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A Categorical Study of Composition Algebras via Group Actions and Triality

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Abstract
Alsaody, S. 2015. A Categorical Study of Composition Algebras via Group Actions and Triality. *Uppsala Dissertations in Mathematics* 88. 45 pp. Uppsala: Department of Mathematics. ISBN 978-91-506-2454-0.

A composition algebra is a non-zero algebra endowed with a strictly non-degenerate, multiplicative quadratic form. Finite-dimensional composition algebras exist only in dimension 1, 2, 4 and 8 and are in general not associative or unital. Over the real numbers, such algebras are division algebras if and only if they are absolute valued, i.e. equipped with a multiplicative norm. The problem of classifying all absolute valued algebras and, more generally, all composition algebras of finite dimension remains unsolved. In dimension eight, this is related to the triality phenomenon. We approach this problem using a categorical language and tools from representation theory and the theory of algebraic groups.

We begin by considering the category of absolute valued algebras of dimension at most four. In Paper I we determine the morphisms of this category completely, and describe their irreducibility and behaviour under the actions of the automorphism groups of the algebras.

We then consider the category of eight-dimensional absolute valued algebras, for which we provide a description in Paper II in terms of a group action involving triality. Then we establish general criteria for subcategories of group action groupoids to be full, and applying this to the present setting, we obtain hitherto unstudied subcategories determined by reflections. The reflection approach is further systematized in Paper III, where we obtain a coproduct decomposition of the category of finite-dimensional absolute valued algebras into blocks, for several of which the classification problem does not involve triality. We study these in detail, reducing the problem to that of certain group actions, which we express geometrically.

In Paper IV, we use representation theory of Lie algebras to completely classify all finite-dimensional absolute valued algebras having a non-abelian derivation algebra. Introducing the notion of quasi-descriptions, we reduce the problem to the study of actions of rotation groups on products of spheres.

We conclude by considering composition algebras over arbitrary fields of characteristic not two in Paper V. We establish an equivalence of categories between the category of eight-dimensional composition algebras with a given quadratic form and a groupoid arising from a group action on certain pairs of outer automorphisms of affine group schemes

Keywords: Composition algebra, division algebra, absolute valued algebra, triality, groupoid, group action, algebraic group, Lie algebra of derivations, classification.

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To the pursuit of purest human thought,
of beauty and of rigour unsurpassed,
of truth once questioned, then affirmed, and taught,
— to Mathematics, forever to last.

List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I S. Alsaody, Morphisms in the Category of Finite-Dimensional Absolute Valued Algebras. Colloq. Math. **125** (2011), 147–174.

II S. Alsaody, Corestricted Group Actions and Eight-Dimensional Absolute Valued Algebras. J. Pure Appl. Algebra **219** (2015), 1519–1547.

III S. Alsaody, An Approach to Finite-Dimensional Real Division Composition Algebras through Reflections. Bull. Sci. math. (2014), dx.doi.org/10.1016/j.bulsci.2014.10.001, in press.

IV S. Alsaody, Classification of the Finite-Dimensional Real Division Composition Algebras having a Non-Abelian Derivation Algebra. arXiv:1404.1896, submitted for publication.

V S. Alsaody, Composition Algebras and Outer Automorphisms of Algebraic Groups. arXiv:1504.01278, submitted for publication.

Contents

1. Prologue

... aber die Natur versteht gar keinen Spaß, sie ist immer wahr, immer ernst, immer strenge; sie hat immer Recht, und die Fehler und Irrtümer sind immer des Menschen.

Johann Wolfgang von Goethe

Mathematics is, in its nature, abstract, general, systematic and precise. The pursuit of mathematics is therefore, arguably, the abstraction, generalization and systematization of human thought. Two elementary examples of this are the use of numbers to represent quantities, and geometric figures to represent shapes. In our first encounter with algebra, we learn how to add, subtract, multiply and divide these numbers, while in early geometry, we explore the geometric figures and develop a concept of length, angle and shape. As we advance, we discover intricate connections between the two domains. The Pythagorean theorem, for instance, provides an algebraic relation between the numbers used to quantify the sides of a right-angled triangle. It is therefore natural to ask the following.

