Numerical solution methods for glacial rebound models

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Abstract

We consider the finite element discretization of the system of partial differential equations describing the stress field and the displacements in a (visco)elastic inhomogeneous layered media in response to a surface load. The underlying physical phenomenon, which is modelled, is glacial advance and recession, and the resulting crustal stress state. We analyse the elastic case in more detail and present discretization error estimates. The so-obtained linear system of equations is solved by an iterative solution method with suitable preconditioning and numerical experiments are presented.

1 Introduction

We consider the problem to compute the stress field $\underline{\sigma}$ and the displacements $\mathbf{u}$ in a (visco)elastic inhomogeneous layered media in response to a surface load.

The underlying physical phenomenon, which is modelled, is glacial advance and recession, and the resulting crustal stress state. Recently, this problem has attracted much attention in the Geophysical community. Various models have been tested and results from numerical simulations have been compared with some analytical approaches in order to check the hypothesis that the stress induced by postglacial readjustment processes has triggered large earthquakes (of magnitude 8) in some regions in the Northern Hemisphere. This hypothesis is supported by a number of large postglacial faults which have been formed about 9000 to 13000 years ago.

An ongoing glacioisostatic recovery is registered in Central Scandinavia. The coastal area of the northern Baltic Sea is rising nearly one centimeter per year. Accumulated rebound would give up to 10 cm per decade, one meter per century, etc. In order for isostatic equilibrium to be achieved, Hudson Bay still needs to rebound as much as 150 m (cf. [29]). The amount of residual rebound, coupled with low and declining rates of recovery, imply that the lithosphere may not completely reach equilibrium within itself before the next glacial period. All evidence at hand indicate that postglacial rebound has to be taken into account in the context of other problems, such as for example predicting safety of nuclear waste repositories.

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The simulations of postglacial rebound are usually performed by first gradually imposing a surface load of certain profile until the maximum height of the icecap is reached and subsequently diminishing the load, computing at each step the corresponding stress field. One important characteristic of the above is the very long time period (several tens of thousands of years) for which the processes have to be simulated. Clearly, this poses high demands on the efficiency and robustness of the numerical solution techniques, in order to enable solving large models (with enough spatial resolution) within a feasible overall simulation time.

Currently, numerical simulations are performed using available commercial finite element packages. Several obstacles have been noticed. First, almost all well-tested finite element packages are engineering-oriented and are designed to solve the stiffness equation

$$\nabla \underline{\sigma} + \mathbf{f} = 0,$$

which turns out to be overly simplified for geophysical applications and does not include phenomena like self-gravitation and isostasy[1], for instance. Secondly, some of the simulation runs took very long execution time (6-10 days, as reported in [14], for example).

In this paper we discretize the model in its original formulation using standard Finite element method (FEM) and solve the arising linear algebraic systems of equations with preconditioned iterative solution methods. The aim is to construct and test various preconditioning techniques, and demonstrate their efficiency on large field benchmark problems.

The paper is organized as follows. Section 2 contains the problem formulation in terms of partial differential equations and boundary conditions. In Section 3 we describe the corresponding weak formulation and a Galerkin a finite element formulation of the problem for the purely elastic case and the properties of the underlying linear algebraic system of equations. Numerical experiments to compute the elastic response of a pre-stressed body under surface load are presented in Section 4 and are compared with the results obtained when modelling the same phenomenon using a commercial Finite Element package. Conclusions and some directions for further research are given in Section 5.

2 Description of the problem

2.1 Governing equations

If a large elastic solid is put in a gravitational field, it becomes gravitationally pre-stressed with pressure $p^{(0)}$. This pressure can be regarded as an initial condition imposed on the problem and does not cause deformations. When a new stress $\underline{\sigma}$ is applied to the body, it

[1]Isostasy is the concept that the elevation of the Earth's surface (over tens of millions of years) seeks a balance between the weight of lithospheric rocks and the buoyancy of asthenospheric "fluid" (nearly-molten rock). Gentle regional movement of the lithosphere occurs in response to short-term (thousands to millions of years) loading and unloading, as by ice, erosion and sediment deposition.

causes strain $\underline{\varepsilon}$ according to the corresponding constitutive relations and the total stress $\mathbf{T}$ becomes (see the derivations in [27] and [28], for example)

$$\mathbf{T} = \underline{\sigma} - p^{(0)}\, I, \tag{1}$$

where I denotes the corresponding unit tensor. When motion with velocity $\mathbf{u}$ occurs, advection of pre-stress takes place, namely

$$\mathbf{T} = \begin{cases} \underline{\sigma} - p^{(0)}\big|_{t_0}\, I, & \text{for a viscous body} \\ \underline{\sigma} - p^{(0)}\big|_{t_0}\, I + \mathbf{u}\cdot\nabla p^{(0)}\, I, & \text{for an elastic body.} \end{cases} \tag{2}$$

Taking next into consideration the equation of motion (Newton's law), namely,

$$\rho^{(0)}\frac{\partial^2 \mathbf{u}}{\partial t^2} = \underline{\nabla}\cdot\mathbf{T} + \mathbf{F}, \tag{3}$$

where $\mathbf{F}$ is the body force and $\rho^{(0)}$ is density, we obtain the equation of motion for pre-stressed body

$$\rho^{(0)}\frac{\partial^2 \mathbf{u}}{\partial t^2} = \begin{cases} \underline{\nabla}\cdot\underline{\sigma} - \underline{\nabla}p^{(0)} + \mathbf{F}, & \text{for a viscous body} \\ \underline{\nabla}\cdot\underline{\sigma} - \underline{\nabla}p^{(0)} + \underline{\nabla}(\mathbf{u}\cdot\nabla p^{(0)}) + \mathbf{F}, & \text{for an elastic body.} \end{cases} \tag{4}$$

Since the body force F is due to gravity, we have

$$\mathbf{F} = \rho\,\mathbf{g} = -\rho\underline{\nabla}\phi, \tag{5}$$

where $\mathbf{g}$ is the gravity force ($\mathbf{g} = [0, 0, -g]^T$), g is the acceleration of gravity, ϕ is the gravitational potential, which is related to the density ρ via Poisson's equation $\nabla^2\phi = 4\pi G\rho$, and G is the universal gravity constant. Here it is assumed that the x_3-axis is vertical and pointing upwards, as indicated in Figure 2.

Neglecting the acceleration terms on the left side of (4), which are negligible for motion near isostatic equilibrium, we obtain the following equation of motion for isostatic processes,

$$\mathbf{0} = \begin{cases} \underline{\nabla}\cdot\underline{\sigma} - \underline{\nabla}p^{(0)} - \rho\underline{\nabla}\phi, & \text{for a viscous body} \\ \underline{\nabla}\cdot\underline{\sigma} - \underline{\nabla}p^{(0)} + \underline{\nabla}(\mathbf{u}\cdot\nabla p^{(0)}) - \rho\underline{\nabla}\phi, & \text{for a elastic body.} \end{cases} \tag{6}$$

Next we consider the terms $\underline{\nabla}p^{(0)}$ and $\rho\underline{\nabla}\phi$. In the initial state of zero motion, Newton's law becomes

$$\mathbf{0} = -\underline{\nabla}p^{(0)} + \mathbf{F}_0 = -\underline{\nabla}p^{(0)} - \rho^{(0)}\underline{\nabla}\phi_0, \quad \text{i.e.,}\ \underline{\nabla}p^{(0)} = -\rho^{(0)}\underline{\nabla}\phi_0. \tag{7}$$

Let now $\rho^{(\Delta)}$, $\phi^{(\Delta)}$ be the deviations from the initial state, i.e.,these can be seen as changes (perturbations) of $\rho^{(0)}$ and $\phi^{(0)}$ and are related to the local incremental fields:

$$\begin{aligned} \rho(\mathbf{r},t) &= \rho^{(0)}(\mathbf{r}) + \rho^{(\Delta)}(\mathbf{r},t) = \rho^{(0)}(\mathbf{r}) - \nabla\cdot(\rho^{(0)}\mathbf{u}), \\ \phi(\mathbf{r},t) &= \phi^{(0)}(\mathbf{r}) + \phi^{(\Delta)}(\mathbf{r},t). \end{aligned} \tag{8}$$

