify that angles and the distance functions are preserved between the models (as we did in the last section) but this requires fairly long calculations that we skip here.

As stated earlier, the main advantage of this model is that it is free of distortions. The hyperbolic line segment between two points on H^n is the shortest path on the surface of a hyperboloid in $\mathbb{L}^{n+1}$ and the angle defined by line segments matches the general definition of an angle in Lorentzian space. The former can be realized by using this definition:

Definition 4.8. The Lorentzian distance between two vectors x and y is the complex number $||x - y||$. The hyperbolic distance between hyperbolic points x and y is the real number $d_H(x, y) = \text{arccosh}(-x \circ y)$.

The hyperbolic distance function is a metric since it satisfies all the necessary conditions. Only the triangle inequality is nontrivial to show but a proof using Lorentz transformations can be found in [Ratcliffe, 1994, page 66]. This definition of distance completes the description of the hyperboloid model.

We conclude the paper by discussing briefly the relation of the model to mathematical physics. $\mathbb{L}^4$ is called *Minkowski space-time*[27] and is the space-time setting for special relativity. The vectors of Minkowski space-time are called *events*. An event x is said to be light-like if $||x|| = 0$; the set of such events forms two hypercones, called light cones, defined by the equation $x_1^2 = x_2^2 + x_3^2 + x_4^4$. The events inside the light cones (when $x_1^2 < x_2^2 + x_3^2 + x_4^4$) are said to be time-like, the events outside the light cones are said to be space-like. This is because we think of the x_1 axis as representing time, and events within the positive ($x_1 > 0$) light cone, also called the future ligth cone, is the set of all events that are affected by the event at the origin (likewise, all events within the past light cone have affected the event at the origin in the past). Thus, any point in space will be affected by the event at the origin sometime in the future as the positive light cone continues to grow. We can set the origin at any point so any event in this model can be considered to determine a pair of light cones which are all just translations of each other. Therefore we have that in special relativity, the speed of light is constant from any frame of reference.

In the physics context, $\mathbb{L}^4$ is called $\mathbb{M}^4$. In Figure 3, we have a simplified three-dimensional version, $\mathbb{M}^3$, with two-dimensional space combined with the third dimension of time, which is marked by t on the graph. The hyperboloid in the figure is asymptotically bound by the the cone, which is the positive light cone of this space-time setting. From the point of view of the observer at the origin, the hyperboloid is a circle that will appear in the

[27]See [Ratcliffe, 1994, page 103].

future with a centre at the origin. Its radius will increase faster than the speed of light.[28]

[28]See [Reynolds, 1993, page 443-444].

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