Question. How can we generalize the concept of numbers, the arithmetic operations thereupon, and the notions of lengths and angles, without these completely losing their intuitive meaning and connections with one another?

The study of composition algebras, which we here propose to undertake, was born in an attempt to properly define and answer this question. While the question may sound deceivingly simple, the theory thus emerging is non-trivial and rich, in its own right as well as in view of its applications to other areas of mathematics and science.

Historically[1], there are several related origins of this study. There is, on the one hand, Hamilton's attempt to establish an algebraic framework for three-dimensional space in a way that generalizes the one-dimensional real line and the two-dimensional complex plane. Of particular importance was the existence of a notion of length and distance compatible with the arithmetic operations, i.e. the existence of a multiplicative absolute value. While the attempt failed, it led Hamilton to discover the four-dimensional algebra of the quaternions in 1843 , which was soon followed by the discovery of

[1] This brief history is a summary of the more extensive accounts provided in [17] and [37].

the eight-dimensional octonion algebra, independently by Graves and Cayley. The quaternions not being commutative, and the octonions neither being commutative nor associative, this opened the door to the study of general algebraic structures not necessarily satisfying properties which we expect from our common use of numbers and arithmetic.

Hamilton's problem is intimately related to the study of compositions of quadratic forms, which were already treated by Gauss in his *Disquisitiones arithmeticae* from 1801. One of the main questions in this area is whether it is possible to write the product of two sums of squares of n variables as a sum of squares of n new variables which depend on the old ones in a structured way, i.e. bilinearly. In 1898, Hurwitz took the major step in proving that this is possible precisely when n equals 1, 2, 4 or 8, i.e. the dimension of the real numbers, complex numbers, quaternions or octonions.

Much has happened since the work of Hurwitz. Composition algebras and real division algebras, both of which generalize the above examples, have appeared in various contexts. Composition algebras were used by E. Cartan, Jacobson and others to study Lie groups and Lie algebras, as in [25]. Cartan also used octonion algebras in [7] to formulate the principle of triality, a remarkable phenomenon which only occurs in dimension eight. Real division algebras are intimately related to the topological problem of the parallelizability of spheres, studied by Hopf, Bott, Milnor and Kervaire in [22], [5] and [27]. Through these papers as well as [1] and [26], the theory of composition algebras and real division algebras was systematized and made accessible for further studies. During the last decades, many efforts have been made to understand the structure of these algebras in depth, such as [3], [18], [19], [32], [35], [10] and [6]. Recently, a categorical approach inspired by representation theory was developed by Dieterich, as summarized in [15]. Division and composition algebras have moreover found applications in coding theory [30] and theoretical and particle physics [31]. Recently, they have also appeared in the solution of partial differential equations [29]. Nevertheless, the structure of these algebras is far from being completely understood. This thesis is intended as a new contribution to the study of finite-dimensional composition algebras, as well as that of division algebras.

An important goal when studying algebraic objects is to *classify* them. This requires determining when two objects are *isomorphic*, i.e. algebraically similar, and thence providing one object from each isomorphism class. For composition algebras and division algebras, this has proved to be a very hard problem which remains largely unsolved, and it has become apparent that a classification, if accomplished, would be overwhelmingly large. Our approach uses a categorical viewpoint and relies on and develops tools from various parts of algebra and geometry. It includes dealing with group actions, algebraic groups and representations of Lie algebras, with the aim of furthering the current understanding of composition algebras.

2. Preliminaries

2.1 Algebras

Algebras provide a framework for generalizing addition and multiplication to arbitrary finite and infinite dimensions. Given a field k (e.g. that of the real numbers), an *algebra over k* or *k-algebra* is a vector space A over k endowed with a bilinear multiplication, i.e. a map

$$\mu : A \times A \to A, \quad (x,y) \mapsto xy := \mu(x,y)$$

which is linear in each argument. In general, the multiplication is not assumed to satisfy any additional properties, such as commutativity, associativity, or the existence of a multiplicative unity. In fact, the algebras we aim to consider rarely enjoy any of these properties. Algebras which do are called commutative, associative, and unital, respectively. For any algebra A and any $a \in A$, the maps $x \mapsto ax$ and $x \mapsto xa$ of left and right multiplication define linear operators $L_a = L_a^A$ and $R_a = R_a^A$, respectively, on A. The algebra A is called a *division algebra* if for each $a \in A \setminus \{0\}$, the maps L_a and R_a are invertible. In finite dimension, this is equivalent to the non-existence of zero-divisors in A. Note that if A is not associative, the division property does not imply the existence of a unity, and it is in general not true that the inverse of right or left multiplication by a non-zero element $a \in A$ is given by right or left multiplication by some $b \in A$.