In the derivation of the modelling equations we also include the equation of continuity (conservation of mass), namely (see, for instance [15]),

$$\rho^{(\Delta)} + \nabla \cdot (\rho^{(0)}\mathbf{u}) = 0. \tag{9}$$

We substitute (8) and (9) into (6) and obtain

$$\begin{aligned}
-\underline{\nabla} p^{(0)} \quad - \quad & \rho\underline{\nabla}\phi = -\underline{\nabla} p^{(0)} - \left(\rho^{(0)} - \nabla \cdot (\rho^{(0)}\mathbf{u})\right)\left(\underline{\nabla}\phi^{0} + \underline{\nabla}\phi^{(\Delta(}\right) && (10)\\
= \quad & -\underline{\nabla} p^{(0)} - \rho^{(0)}\underline{\nabla}\phi^{(0)} + \nabla \cdot (\rho^{(0)}\mathbf{u})\phi^{(0)} - \rho^{(0)}\underline{\nabla}\phi^{(\Delta)} + \nabla \cdot (\rho^{(0)}\mathbf{u})\underline{\nabla}\phi^{(\Delta)} && (11)\\
= \quad & -\underline{\nabla} p^{(0)} + \underline{\nabla} p^{(0)} - \rho^{(0)}\underline{\nabla}\phi^{(\Delta)} - \rho^{(\Delta)}\underline{\nabla}\phi^{(0)} - \rho^{(\Delta)}\underline{\nabla}\phi^{(\Delta)} && (12)\\
= \quad & -\rho^{(0)}\underline{\nabla}\phi^{(\Delta)} + \nabla \cdot (\rho^{(0)}\mathbf{u})(\underline{\nabla}\phi^{(0)} + \underline{\nabla}\phi^{(\Delta)}) && (13)\\
= \quad & -\rho^{(0)}\underline{\nabla}\phi^{(\Delta)} + \rho^{(0)}\nabla \cdot \mathbf{u}\underline{\nabla}\phi && (14)\\
= \quad & -\rho^{(0)}\underline{\nabla}\phi^{(\Delta)} - \rho^{(0)}g^{(0)}\mathbf{e}_d\nabla \cdot \mathbf{u}. && (15)
\end{aligned}$$

Therefore, relations (6) become

$$\mathbf{0} = \begin{cases} \underline{\nabla} \cdot \underline{\sigma} - \rho^{(0)}g^{(0)}\mathbf{e}_d\nabla \cdot \mathbf{u} - \rho^{(0)}\underline{\nabla}\phi^{\Delta}, & \text{for viscous body} \\ \underline{\nabla} \cdot \underline{\sigma} + \underline{\nabla}(\mathbf{u} \cdot \nabla p^{(0)}) - \rho^{(0)}g^{0}\mathbf{e}_d\nabla \cdot \mathbf{u} - \rho^{(0)}\underline{\nabla}\phi^{\Delta}, & \text{for elastic body.} \end{cases} \tag{16}$$

From now on we consider only the second equation in (16), namely, for the elastic case only,

$$\underbrace{\underline{\nabla} \cdot \underline{\sigma}}_{(A)} + \underbrace{\nabla(\mathbf{u} \cdot \nabla p^{(0)})}_{(B)} - \underbrace{\rho^{(0)}g^{(0)}\mathbf{e}_d\nabla \cdot \mathbf{u}}_{(C)} - \underbrace{\rho^{(0)}\underline{\nabla}\phi^{(\Delta)}}_{(D)} = \mathbf{0} \quad \text{in } \Omega \subset \mathbb{R}^d, d = 2, 3. \tag{17}$$

Equation (17) is the material incremental momentum equation for quasi-static infinitesimal perturbations of a stratified, compressible fluid Earth, initially in hydrostatic equilibrium, subject to gravitational forces but neglecting internal forces (cf. [12]) and is the governing balance equation describing the geophysical problem of postglacial rebound.

We recall, that $\underline{\sigma}$ is the Cauchy stress tensor, $\mathbf{u} = [u_i]_{i=1}^d, d = 2, 3$ is the displacement vector in two or three dimensional space, $p = -\frac{1}{3}trace(\underline{\sigma})$ is pressure, ρ is density and g is gravitational acceleration.

Consider now equation (17) in more detail. Term (A) describes the force from spatial gradients in stress. Terms (C) and (D) describe perturbations of the gravitational force and gravitational acceleration due to changes of density. Term (C) is referred to as the *buoyancy force*. Equation (17) is usually simplified by the assumption for non-self-gravitating Earth. In this case one can neglect the changes in the gravitational field and, thus, neglect term (D). Further simplification can be obtained by the assumption that the Earth layer is incompressible, i.e., ρ is constant. The latter would mean that $\rho^{(\Delta)} = 0$ and the term (C) would vanish also. For our considerations, only the first simplification is done and we consider further the model, applicable for compressible and incompressible Earth. Thus, the momentum equation (17) becomes

$$\underline{\nabla} \cdot \underline{\sigma} + \underline{\nabla}(\mathbf{u} \cdot \nabla p^{(0)}) - \rho^{(0)} g^{(0)} \nabla \cdot \mathbf{u} = \mathbf{0} \quad \text{in } \Omega \subset \mathbb{R}^d, d = 2, 3 \tag{18}$$

in tensor notations, or $\sigma_{ij,j} + (p^0_{,j} u_j)_{,i} - \rho^{(0)} g^{(0)} (u_j)_{,i} = 0$ elementwise.

Term (B) represents the so-called *advection of pre-stress* and describes how the hydrostatic background (initial) stress is carried by the moving material. It is observed (see [18] for instance) that in almost all elastic problems of geodynamic interest the effect of the pre-stress advection term turns out to be very modest. In the viscoelastic theory of isostacy, however, the incorporation of this term has proven to be crucial for the successful modelling of the underlying processes, the reason being that through this term deformation-induced buoyancy restoring force is introduced, which is central to the physics of the glacial isostatic adjustment process.

In the consideration below and for the numerical simulations one more simplification is used, namely, it is assumed that the advection term describes the advection in the direction of the gravity field only. Therefore, using the relation $\underline{\nabla} p^{(0)} = \rho^{(0)} g^0 \mathbf{e}_d$ we finally rewrite term (B) in the form

$$\rho^{(0)} g^{(0)} \nabla(\mathbf{u} \cdot \mathbf{e}_d), \tag{19}$$

where $\mathbf{e}_d = [0, 0, -1]^T$ is a unit vector in the radial direction. Thus, $\mathbf{u} \cdot \mathbf{e}_d$ is the vertical component (u_d) of the displacement field.

Incorporating the above simplifications with respect to terms (B), (C) and (D), we obtain the following form of the governing equilibrium equation

$$\begin{aligned} &\nabla \underline{\sigma} + \rho^{(0)} g^{(0)} \Big(\nabla(\mathbf{u} \cdot \mathbf{e}_d) - \mathbf{e}_d \nabla \cdot \mathbf{u} \Big) = \mathbf{0} \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, d = 2, 3, \\ &\mathbf{u} = \mathbf{0}, \quad \mathbf{x} \in \Gamma_D, \qquad \sum_{j=1}^{d} \sigma_{ij} \mathbf{n}_j = \gamma_j, \quad \mathbf{x} \in \Gamma_N = \partial\Omega \backslash \Gamma_D. \end{aligned} \tag{20}$$

Here $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is considered to be an open connected and bounded domain with a boundary $\partial\Omega$, which is split into Γ_D and Γ_N in a time-independent manner, and Γ_D has a strictly positive measure.

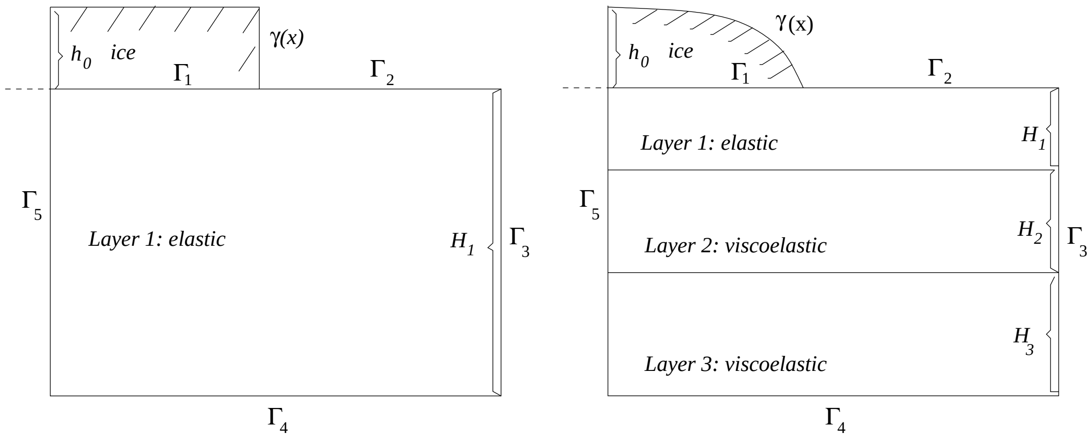
2.2 Constitutive relations

We assume small deformations, i.e. strain $\underline{\varepsilon}$ and displacements $\mathbf{u}$ are related as

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \ 1 \le i, j \le d. \tag{21}$$

In order to completely formulate the problem in terms of displacements, we use the constitutive relation between strain and stress. Consider the following two cases.

Case 1: Consider a purely elastic Earth model, as illustrated in Figure 1(a).



(a) Elastic layer over a fluid mantle

(b) Earth model with several homogeneous viscoelastic layers

Figure 1: Two-dimensional flat Earth models

Case 2: Consider a layered viscoelastic Earth model, as shown in Figure 1(b), where both elastic and viscoelastic materials are present.

For both cases, the governing equation to be solved is (20), however with different constitutive relations. In Layer 1 the material is elastic and stress and strain are related as for linear elastic anisotropic materials

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}, \tag{22}$$

where ε is the strain tensor and c_{ijkl} is the elasticity tensor.

In Case 2 the material is assumed to be elastic on one part of the domain (Layer 1) and viscoelastic in the rest of Ω (Layers 2 and 3) obeying a constitutive relation of the form (see, for instance [16])

$$\sigma_{ij}(\mathbf{x}, t) = \underbrace{c_{ijlk}\varepsilon_{kl}}_{(E)} + \underbrace{\int\limits_{t_0}^{t} b_{ijkl}(t-s)\varepsilon_{kl}(s)\, ds.}_{(V)} \tag{23}$$

Relation (23) is referred to as Hooke's law with memory. It involves two different tensors representing the stress relaxation function and describes rocks with long memory for which the state of stress at time t depends on the deformation at time t (term (E)) but also on the deformations at times prior to t (term (V)). The tensor c_{ijkl} measures the elastic response and the tensor b_{ijkl} is the tensor of stress relaxation functions.

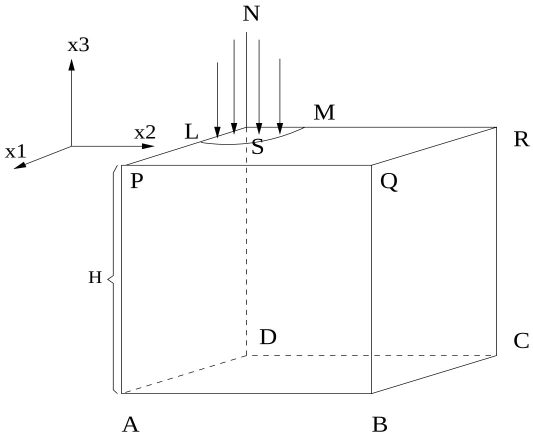


Figure 2: Boundary conditions

In Earth models, the so-called Maxwell-viscoelastic relation is often used (cf. [13]), where

$$\int_{t_0}^{t} b_{ijkl}(t-s)\varepsilon_{kl}(s)\,ds = \int_{0}^{t} b(t-s)\frac{\partial}{\partial\tau}\left(\frac{\partial u_i(s)}{\partial x_j}+\frac{\partial u_j(s)}{\partial x_i}\right)ds$$

with $b(t) = \mu e^{t/\tau_M}$, where μ is the shear modulus (the first Lamé constant) and τ_M is the so-called Maxwell time[2].

2.3 Boundary conditions

We assume that the load is plain-symmetric in 3D (axisymmetric in 2D) both in x- and y-directions, and we will consider only a quarter of the true domain of interest, as illustrated in Figure 2. The surface load is acting on the surface $LMS, (z = 0)$. The load is imposed as a Heaviside function with different profiles, for instance of boxcar, elliptic or parabolic shape (see [13]). Boxcar and elliptic profiles in 2D are illustrated in Figure 1(a) and 1(b). On the free from load surface $PQRS\backslash LMS$, the normal stress σ_{33} and shear stress σ_{13} are both zero. In terms of displacements, homogeneous Neumann boundary conditions are imposed there, i.e., $\partial u_i/\partial\mathbf{n} = 0, i = 1,2,3$ and $\mathbf{n}$ is the outer unit normal vector ($\mathbf{n} = [0,0,1]$ in this particular case).

Faces $CDSR, ADSP$ are symmetry planes. Across solid-solid boundaries, namely all side surfaces $ABQP, BCRQ$ the boundary conditions are determined from the continuity of stress and displacements conditions, i.e., $\partial u_i/\partial\mathbf{n} = 0, i = 1,2,3$ and on the edge DS the displacements in x- and y-directions are imposed to be zero due to the assumed symmetries.

Across the solid-fluid boundary $ABCD$ at $z = -H$ the normal stress is continuous: $\sigma_{33}\big|_{z=-H} = \rho_f g^{(0)} u_d\big|_{z=-H}$, where ρ_f is the density of the fluid and shear stress σ_{13} is zero.

[2]Maxwell time τ_M is defined as the ratio η/μ, where η is the dynamic viscosity. The general definition of dynamic viscosity is $[shear\_stress] = \eta[strain\_rate]$. Thus, for linear elasticity $\eta = 1/2$.

In terms of displacements, the boundary conditions at $z = -H$ are of Robin (mixed) type, i.e., $\partial u_d/\partial \mathbf{n} = \rho_f g^{(0)} u_d$ and $\partial u_i/\partial \mathbf{n} = 0, i = 1, 2$.

3 An elastic model

3.1 Variational formulation

The equilibrium state of an isotropic pre-stressed elastic material body, subject to body forces $\mathbf{f}$ and surface forces γ, described by relations (20), can be rewritten as follows:

$$-\sum_{j=1}^{d} \frac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} + \rho g \frac{\partial u_d}{\partial x_i} - \rho g \left(\sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j} \right) \mathbf{e}_d^{(i)} = f_i, \quad \mathbf{x} \in \Omega,\ i = 1, \cdots, d \tag{24}$$

$$\sum_{j=1}^{d} \sigma_{ij}(\mathbf{u}) = \gamma_j, ; \quad \mathbf{x} \in \Gamma_N,$$

$$\mathbf{u} = \mathbf{0}, ; \quad \mathbf{x} \in \Gamma_D.$$

Here $\mathbf{e}_d^{(i)}$ are the components of the vector $\mathbf{e}_d$.

The Hooke's law for linear elasticity and isotropic material reads as

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda \left(\sum_{s=1}^{d} \varepsilon_{ss} \right) \delta_{ij} \tag{25}$$

or, in tensor form,

$$\underline{\sigma} = 2\mu\underline{\varepsilon} + \lambda trace(\underline{\varepsilon}).$$

Using (21), it can be easily seen that $\nabla \cdot \mathbf{u} = trace(\underline{\varepsilon}) = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} trace(\underline{\sigma})$. The equilibrium conditions for the displacements can be formulated as a constrained optimization problem, where the displacements $\mathbf{u}$ at equilibrium minimize the following energy functional

$$\begin{aligned} \Phi(\mathbf{u}) = & \int_\Omega \Big[\mu \sum_{i,j=1}^{d} (\varepsilon_{ij}(\mathbf{u}))^2 + \tfrac{\lambda}{2} (trace(\underline{\varepsilon}))^2 + \\ & \qquad \rho\, g \left(\underline{\nabla}(\mathbf{u} \cdot \mathbf{e}_d) \cdot \mathbf{u} - (\nabla \cdot \mathbf{u})(\mathbf{e}_d \cdot \mathbf{u}) \right) - \mathbf{f} \cdot \mathbf{u} \Big] \, d\Omega - \int_{\Gamma_N} \underline{\gamma} \cdot \mathbf{u} \, d\Gamma \\ = & \int_\Omega \Big[\mu \sum_{k=1}^{d} (\nabla u_k)^2 - \tfrac{\mu}{2} (\nabla \times \mathbf{u})^2 + \tfrac{\lambda}{2} (\underline{\nabla} \cdot \mathbf{u})^2 + \\ & \qquad \rho\, g \left(\underline{\nabla}(\mathbf{u} \cdot \mathbf{e}_d) \cdot \mathbf{u} - (\nabla \cdot \mathbf{u})(\mathbf{e}_d \cdot \mathbf{u}) \right) - \mathbf{f} \cdot \mathbf{u} \Big] \, d\Omega - \int_{\Gamma_N} \underline{\gamma} \cdot \mathbf{u} \, d\Gamma \end{aligned} \tag{26}$$

defined for all admissible displacements $\mathbf{u}$ in the function space $V = \{\mathbf{u} \in H^1(\Omega)^d; \mathbf{u} = 0 \text{ on } \Gamma_D, \int_\Omega \underline{\nabla} \times \mathbf{u} \, d\Omega = 0\}$.

From (26) an analogous form as the Lamé-Navier equations of elasticity is obtained, namely,

$$\begin{aligned} -2\mu\Delta\mathbf{u} - \lambda\underline{\nabla}(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u}) - \rho g\left(\underline{\nabla}\left(\mathbf{u}\cdot\mathbf{e}_d\right) - \mathbf{e}_d\nabla\cdot\mathbf{u}\right) &= \mathbf{f} \quad \text{in } \Omega \\ \lambda(\nabla\cdot\mathbf{u})\mathbf{v} + 2\mu\underline{\varepsilon}\cdot\mathbf{v} &= \underline{\gamma} \quad \text{on } \Gamma_N \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_D \end{aligned} \tag{27}$$

where $\mu = \dfrac{E}{2(1+\nu)}$ and $\lambda = \mu\dfrac{2\nu}{1-2\nu}$ are the Lamé coefficients and E and ν are Young's modulus and Poisson's ratio, respectively.

As can be seen, when $\nu \to \frac{1}{2}$, then $\lambda \to \infty$ and the functional $\Phi(\mathbf{u})$ becomes very sensitive to small perturbations of the displacement field $\mathbf{u}$. A usual approach to handle (nearly) incompressible materials is to first regularize the problem by introducing the scaled (hydrostatic) pressure p as an auxiliary variable,

$$p = \frac{\lambda}{\mu}\nabla\cdot\mathbf{u} = \frac{\nu}{\mu(1-\nu)}trace(\underline{\sigma}) \tag{28}$$

and consider the following coupled differential equation problem

$$\left\{\begin{array}{rll} -2\mu\Delta\mathbf{u} - \mu\nabla\times(\nabla\times\mathbf{u}) - \rho g\left(\underline{\nabla}\left(\mathbf{u}\cdot\mathbf{e}_d\right) - \mathbf{e}_d\nabla\cdot\mathbf{u}\right) & -\mu\underline{\nabla}p & = \mathbf{f} \\ \mu\nabla\cdot\mathbf{u} & -\frac{\mu^2}{\lambda}p & = \mathbf{0} \end{array}\right. \tag{29}$$

with boundary conditions as given in (27).