2.1.1 Composition Algebras

To add geometric structure to the algebras, and to provide a framework for computing lengths and angles in a generalized sense, the algebras are endowed with compatible quadratic forms which behave reasonably, as we shall now explain. A *quadratic form* on a vector space V over a field k is a map $q : V \to k$ such that $q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in k$ and $v \in V$, and the map

$$b_q : V \times V \to k, \quad b_q(v,w) = q(v+w) - q(v) - q(w),$$

is bilinear. The symmetric bilinear form b_q, called the *polar* of q, satisfies $b_q(v,v) = 2q(v)$ for all $v \in V$. Thus if the characteristic of k is not two, the form q is determined by b_q. An element $v \in V$ is called *isotropic* if $q(v) = 0$, and two elements $v,w \in V$ are said to be *orthogonal* if $b_q(v,w) = 0$. For each subset $S \subseteq V$ we define the *orthogonal complement of S* to be the set $S^\perp$ of all $v \in V$ which are orthogonal to each $s \in S$. The quadratic form q is called

(i) *non-degenerate* if $V^\perp \cap \ker(q) = \{0\}$,
(ii) *strictly non-degenerate* if b_q is non-degenerate, i.e. if $V^\perp = \{0\}$, and
(iii) *anisotropic* if $\ker(q) = \{0\}$.

Remark 2.1.1. In characteristic different from 2, the equality $2q(v) = b_q(v,v)$ implies that $V^\perp \subseteq \ker(q)$, and therefore (1) and (2) are equivalent. Strictly non-degenerate forms may be anisotropic or not, in which case they are called isotropic.

A *quadratic space* is a pair (V,q) where V is a vector space and q a quadratic form on V. A linear map $\varphi : (V,q) \to (V',q')$ between two quadratic spaces is called *orthogonal* or an *isometry* if it respects the quadratic structure in the sense that $q' \circ \varphi = q$. If q is non-degenerate, then every isometry from (V,q) to a quadratic space (V',q') is injective. In particular, orthogonal operators on finite-dimensional quadratic spaces are invertible whenever the quadratic form is non-degenerate. We will discuss these further in Section 2.2.

Returning to algebras, a quadratic form on an algebra A is called *multiplicative* if $q(ab) = q(a)q(b)$ for all $a,b \in A$. We now have all the notions we need to speak about composition algebras.

Definition 2.1.2. A *composition algebra* over a field k is a non-zero k-algebra endowed with a strictly non-degenerate, multiplicative quadratic form.

Example 2.1.3. Over the real numbers, the classical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ of real numbers, complex numbers, quaternions and octonions are unital composition algebras of dimension 1, 2, 4, and 8, respectively.

Example 2.1.4. Any field of characteristic not two forms a one-dimensional unital composition algebra over itself, with the square map as quadratic form. In characteristic 2, the equality $2q(v) = b_q(v,v)$ implies that a quadratic form on a one-dimensional space cannot be strictly non-degenerate, and thus one-dimensional composition algebras over fields of characteristic 2 do not exist.

Some authors do not require the non-degeneracy of the quadratic form to be strict in the definition of composition algebras. By the above remark, this only matters if the characteristic of the field is 2, and would allow the field itself and certain purely inseparable extensions as composition algebras.

Example 2.1.5. Over any field k, one can always construct a unital composition algebra in each of the dimensions 2, 4 and 8. In dimension 2, the space $k \times k$ with componentwise multiplication and quadratic form given by $(x,y) \mapsto xy$ is a unital commutative associative composition algebra. In dimension 4, the matrix algebra $\mathrm{M}_2(k)$ is a unital associative composition algebra with the determinant as quadratic form. In dimension eight, the vector-matrix algebra due to Zorn [41] is a unital composition algebra which is neither commutative nor associative. All these algebras share the property that the quadratic form is isotropic, and they are therefore called *split*.