In the sequel we consider a slightly more general formulation of the problem (29), namely, we consider an advection term of the form

$$-\underline{\nabla}(\mathbf{u}\cdot\mathbf{b}) + \mathbf{c}\nabla\cdot\mathbf{u}, \tag{30}$$

where $\mathbf{b}$ and $\mathbf{c}$ are coefficient vectors.

Remark 3.1 *As is well known, from the properties of the operator $\underline{\nabla}$ we have that for any two differentiable vector functions $\mathbf{f}$ and $\mathbf{g}$ there holds*

$$\underline{\nabla}(\mathbf{f}\cdot\mathbf{g}) = \underbrace{(\mathbf{f}\cdot\underline{\nabla})\mathbf{g}}_{(a)} + \underbrace{(\mathbf{g}\cdot\underline{\nabla})\mathbf{f}}_{(b)} + \underbrace{\mathbf{f}\times(\underline{\nabla}\times\mathbf{g})}_{(c)} + \underbrace{\mathbf{g}\times(\underline{\nabla}\times\mathbf{f})}_{(d)}.q \tag{31}$$

From (31) we see that the term $\underline{\nabla}(\mathbf{u}\cdot\mathbf{b})$ is of more general form as compared to the first-order term in the linearized Navier-Stokes equations, for instance, which latter is of the form (b).

In the case when $\mathbf{b}(=\mathbf{g})$ is a constant vector, terms (a) and (c) in (31) vanish.

The target problem reads now as follows.

$$\left\{\begin{array}{rll} -2\mu\Delta\mathbf{u} - \mu\nabla\times(\nabla\times\mathbf{u}) - \underline{\nabla}(\mathbf{u}\cdot\mathbf{b}) + \mathbf{c}\nabla\cdot\mathbf{u} & -\mu\underline{\nabla}p & = \mathbf{f} \\ \mu\nabla\cdot\mathbf{u} & -\frac{\mu^2}{\lambda}p & = \mathbf{0} \end{array}\right. \tag{32}$$

3.2 Variational formulation

The corresponding variational formulation is defined in terms of the Sobolev spaces $\mathbf{V} = (H_0^1(\Omega))^d$, $d = 2, 3$ equipped with the norm $\|\cdot\|_{\mathbf{V}} = \|\cdot\|_1$ and $P = \{p \in L^2(\Omega);\ \int_\Omega \mu\, p\, d\Omega = 0\}$, equipped with the norm $\|\cdot\|_P = \|\cdot\|_0$. It leads to the following mixed variable problem:
Seek $\mathbf{u} \in \mathbf{V}$ and $p \in P$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) - c(p,q) = 0, & \forall q \in P, \end{cases} \tag{33}$$

where

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) &= \int_\Omega \left[2\mu \sum_{k=1}^{d} (\underline{\nabla} u_k)\cdot(\underline{\nabla} v_k) - \mu(\underline{\nabla}\times\mathbf{u})\cdot(\underline{\nabla}\times\mathbf{v}) - \underline{\nabla}(\mathbf{u}\cdot\mathbf{b})\cdot\mathbf{v} + (\nabla\cdot\mathbf{u})(\mathbf{c}\cdot\mathbf{v}) \right] d\Omega \\
b(\mathbf{u},p) &= \int_\Omega \mu(\underline{\nabla}\cdot\mathbf{u})p\, d\Omega = -\int_\Omega \mu\underline{\nabla}(p)\cdot\mathbf{u} \\
c(p,q) &= \int_\Omega \frac{\mu^2}{\lambda} p\, q\, d\Omega \\
(\mathbf{u},\mathbf{f}) &= \int_\Omega (\mathbf{u}\cdot\mathbf{f}) d\Omega \\
\langle \mathbf{u}, \underline{\gamma} \rangle &= \int_{\Gamma_N} \mathbf{u}\cdot\underline{\gamma}\, d\Gamma
\end{aligned}$$

For the above defined bilinear form $b(\mathbf{u},p)$, the following inf-sup (or Ladyzhenskaya-Babuška-Brezzi) condition holds

$$\sup_{\mathbf{u}\in\mathbf{V}\setminus\{\mathbf{0}\}} \frac{b(\mathbf{u},p)}{\|\mathbf{u}\|_{\mathbf{V}}} \geq \gamma\|p\|_P, \forall p \in P, \mu(\mathbf{x}) \geq \mu_0 > 0, \nu_0 \leq \nu(\mathbf{x}) \leq 1, \tag{34}$$

where $\|p\|_P = \left(\int_\Omega p\, q\, d\Omega \right)$.

In the derivations to follow we use the boundedness of the bilinear forms $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$,

$$\begin{aligned} |b(\mathbf{u},p)| \leq C^{(b)}\|\mathbf{u}\|_1\|p\|_0, &\quad \text{where } C^{(b)} = \max_\Omega \mu \\ 0 \leq c(p,q) \leq C^{(c)}\|p\|_0\|q\|_0, &\quad \text{where } C^{(c)} = \max_\Omega(\mu/\lambda). \end{aligned} \tag{35}$$

We note that $c(p,q) = 0$ for incompressible materials ($\lambda = \infty$).

The standard way to ensure existence and uniqueness of the solution of a variational problem is to show boundedness and coercivity of the bilinear form $a(\mathbf{u},\mathbf{v})$ and then apply the Lax-Milgram lemma.

Boundedness: There exists C_1 such that $|a(\mathbf{u},\mathbf{v})| \leq C_1\|\mathbf{u}\|_{\mathbf{V}}\|\mathbf{v}\|_{\mathbf{V}},\ \forall \mathbf{u},\mathbf{v} \in \mathbf{V}$

Coercivity: There exists C_2 such that $a(\mathbf{u},\mathbf{u}) \geq C_2\|\mathbf{u}\|_V^2,\ \forall \mathbf{u} \in V$

We address next the boundedness and coercivity of the bilinear form $a(\cdot,\cdot)$. We define the auxiliary bilinear forms $\widehat{a}(\cdot,\cdot)$, $\widetilde{a}(\cdot,\cdot)$ and $\widetilde{\widetilde{a}}(\cdot,\cdot)$ as

$$\begin{aligned}
\widehat{a}(\mathbf{u},\mathbf{v}) &= \int_\Omega 2\mu \sum_{k=1}^{d} (\underline{\nabla} u_k)\cdot(\underline{\nabla} v_k) d\Omega,\\
\widetilde{a}(\mathbf{u},\mathbf{v}) &= \int_\Omega \left[2\mu \sum_{k=1}^{d} (\underline{\nabla} u_k)\cdot(\underline{\nabla} v_k) - \mu(\underline{\nabla}\times\mathbf{u})\cdot(\underline{\nabla}\times\mathbf{v})\right] d\Omega,\\
\widetilde{\widetilde{a}}(\mathbf{u},\mathbf{v}) &= \int_\Omega \Big(-\underline{\nabla}(\mathbf{u}\cdot\mathbf{b})\mathbf{v} + (\nabla\cdot\mathbf{u})(\mathbf{c}\cdot\mathbf{v})\Big) d\Omega.
\end{aligned}$$

By construction, $a(\mathbf{u},\mathbf{v}) = \widetilde{a}(\mathbf{u},\mathbf{v}) + \widetilde{\widetilde{a}}(\mathbf{u},\mathbf{v})$. We make now the following assumptions for the coefficient vectors $\mathbf{b}(\mathbf{x}) \in \mathbb{R}^d$ and $\mathbf{c}(\mathbf{x}) \in \mathbb{R}^d$, namely, that there exist constants α_1, α_2 and β, independent on $\mathbf{u}$ and $\mathbf{v}$, such that there holds

$$|b_i(\mathbf{x})| \le \alpha_1 \quad i = 1,\cdots,d \tag{36}$$
$$|\nabla\cdot\mathbf{b}| \le \alpha_2 \tag{37}$$
$$|\mathbf{c}| \le \beta \tag{38}$$

We assume also that the problem (33) possesses a solution and aim to show coercivity (in some weak form) and to derive quasi-optimal error bounds for the corresponding Galerkin finite element method.

First we note, that the dominating part $\widetilde{a}(\mathbf{u},\mathbf{v})$ of $a(\mathbf{u},\mathbf{v})$ is bounded and coercive. The latter can be seen by using Korn's inequality (cf. for example, [3]). It is known that there exists a constant $K(\Omega)$, which may depend only on the domain Ω and on the boundary conditions, such that

$$K(\Omega)\widehat{a}(\mathbf{u},\mathbf{u}) \le \widetilde{a}(\mathbf{u},\mathbf{u}) \le 2\,\widehat{a}(\mathbf{u},\mathbf{u}) \;\forall \mathbf{u}\in\mathbf{V}. \tag{39}$$

Therefore, there exist constants K_1 and K_2, which depend only on Ω and on the boundary conditions, such that

$$|\widetilde{a}(\mathbf{u},\mathbf{v})| \le K_1\|\mathbf{u}\|_1\|\mathbf{v}\|_1 \tag{40}$$
$$\widetilde{a}(\mathbf{u},\mathbf{u}) \ge K_2\|\mathbf{u}\|_1^2. \tag{41}$$

We consider now the first-order terms in $\widetilde{\widetilde{a}}(\mathbf{u},\mathbf{v})$. The following estimates hold.

$$\begin{aligned}
\left|\int_\Omega \underline{\nabla}(\mathbf{u}\cdot\mathbf{b})\mathbf{v} d\Omega\right| &\le \sum_{k=1}^{d}\int_\Omega \left|\frac{\partial}{\partial x_k}(u_k\, b_k) v_k\right| d\Omega\\
&\le \sum_{k=1}^{d}\int_\Omega \left|\frac{\partial u_k}{\partial x_k} b_k\, v_k\right| d\Omega + \sum_{k=1}^{d}\int_\Omega \left|u_k\frac{\partial b_k}{\partial x_k} v_k\right| d\Omega\\
&\le \alpha_1\, d\|\mathbf{u}\|_1\|\mathbf{v}\|_0 + \alpha_2\, d\|\mathbf{u}\|_0\|\mathbf{v}\|_0.
\end{aligned} \tag{42}$$

$$\begin{aligned}
\left|\int_\Omega (\underline{\nabla}\cdot\mathbf{u})(\mathbf{c}\cdot\mathbf{v}) d\Omega\right| &\le \sum_{k=1}^{d}\int_\Omega |(\underline{\nabla}\cdot\mathbf{u})c_k\, v_k|\, d\Omega\\
&\le \beta\, d\|\mathbf{u}\|_1\|\mathbf{v}\|_0.
\end{aligned} \tag{43}$$

Therefore,

$$\begin{array}{rcl}\left|\widetilde{\widetilde{a}}(\mathbf{u},\mathbf{v})\right| & \leq & d(\alpha_1\|\mathbf{u}\|_1+\alpha_2\|\mathbf{u}\|_0+\beta\|\mathbf{u}\|_1)\|\mathbf{v}\|_0 \\ & = & d(\alpha_1+\beta)\|\mathbf{u}\|_1\|\mathbf{v}\|_0+d\alpha_2\|\mathbf{u}\|_0\|\mathbf{v}\|_0 \\ & \leq & \sigma\|\mathbf{u}\|_1\|\mathbf{v}\|_0,\end{array} \tag{44}$$

where $\sigma = d(\alpha_1+\alpha_2+\beta)$. We use now Young's inequality, ($ab \leq \frac{\varepsilon}{2}|a|^2+\frac{1}{2\varepsilon}|b|^2$) as follows

$$\|\mathbf{u}\|_1\|\mathbf{v}\|_0 \leq \frac{\varepsilon}{2}\|\mathbf{u}\|_1^2+\frac{1}{2\varepsilon}\|\mathbf{v}\|_0^2,\ \forall\varepsilon>0. \tag{45}$$

Combining (44) and (45) we obtain

$$\left|\widetilde{\widetilde{a}}(\mathbf{u},\mathbf{v})\right| \leq \sigma\frac{\varepsilon}{2}\|\mathbf{u}\|_1^2+\sigma\frac{1}{2\varepsilon}\|\mathbf{v}\|_0^2. \tag{46}$$

We want to prove a relation of the type (Gårding inequality) $a(\mathbf{u},\mathbf{u}) \geq C^{(1)}\|\mathbf{u}\|_1^2 - C^{(2)}\|\mathbf{u}\|_0^2$, where $C^{(1)}>0$ holds for all $\mathbf{u}\in\mathbf{V}$. We show below that this is true for a particular choice of the parameter ε in (46).

$$\begin{array}{rcl}a(\mathbf{u},\mathbf{u}) & = & \widetilde{a}(\mathbf{u},\mathbf{u})+\widetilde{\widetilde{a}}(\mathbf{u},\mathbf{u}) \\ & \geq & K_2\|\mathbf{u}\|_1^2-\left|\widetilde{\widetilde{a}}(\mathbf{u},\mathbf{u})\right| \\ & \geq & K_2\|\mathbf{u}\|_1^2-\sigma\frac{\varepsilon}{2}\|\mathbf{u}\|_1^2-\sigma\frac{1}{2\varepsilon}\|\mathbf{v}\|_0^2 \\ & = & \left(K_2-\sigma\frac{\varepsilon}{2}\right)\|\mathbf{u}\|_1^2-\sigma\frac{1}{2\varepsilon}\|\mathbf{v}\|_0^2.\end{array} \tag{47}$$

We choose now ε such that $C^{(1)} \equiv K_2-\sigma\frac{\varepsilon}{2}>0$, i.e.,

$$0<\varepsilon<\frac{2K_2}{\sigma}.$$

For the latter choice of ε, $C^{(1)}>0$ and $C^{(2)}\equiv\sigma\frac{1}{2\varepsilon}>0$.

Thus, we obtain that for all $\mathbf{u}\in\mathbf{V}$ there holds

$$a(\mathbf{u},\mathbf{u}) \geq C^{(1)}\|\mathbf{u}\|_1^2-C^{(2)}\|\mathbf{u}\|_0^2, \tag{48}$$

where $C^{(1)}>0$ and $C^{(2)}>0$ do not depend on $\mathbf{u}$.

To show boundedness of $a(\mathbf{u},\mathbf{v})$, we denote $C^{(a)}=2K_1+\sigma$ and using the relation $\|\cdot\|_0\leq\|\cdot\|_1$, and we find

$$\begin{array}{rcl}|a(\mathbf{u},\mathbf{v})| & \leq & 2|\widehat{a}(\mathbf{u},\mathbf{v})|+\sigma\|\mathbf{u}\|_1\|\mathbf{v}\|_0 \\ & \leq & 2K_1\|\mathbf{u}\|_1\|\mathbf{v}\|_1+\sigma\|\mathbf{u}\|_1\|\mathbf{v}\|_0 \\ & \leq & C^{(a)}\|\mathbf{u}\|_1\|\mathbf{v}\|_1.\end{array} \tag{49}$$

Remark 3.2 *For incompressible materials, the term $\mathbf{c}\nabla\cdot\mathbf{u}$ becomes zero. When in addition $\mathbf{b}=\mathbf{e}_d$, then $\nabla\cdot\mathbf{b}=0$ and we see that in this case the bilinear form $a(\mathbf{u},\mathbf{v})$ is coercive.*

3.3 Finite element discretization and error estimates

Let $\mathbf{V}^h$ and P^h be finite element subspaces of $\mathbf{V}$ and P correspondingly, and $\mathbf{u}_h$, $\mathbf{v}_h$, p_h and q_h be the discrete counterparts of $\mathbf{u}$, $\mathbf{v}$, p and q. The discrete formulation of (33) reads then as follows:
Seek $\mathbf{u}_h$ and p_h, such that relations (50) hold for all $\mathbf{v}_h \in \mathbf{V}^h$ and for all $q_h \in P^h$.

$$\begin{array}{ll} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}^h, \\ b(\mathbf{u}_h, q_h) - c(p_h, q_h) = 0, & \forall q_h \in P^h. \end{array} \tag{50}$$

As is well known, in order to obtain a stable discrete formulation, the finite element spaces $\mathbf{V}^h$ and P^h cannot be arbitrarily chosen. They have to form a stable pair, i.e., such that the discrete analog of the inf-sup condition should hold, namely,

$$\sup_{\mathbf{u}_h \in \mathbf{V}^h} \frac{b(\mathbf{u}_h, p_h)}{\|\mathbf{u}_h\|_{\mathbf{V}^h}} \geq \gamma_h \|p_h\|_{P^h}, \geq \gamma_0 \|p_h\|_{P^h}, \forall p_h \in P^h, \tag{51}$$

for some positive constant $\gamma_0 > 0$, which for practical purposes should not be very small.

The interpretation of the discrete LBB condition (51) is that if the LBB constant γ_0 is independent on the discretization parameter h, then the rate of convergence of the FE solution $\mathbf{u}_h$ to the solution of the continuous variational problem is bounded uniformly with respect to the problem parameters E and ν.

There exist a variety of stable finite element pairs. A preferred choice is $\mathbf{u}_h \in \pi_2^h$ (componentwise) and $p_h \in \pi_1^h$, where π_2^h and π_1^h are the spaces of piecewise quadratic and piecewise linear polynomials. The discretization error for $\mathbf{u}$ and p is shown to be

$$\|\mathbf{u} - \mathbf{u}_h\|_V + \|p - p_h\|_P \leq const \left(\inf_{\mathbf{v}_h \in \mathbf{V}^h} \|\mathbf{u} - \mathbf{v}_h\|_V + \inf_{q_h \in P^h} \|p - q_h\|_P \right),$$

for any elements $\mathbf{v}_h \in \mathbf{V}^h$ and $q_h \in P^h$.