Unital composition algebras are called *Hurwitz algebras* in honour of Hurwitz who, in his paper [23] from 1898, took the major step toward proving that over any field, finite-dimensional unital composition algebras must have dimension 1, 2, 4 or 8. Hurwitz algebras of dimension 1 (in characteristic not two) and 2 are commutative, while those of dimension 4, known as *quaternion algebras*, are associative but not commutative. Hurwitz algebras of dimension eight are called *octonion algebras* and are neither commutative nor associative. They are however *alternative*, i.e. each subalgebra generated by two elements is associative. For a detailed study of the structure of Hurwitz algebras, the reader is referred to [37], which also contains a description of the *Cayley–Dickson process*, by means of which any Hurwitz algebra of even dimension $2d$ can be constructed from a d-dimensional one.

Remark 2.1.6. In this thesis, we will be exclusively concerned with finite-dimensional algebras. Infinite-dimensional composition algebras do exist, and an example over the real numbers is given in [38]. However, Kaplansky proved in [26] that every Hurwitz algebra is finite-dimensional.

As for not necessarily unital composition algebras, in his work [26] from 1953, Kaplansky extended the dimension condition to all finite-dimensional composition algebras. Given a finite-dimensional algebra A and invertible linear operators f and g on A, the *isotope* $A_{f,g}$ of A is defined as the algebra with underlying vector space A, and multiplication given by

$$x \cdot y = f(x)g(y),$$

where juxtaposition denotes the multiplication of A. Despite the notation, the maps f and g are not uniquely determined by $A_{f,g}$. If A is a composition algebra and f and g are isometries, the *orthogonal isotope* $A_{f,g}$ is a composition algebra with the same quadratic form as A. Kaplansky's method was then to construct, for each finite-dimensional composition algebra A, the isotope $H = A_{(R_e^A)^{-1},(L_e^A)^{-1}}$ with $e \in A$ satisfying $q(e) = 1$. Such an element always exists, and H is a Hurwitz algebra. Then A becomes an orthogonal isotope $H_{f,g}$ of H with $f = R_e^A$ and $g = L_e^A$. This proves the following.

Theorem 2.1.7. *(Kaplansky, 1953) each finite-dimensional composition algebra over a field is an orthogonal isotope of a Hurwitz algebra. In particular, the dimension of a finite-dimensional composition algebra is 1, 2, 4 or 8.*

The above examples show that composition algebras exist in each such dimension in characteristic not two.

The interplay between the multiplication and the quadratic form of a finite-dimensional composition algebra raises some subtle questions. To begin with, one may wonder precisely which quadratic forms may occur as quadratic forms of composition algebras. As shown in [28], these are precisely the Pfister forms in the appropriate dimension (in characteristic two required to be strictly non-degenerate). In characteristic different from two, a *p-Pfister form* is a quadratic form on a vector space of dimension 2^p, given, in some basis, as

$$\langle 1,-\alpha_1\rangle \otimes \cdots \otimes \langle 1,-\alpha_p\rangle$$

for some $\alpha_1,\ldots,\alpha_p \in k^*$, where for each $\alpha \in k^*$

$$\langle 1,-\alpha\rangle(x_1,x_2) = x_1^2 - \alpha x_2^2.$$

The corresponding form in characteristic two is slightly more involved.

Another question is to what extent the algebra multiplication determines the quadratic form, and vice versa. As it turns out, the multiplication determines the quadratic form completely, and in fact any algebra isomorphism between two finite-dimensional composition algebras is an isometry, as shown in [33]. In the other direction, it is proved in [37] that two Hurwitz algebras are isomorphic if and only if their quadratic forms are isometric. Thus each isometry class of finite-dimensional composition algebras contains precisely one isomorphism class $\mathscr{H}$ of Hurwitz algebras, but in general several isomorphism classes of non-unital composition algebras. By Kaplansky's result these are however all isotopes of algebras in $\mathscr{H}$.