3.3.1 Error estimates

As shown above, the bilinear form $a(\mathbf{u}, \mathbf{v})$ is not coercive in general. Following [1], we derive quasi-optimal error bounds for the Galerkin method, applied to the problem under consideration.

Assume the following assumptions hold.

(A1) $|a(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\|_1 \|\mathbf{v}\|_1$

(A2) Let $\mathbf{V}^{h_N} \subset \mathbf{V}$, $h_N = 1/(N^d)$, $N = 1, 2, \cdots$, be a sequence of finite dimensional subspaces of $\mathbf{V}$. (For notational simplicity we omit the subscript N.) Let there exist a sequence of positive numbers $\{\delta_h\}_{N=1}^{\infty}$, such that $\lim_{N \to \infty} \delta_h = 0$ and that for every $\mathbf{e} \in \mathbf{V}^h$ and $z \in P^h$, satisfying

$$\begin{array}{ll} a(\mathbf{e}, \mathbf{v}) + b(\mathbf{v}, z) = \mathbf{0}, & \forall\, \mathbf{v} \in \mathbf{V}^h \\ b(\mathbf{e}, q) - c(z, q) = 0, & \forall\, q \in P^h \end{array}$$

there holds that

$$\|\mathbf{e}\|_0 \leq \delta_N \|\mathbf{e}\|_1.$$

Let now $\mathbf{u} \in \mathbf{V}$ and $p \in P$ be given. Let $\mathbf{V}^h$ and P^h be finite-dimensional subspaces of $\mathbf{V}$ and P correspondingly, and assume that there exist $\mathbf{u}_h^* \in \mathbf{V}_h$ and $p_h^* \in P^h$, such that

$$\begin{array}{ll} a(\mathbf{u}-\mathbf{u}_h^*, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_h^*) = \mathbf{0}, & \forall \mathbf{v}_h \in \mathbf{V}^h \\ b(\mathbf{u}-\mathbf{u}_h^*, q_h) - c(p - p_h^*, q_h) = 0, & \forall q_h \in P^h. \end{array} \tag{52}$$

Then, choosing $\mathbf{v}_h = \mathbf{u} - \mathbf{u}_h^* = \mathbf{u} - \mathbf{u}_h + \mathbf{u}_h - \mathbf{u}_h^*$ and $q_h = p - p_h^* = p - p_h + p_h - p_h^*$, and subtracting the two equations in (52) we arrive at

$$\begin{array}{rl} a(\mathbf{u}_h - \mathbf{u}_h^*, \mathbf{u}_h - \mathbf{u}_h^*) + c(p_h - p_h^*, p_h - p_h^*) = & a(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_h - \mathbf{u}_h^*) + b(\mathbf{u}_h - \mathbf{u}_h^*, p - p_h) \\ & -b(\mathbf{u} - \mathbf{u}_h, p_h - p_h^*) + c(p - p_h, p_h - p_h^*) \end{array} \tag{53}$$

which holds for any $\mathbf{u}_h \in \mathbf{V}^h$ and any $p_h \in P^h$. We assume now that either

$$\gamma \|p_h\|_0 \leq \sup_{\mathbf{u}_h} \frac{b(\mathbf{u}_h, p_h)}{\|\mathbf{u}_h\|_1} \quad \text{or} \quad \alpha \|p_h\|_0^2 \leq c(p_h, p_h), \quad \alpha > 0,$$

is satisfied. The former option holds true for LBB-stable discretizations and the latter option holds true for stabilized discretizations.

We combine the assumption (A2) with $\mathbf{e} = \mathbf{u}_h - \mathbf{u}_h^*$ and (48):

$$\begin{array}{l} a(\mathbf{u}_h - \mathbf{u}_h^*, \mathbf{u}_h - \mathbf{u}_h^*) + c(p - p_h^*, p - p_h^*) \geq a(\mathbf{u}_h - \mathbf{u}_h^*, \mathbf{u}_h - \mathbf{u}_h^*) \geq \\ C^{(1)} \|\mathbf{u}_h - \mathbf{u}_h^*\|_1^2 - C^{(2)} \|\mathbf{u}_h - \mathbf{u}_h^*\|_0^2 \geq (C^{(1)} - C^{(2)} \delta_h^2) \|\mathbf{u}_h - \mathbf{u}_h^*\|_1^2 \end{array} \tag{54}$$

Next we assume that a discrete LBB condition holds true (given in (51)) and obtain the following relations:

$$\begin{array}{rcl} \|p - p_h^*\|_0 & \leq & \|p - p_h\|_0 + \|p_h - p_h^*\|_0 \\ & \leq & \|p - p_h\|_0 + \frac{1}{\gamma} \sup\limits_{\mathbf{v}_h} \frac{b(\mathbf{v}_h, p_h - p_h^*)}{\|\mathbf{v}_h\|_1} \\ & \leq & \|p - p_h\|_0 + \frac{1}{\gamma} \sup\limits_{\mathbf{v}_h} \frac{b(\mathbf{v}_h, p_h - p)}{\|\mathbf{v}_h\|_1} + \frac{1}{\gamma} \sup\limits_{\mathbf{v}_h} \frac{b(\mathbf{v}_h, p - p_h^*)}{\|\mathbf{v}_h\|_1} \\ & \leq & \|p - p_h\|_0 + \frac{C^{(b)}}{\gamma} \|p - p_h\|_0 + \frac{1}{\gamma} \sup\limits_{\mathbf{v}_h} \frac{a(\mathbf{u} - \mathbf{u}_h^*, \mathbf{v}_h)}{\|\mathbf{v}_h\|_1} \\ & \leq & \left(1 + \frac{C^{(b)}}{\gamma}\right) \|p - p_h\|_0 + \frac{C^{(a)}}{\gamma} \|\mathbf{u} - \mathbf{u}_h^*\|_1, \end{array} \tag{55}$$

where we have used the first part of (52). From the boundedness estimates, applied to

(53), and (55) we obtain

$$\begin{array}{rl} & a(\mathbf{u}_h-\mathbf{u}_h^*,\mathbf{u}_h-\mathbf{u}_h^*)+c(p_h-p_h^*,p_h-p_h^*) \\ \leq & C^{(a)}\|\mathbf{u}_h-\mathbf{u}_h^*\|_1\|\mathbf{u}-\mathbf{u}_h\|_1+C^{(b)}\|\mathbf{u}-\mathbf{u}_h\|_1\|p_h-p_h^*\|_0 \\ & +C^{(b)}\|\mathbf{u}_h-\mathbf{u}_h^*\|_1\|p-p_h\|_0+C^{(c)}\|p_h-p_h^*\|_0\|p-p_h\|_0 \\ \leq & C^{(a)}\|\mathbf{u}-\mathbf{u}_h\|_1^2+C^{(a)}\|\mathbf{u}-\mathbf{u}_h^*\|_1\|\mathbf{u}-\mathbf{u}_h\|_1 \\ & +2C^{(b)}\|\mathbf{u}-\mathbf{u}_h\|_1\|p-p_h\|_0+C^{(b)}\|\mathbf{u}-\mathbf{u}_h\|_1\|p-p_h^*\|_0 \\ & +C^{(b)}\|\mathbf{u}-\mathbf{u}_h^*\|_1\|p-p_h\|_0+C^{(c)}\|p-p_h\|_0^2+C^{(c)}\|p-p_h\|_0\|p-p_h^*\|_0 \\ \leq & \left[\alpha_1C^{(a)}+\alpha_2C^{(b)}+\alpha_3\frac{C^{(a)}C^{(b)}}{\gamma}+\alpha_4\frac{C^{(a)}C^{(c)}}{\gamma}\right]\|\mathbf{u}-\mathbf{u}_h^*\|_1^2 \\ & +\left[C^{(a)}+\frac{C^{(a)}}{4\alpha_1}+C^{(b)}+\frac{1}{2}C^{(b)}\left(1+\frac{C^{(b)}}{\gamma}\right)+\frac{C^{(a)}C^{(b)}}{4\alpha_3\gamma}\right]\|\mathbf{u}-\mathbf{u}_h\|_1^2 \\ & +\left[C^{(b)}+C^{(c)}+\frac{C^{(b)}}{4\alpha_2}+\frac{1}{2}(C^{(b)}+2C^{(c)})\left(1+\frac{C^{(b)}}{\gamma}\right)+\frac{C^{(a)}C^{(c)}}{4\alpha_4\gamma}\right]\|p-p_h\|_0^2. \end{array} \tag{56}$$

Combining (54) and (56) we obtain

$$\begin{array}{rl} & \left[C^{(1)}-C^{(2)}\delta_h^2-\alpha_1C^{(a)}-\alpha_2C^{(b)}-\alpha_3\frac{C^{(a)}C^{(b)}}{\gamma}-\alpha_4\frac{C^{(a)}C^{(c)}}{\gamma}\right]\|\mathbf{u}-\mathbf{u}_h^*\|_1^2 \\ \leq & \left[C^{(a)}+\frac{C^{(a)}}{4\alpha_1}+C^{(b)}+\frac{1}{2}C^{(b)}\left(1+\frac{C^{(b)}}{\gamma}\right)+\frac{C^{(a)}C^{(b)}}{4\alpha_3\gamma}\right]\|\mathbf{u}-\mathbf{u}_h\|_1^2 \\ & +\left[C^{(b)}+C^{(c)}+\frac{C^{(b)}}{4\alpha_2}+\frac{1}{2}(C^{(b)}+2C^{(c)})\left(1+\frac{C^{(b)}}{\gamma}\right)+\frac{C^{(a)}C^{(c)}}{4\alpha_4\gamma}\right]\|p-p_h\|_0^2. \end{array} \tag{57}$$

Here $\alpha_1\cdots\alpha_4$ are arbitrary positive constants. We see that if we choose $\alpha_1=\frac{C^{(2)}\delta_h^2}{4C^{(a)})}$, $\alpha_2=\frac{C^{(2)}\delta_h^2}{4C^{(b)})}$, $\alpha_3=\frac{C^{(2)}\delta_h^2\gamma}{4C^{(a)}C^{(b)}}$ and $\alpha_4=\frac{C^{(2)}\delta_h^2\gamma}{4C^{(a)}C^{(c)}}$ we obtain

$$\begin{array}{rl} & (C^{(1)}-2C^{(2)}\delta_h^2)\|\mathbf{u}-\mathbf{u}_h^*\|_1^2 \\ \leq & \left[C^{(1)}+C^{(a)}+\frac{{C^{(a)}}^2}{C^{(2)}\delta_h^2}+\frac{C^{(b)}}{2}\left(3+\frac{C^{(b)}}{\gamma}\right)+\frac{{C^{(a)}}^2}{C^{(2)}\delta_h^2}\left(1+\frac{{C^{(b)}}^2}{\gamma^2}\right)\right]\|\mathbf{u}-\mathbf{u}_h\|_1^2 \\ & +\left[C^{(b)}+\frac{1}{2}(C^{(b)}+2C^{(c)})\left(1+\frac{C^{(b)}}{\gamma}\right)+\frac{1}{C^{(2)}\delta_h^2}({C^{(b)}}^2+\frac{{C^{(a)}}^2{C^{(c)}}^2}{\gamma^2})\right]\|p-p_h\|_0^2. \end{array} \tag{58}$$

We finally obtain the following error estimate, which holds for all $\mathbf{u}_h\in\mathbf{V}_h$, $p_h\in P$ and $N>N_0$:

$$\|\mathbf{u}-\mathbf{u}_h^*\|_1+\|p-p_h^*\|_0\leq C_1\|\mathbf{u}-\mathbf{u}_h\|_1+C_2\|p-p_h\|_0 \tag{59}$$

and the constants C_1, C_2 do not depend on the discretization parameter h.

Further (see [1], for instance), it can be shown that the discrete solution $\mathbf{u}_h^*$, p_h^* exists and it is unique for some sufficiently large N (small h), $N>N_0$.

Remark 3.3 *For small values of γ and δ_h we see from the latter derivations that the constants C_1 and C_2 can become large and the so-obtained error estimates become quite pessimistic. In practice, for the particular solution the constraints can take more favourable values when the solution is smooth and/or is not near incompressibility.*

3.3.2 Equal order discretization for displacements and pressure

Instead of using stable pairs of finite element spaces, one can use equal order finite elements for displacements and pressure, and some stabilized version of the discrete problem (50). For example, a stabilized and consistent formulation can be obtained in the following manner. We take divergence of the first equation in (29), use the fact that divergence of *curl* of any vector function is equal to zero, and add the resulting equation to the second equation in (29), multiplied by a stabilization parameter σ_h. Formally we have the following sequence of transformations.

$$\begin{aligned} -2\mu\nabla\cdot\Delta\mathbf{u} - \mu\nabla\cdot(\nabla\times(\nabla\times\mathbf{u})) - \xi(\mathbf{u},\mathbf{b},\mathbf{c}) - \mu\nabla\cdot\underline{\nabla}p = &\ \nabla\cdot\mathbf{f} \\ -2\mu\Delta(\nabla\cdot\mathbf{u}) - \mu\Delta p - \xi(\mathbf{u},\mathbf{b},\mathbf{c}) = &\ \nabla\cdot\mathbf{f} \\ -\mu\left(1+\tfrac{2\mu}{\lambda}\right)\Delta p = &\ \nabla\cdot\mathbf{f} + \xi(\mathbf{u},\mathbf{b},\mathbf{c}) \\ \mu\nabla\cdot\mathbf{u} - \sigma_h\,\mu\left(1+\tfrac{2\mu}{\lambda}\right)\Delta p - \tfrac{\mu^2}{\lambda}p = &\ \sigma_h\nabla\cdot\mathbf{f} + \sigma_h\xi(\mathbf{u},\mathbf{b},\mathbf{c}), \end{aligned}$$

where $\xi(\mathbf{u},\mathbf{b},\mathbf{c}) = \nabla\cdot(\underline{\nabla}\mathbf{u}\cdot\mathbf{b}) - \nabla\cdot\mathbf{c}\nabla\cdot\mathbf{u}$. Then we consider the problem

$$\begin{aligned} -2\mu\Delta\mathbf{u} - \mu\nabla\times(\nabla\times\mathbf{u}) - \underline{\nabla}\mathbf{u}\cdot\mathbf{b} + \mathbf{c}\nabla\cdot\mathbf{u} - \mu\underline{\nabla}p = &\ \mathbf{f} \\ \mu\nabla\cdot\mathbf{u} - \sigma_h\,\mu\left(1+\tfrac{\mu}{\lambda}\right)\Delta p - \tfrac{\mu^2}{\lambda}p = &\ \sigma_h\nabla\cdot\mathbf{f} + \sigma_h\xi(\mathbf{u},\mathbf{b},\mathbf{c}) \end{aligned} \tag{60}$$

By using a similar technique as in [5] one can show that discrete LBB condition holds for problem (60) discretized by standard piecewise linear finite elements. The choice of the stabilization parameter $\sigma_h = O(h^2)$ can be justified as in [4]. There, a defect-correction algorithm is described in order to handle the term $\sigma_h\xi(\mathbf{u},\mathbf{b},\mathbf{c})$.

3.4 Preconditioners for saddle point systems

The finite element discretization of (50) leads to a linear algebraic system

$$\mathcal{A}\begin{bmatrix}\mathbf{u}_h\\ p_h\end{bmatrix} \equiv \begin{bmatrix}M & B^T\\ B & -C\end{bmatrix}\begin{bmatrix}\mathbf{u}_h\\ p_h\end{bmatrix} = \begin{bmatrix}\mathbf{r}_h\\ \mathbf{s}_h\end{bmatrix} \tag{61}$$

The system matrix $\mathcal{A}$ admits a saddle point form and is unsymmetric indefinite. The nonsymmetricity is due to the discretized first order terms in the block M.

We consider the following two approaches to precondition $\mathcal{A}$ by a matrix $\mathcal{B}$, namely, *block-triangular* preconditioner (62), and *indefinite preconditioner*, (63).

3.4.1 Block-triangular (one-sided) preconditioner

$$\mathcal{B}_1 = \begin{bmatrix}D_1 & 0\\ B & -D_2\end{bmatrix} \tag{62}$$

Following the derivations in [6], assuming that M, D_1 and D_2 are nonsingular, we consider the generalized eigenvalue problem

$$\mathcal{A}\mathbf{x} = \lambda\mathcal{B}_1\mathbf{x} \quad \text{or} \quad (\mathcal{A}-\mathcal{B}_1)\mathbf{x} = (\lambda-1)\mathcal{B}_1\mathbf{x}$$

which takes the form

$$\begin{bmatrix} D_1^{-1}(M - D_1) & D_1^{-1}B^T \\ D_2^{-1}BD_1^{-1}(M - D_1) & D_2^{-1}(BD_1^{-1}B^T + C) - I_2 \end{bmatrix} \mathbf{x} = (\lambda - 1)\mathbf{x}.$$

The latter relation shows how to choose D_1 and D_2 efficiently, namely D_1 has to be a good preconditioner to M and D_2 should be a good preconditioner to the approximated negative Schur complement of $\mathcal{A}$, $S = C + BD_1^{-1}B^T$.

3.4.2 Indefinite preconditioner

$$\mathcal{B}_2 = \begin{bmatrix} D_1 & B^T \\ B & -R \end{bmatrix} \tag{63}$$

The explicit form of the inverse of $\mathcal{B}_2$ is

$$\begin{bmatrix} D_1^{-1} - D_1^{-1}B^TS^{-1}BD_1^{-1} & D_1^{-1}B^TS^{-1} \\ S^{-1}BD_1^{-1} & -S^{-1} \end{bmatrix}$$

where $S = R + BD_1^{-1}B^T$. Consider again the generalized eigenvalue problem $(\mathcal{A} - \mathcal{B}_2)\mathbf{x} = (\lambda - 1)\mathcal{B}_2\mathbf{x}$. We obtain

$$\mathcal{B}_2^{-1}(\mathcal{A} - \mathcal{B}_2) = \begin{bmatrix} D_1^{-1}(M - D_1) - D_1^{-1}B^TS^{-1}BD_1^{-1}(M - D_1) & D_1^{-1}B^TS^{-1}(R - C) \\ S^{-1}BD_1^{-1}(M - D_1) & -S^{-1}(R - C). \end{bmatrix}$$

Thus, by choosing D_1 close enough to M and R to C, we can cluster the eigenvalues of $\mathcal{B}_2^{-1}\mathcal{A}$ around $(1, 0)$ in the complex plane.