In this regard it is worth noting that from the theory of quadratic forms it follows that for each $d \in \{2,4,8\}$, there is precisely one isomorphism class of split Hurwitz algebras of dimension d. A representative of each of these isomorphism classes was given in Example 2.1.5. The algebras which are not split are characterized by the following fact from [40].

Proposition 2.1.8. *A finite-dimensional composition algebra is a division algebra if and only if its quadratic form is anisotropic.*

This motivates the study of algebras which are division algebras and composition algebras. Over the real numbers, such algebras have a particularly nice structure.

2.1.2 Absolute Valued Algebras

An *absolute valued algebra* is a non-zero real algebra endowed with a multiplicative norm. In finite dimension, the absolute valued algebras are precisely those real algebras which are division algebras as well as composition algebras. The quadratic form is then given by the square of the norm. Thus absolute valued algebras form the intersection of important classes of algebras.

In view of the above result by Kaplansky, finite-dimensional absolute valued algebras have dimension 1, 2, 4 or 8. This also follows from the fact that finite-dimensional real division algebras only exist in these dimensions. The elaborate proof of this statement is due to Hopf [22], Bott–Milnor [5] and Kervaire [27], and was accomplished using arguments from algebraic topology and K-theory. Historically, however, the proof of the dimension statement for absolute valued algebras is due to Albert and was effected in [1] in 1947, thus preceding, on the one hand, the work of Kaplansky and, on the other hand, that of Bott, Milnor and Kervaire. Albert in fact showed the following.

Theorem 2.1.9. *(Albert, 1947) Up to isomorphism, the only finite-dimensional absolute valued algebras with a unity are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, and every finite-dimensional absolute valued algebra is isomorphic to an orthogonal isotope of some $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

Remark 2.1.10. Infinite-dimensional absolute valued algebras exist and may or may not be composition algebras, and examples of both are given in [35]. However, as in the case of composition algebras, the existence of a unity implies finite dimension, which was proved in [38].

Absolute valued algebras of dimension 1 are classified up to isomorphism by $\{\mathbb{R}\}$. In dimension 2, the classification consists of the four isotopes

$$\mathbb{C} \quad \mathbb{C}_{\mathrm{Id},\kappa}, \quad \mathbb{C}_{\kappa,\mathrm{Id}} \quad \text{and} \quad \mathbb{C}_{\kappa,\kappa} \tag{2.1}$$

of $\mathbb{C}$, where κ denotes the standard involution on $\mathbb{C}$, i.e. complex conjugation. Another way of stating this is that two-dimensional absolute valued algebras are classified by their double sign, in the sense of Darpö and Dieterich, who showed in [11] that if A is a finite-dimensional real division algebra, then the sign of the determinant of left multiplication by a non-zero element in A is independent of the element, and the same holds for right multiplication. Thus to each such algebra one can assign a pair (i, j), where i (resp. j) is the sign of the determinant of L_a (resp. R_a) for an arbitrary $a \neq 0$ in A, and this pair is invariant under isomorphisms. A similar notion exists for composition algebras over fields of characteristic not two, as discussed in [19].

In dimension four, each of the four double sign blocks is classified by a three-parameter family of algebras. The classification was accomplished in [20], using group action categories, which we will discuss in Section 2.3. In dimension eight, an isomorphism condition between orthogonal isotopes of $\mathbb{O}$ was given in [6]: the map

$$\varphi : \mathbb{O}_{f,g} \to \mathbb{O}_{f',g'}$$

is an isomorphism if and only if φ is a proper isometry and satisfies

$$f' = \varphi_1 f \varphi^{-1} \quad \text{and} \quad g' = \varphi_1 g \varphi^{-1} \tag{2.2}$$

for some pair (φ_1, φ_2) of triality components of φ. This leads us to the study of proper isometries and triality, which we will introduce next.

2.2 Related Structures

As the theorems of Albert and Kaplansky indicate, the study of composition algebras is closely related to that of orthogonal groups. Historically, composition algebras have been used to understand the structure of various groups related to these. Our approach will in a sense be the converse of this, as we will base our study of composition algebras on actions of orthogonal groups and their subgroups, as well as on the representation theory of their corresponding Lie algebras. It is also in this context that triality makes a natural appearance.