Let seek S in a factorized form $S = N_1N_2$. The indefinite preconditioner can then be applied in a factorized form

$$\mathcal{B}_2 = \begin{bmatrix} D_1 & 0 \\ B & N_1 \end{bmatrix} \begin{bmatrix} I_1 & D_1^{-1}B^T \\ 0 & -N_2 \end{bmatrix}.$$

The preconditioner $\mathcal{B}_2$ is analysed and tested in [5, 6] for symmetric saddle point matrices and has shown to be slightly outperforming the one-sided preconditioner in the latter case.

4 Numerical experiments for the elastic model

Problem 4.1 Consider a 2D model of flat Earth subject to a Heavyside load (Figure 3) with mechanical and geometrical properties as shown in Table 1.

The domain is discretized using quasi-regular rectangular mesh and bilinear basis functions.

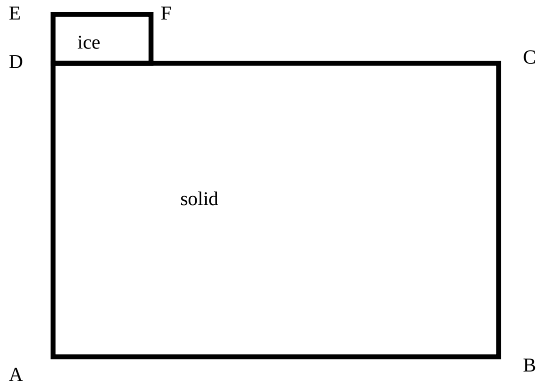


Figure 3: Geometry of the model

Geometrical data		Mechanical parameters	
l_{AB}	10 000 km	E	400 GPa
l_{AD}	4 000 km	ν	0.5
l_{EF}	1 000 km	g	9.82 $m\ s^{-2}$
l_{ED}	2 km	ρ_{ice}	981 $kg\ m^{-3}$
-	-	ρ_{solid}	3000 $kg\ m^{-3}$

Table 1: Mechanical and geometrical data

4.1 Forming the preconditioners

Two different strategies for approximation of the Schur complement of $\mathcal{A}$, denoted S_d and S_m, are tested. In S_d, the M-block is approximated with its diagonal and the negative Schur complement is formed as $-S_d = C + Bdiag(M)^{-1}B^T$. In the second approach $-S_m = C + M_p$, where M_p is the mass matrix for p ($M_p^{ij} = \int_\Omega v_i v_j \, d\Omega$, $v \in P$). For both approximations, in contrast to the exact Schur complement which in general is a dense matrix, the sparsity of C is preserved.

For the block-triangular preconditioner $\mathcal{B}_1$, D_1 is chosen as an incomplete LU factorization of M and D_2 as an incomplete Cholesky factorizations of S_i, $i = d, m$. Although M is nonsymmetric, the Schur complement approximations are symmetric.For $\mathcal{B}_2$, D_1 is chosen in the same way and N_1 and N_2 in $\mathcal{B}_2$ are chosen as incomplete Cholesky factors of the Schur complement approximation.

4.2 Results

The geometry in Figure 3 is discretized with regular quadrilateral finite elements with bilinear basis functions. All code is written in MATLAB and the experiments are performed on a Sun Ultra-Sparc III 900 MHz processor running under Sun Solaris 9.

As iterative scheme the generalized conjugate gradient method minimizing residual (GCG-MR), preconditioned with either $\mathcal{B}_1$ or $\mathcal{B}_2$ is employed. The iteration is terminated when the norm of the residual is decreased six orders of magnitude compared to the initial residual.

In Tables 2 and 3, ilu(M,q) denotes incomplete LU factorization of M with threshold q, and cholinc(S,t) denotes incomplete Cholesky factorization of S with threshold t. Choosing thresholds as zero ($q = 0$, $t = 0$) corresponds to a complete factorization. When the problem size grows larger than $N = 3267$, the complete factorizations requires to much effort and is not computed.

	$\mathcal{B}_2$			$\mathcal{B}_1$		
	cholinc(S,t)					
t	0	0.001	0.01	0	0.001	0.01
ilu(M,q)	$N = 867$					
q = 0	15	14	13	15	15	14
q = 0.001	16	16	14	16	16	15
q = 0.01	24	23	22	23	23	22
ilu(M,q)	$N = 3267$					
q = 0	17	17	15	18	17	15
q = 0.001	23	22	18	23	22	20
q = 0.01	43	42	40	37	38	40
ilu(M,q)	$N = 12675$					
q = 0	-	-	-	-	-	-
q = 0.001	38	34	34	39	38	38
q = 0.01	70	69	77	61	73	88
ilu(M,q)	$N = 49923$					
q = 0	-	-	-	-	-	-
q = 0.001	-	69	65	-	71	69
q = 0.01	-	162	127	-	176	141

Table 2: Iteration counts for $S = C + Bdiag(M)^{-1}B^T$

From Tables 2 and 3 it is evident that neither the choice of Schur complement approximation or its factorization affects the iteration count significantly, since the number of iterations are nearly constant along the rows in the two columns.

The choice of factorization of M is more crucial. Tables 2 and 3 show a dependence between the iteration count and the choice of threshold q. From the tables, it is seen that the iteration count grows nearly proportional to the square root of the problem size.

5 Remarks and intentions for a future research

1. The matrix block M is in general unsymmetric, not positive definite and not an M-matrix. It can be close to singular and very ill-conditioned. To improve the conditioning in the latter case, the following technique can be applied (see, for example

	$\mathcal{B}_2$			$\mathcal{B}_1$		
	choline(S,t)					
t	0	0.001	0.01	0	0.001	0.01
ilu(M,q)	$N = 867$					
q = 0	14	14	14	15	15	15
q = 0.001	16	16	16	16	16	16
q = 0.01	23	23	23	23	23	23
ilu(M,q)	$N = 3267$					
q = 0	15	15	16	16	16	16
q = 0.001	20	20	20	21	21	21
q = 0.01	41	41	42	40	40	42
ilu(M,q)	$N = 12675$					
q = 0	-	-	-	-	-	-
q = 0.001	33	33	36	37	37	39
q = 0.01	68	71	80	74	78	89
ilu(M,q)	$N = 49923$					
q = 0	-	-	-	-	-	-
q = 0.001	-	68	72	-	71	72
q = 0.01	-	162	127	-	176	141

Table 3: Iteration counts for $S = C + M_p$

[11]. Consider

$$\widetilde{\mathcal{A}} = \begin{bmatrix} M + B^TWB & B^T \\ B & -C \end{bmatrix},$$

where W is some nonzero square matrix of order m.

Clearly, the solution of $\begin{bmatrix} M & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$ coincides with that of

$$\begin{bmatrix} M + B^TWB & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 + B^TW\mathbf{f}_2 \\ \mathbf{f}_2 \end{bmatrix}$$

For the special choice $W = \frac{1}{\varepsilon}I$ we obtain

$$\begin{bmatrix} M + \frac{1}{\varepsilon}B^TB & B^T \\ B & -C \end{bmatrix}$$

This transformation is particularly useful if M is indefinite since $M + \frac{1}{\varepsilon}B^TB$ can be made positive definite for small enough value of ε.

The question how to precondition M efficiently is not fully answered. One possible approach to construct a robust preconditioner for M is to modify the corresponding element stiffness matrices, as done in [7].

2. The matrix B is in many cases rank deficient. This invalidates some techniques to approximate the Schur complement, such as the one suggested in [9]

$$S^{-1} = (BB^T)^{-1}BMB^T(BB^T)^{-1}.$$

Some condition number estimates in [11] are derived assuming full column rank of B^T and therefore are not applicable anymore.

3. To precondition the matrix block M, a suitable version of the AMLI-preconditioner is a promising approach. The intention is to use some ideas from [7] and construct a stabilized additive AMLI-preconditioner, based on hierarchical basis functions technique and modification of the element stiffness matrix.

4. Another preconditioning technique, the so-called Approximate Subspace Projection (ASP) method, based on the promising work in [17], can be applied to construct a good preconditioner to M.

5. Current experience has shown that the pressure mass matrix acts as a reasonably good approximation of the block BMB^T in the Schur complement. Further study is needed to seek alternative choices how to construct and precondition S for the preconditioners (62) and (63), for instance to build up S by assembling of locally computed element Schur complement matrices.

6. Regarding the viscoelastic case, two approaches can be utilized:

   (a) Apply an operator splitting and use numerical integration schemes to treat the integral term in (23), (used for instance, in [21]). Doing so, the preconditioners developed for the purely elastic case are directly applicable.

   (b) Use a FE method in time and space (cf. [25], [20], [26], for instance). This approach permits also more rigorous theoretical analysis with respect to error estimates and convergence. When combining time and space, multilevel methods become also applicable for solving time-dependent problems. One solves then for the whole time window globally, allowing in this way for various meshsizes in both time and space over the whole domain.

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