2.2.1 Groups and Group Schemes

Orthogonal operators on a finite-dimensional quadratic space (V,q) with q strictly non-degenerate form a subgroup of the general linear group of V. In this section we will give a brief introduction to orthogonal and related groups. A more elaborate discussion on the topic is found in [28]. We write $\mathrm{GO}(q)$ for the group of similarities of q, i.e. linear operators f on V for which there exists a non-zero scalar $\mu(f)$ with $q(f(x)) = \mu(f)q(x)$ for all $x \in V$. This defines a group homomorphism $\mu : \mathrm{GO}(q) \to k^*$, the kernel of which is precisely the orthogonal group $\mathrm{O}(q)$ consisting of all isometries with respect to q, and this gives rise to the short exact sequence

$$1 \to \mathrm{O}(q) \to \mathrm{GO}(q) \xrightarrow{\mu} k^* \to 1$$

of groups. There is also a homomorphism $\iota : k^* \to \mathrm{GO}(q)$, mapping a scalar to the corresponding multiple of the identity. The projective similarity group $\mathrm{PGO}(q)$ is the quotient group $\mathrm{GO}(q)/\iota(k^*)$, which fits into the exact sequence

$$1 \to k^* \to \mathrm{GO}(q) \to \mathrm{PGO}(q) \to 1.$$

When dealing with a Euclidean space of dimension d with its norm form n, as is the case with absolute valued algebras, we write O_d for $\mathrm{O}(n)$ and SO_d for $\mathrm{O}^+(n)$, the subgroup of O_d consisting of proper isometries, i.e. isometries having determinant one. Proper similarities and isometries can be defined over arbitrary fields and give us the group $\mathrm{GO}^+(q)$ of proper similarities, from which one obtains the groups $\mathrm{O}^+(q)$ and $\mathrm{PGO}^+(q)$ via the above sequences. The definition of being proper, which makes sense even in characteristic 2, can be given in terms of induced maps on Clifford algebras, as described in [28]. Namely, each $f \in \mathrm{GO}(q)$ gives rise to an operator $C(f)$ on the even Clifford algebra $C_0(V,q)$, the restriction of which to the centre of $C_0(V,q)$ has order at most two. The operator f is then called proper precisely if the restriction of $C(f)$ to the centre is trivial. This agrees with the more classical definitions via determinants in characteristic not two.

As we have seen, isomorphisms of composition algebras are isometries. This in particular implies that the automorphism group of a composition algebra with quadratic form q is a subgroup of $\mathrm{O}(q)$. One example, which we shall often come back to, is the automorphism group of an octonion algebra. Given an octonion algebra C with quadratic form q, the group $\mathrm{Aut}(C)$ is a proper subgroup of $\mathrm{O}^+(q)_1$, the group of all proper isometries on C fixing the unity. More precisely, it is a connected, simple algebraic group of type G_2, and thus has dimension 14.

The above mentioned groups are all groups of rational points of affine group schemes. An *affine scheme* over a field k is a functor from the category k-Alg of all unital commutative associative k-algebras to the category of sets, which is *representable*, i.e. naturally isomorphic to the functor $\mathrm{Hom}_{k\text{-Alg}}(A_0, \_)$ for some $A_0 \in k$-Alg. One then says that A_0 represents the functor. An *affine group scheme* is a functor from k-Alg to the category of groups, whose composition with the forgetful functor into the category of sets is representable. An *algebraic group* is an affine group scheme which is algebraic, i.e. represented by a finitely generated algebra, and smooth (see e.g. [28]). It is known that an algebra represents an affine group scheme if and only if it is a *Hopf algebra*, and the study of affine group schemes is in some sense dual to that of Hopf algebras. The theory of affine group schemes is established in [13], [39], and [28], and the reader is referred there for a detailed account.

2.2.2 Triality

The *Principle of Triality* was first discovered in 1925 by Elie Cartan, and formulated in [7] in terms of the real division algebra $\mathbb{O}$ of the octonions, and its Euclidean norm form n. In concrete terms, it is the statement that for any $\varphi \in \mathrm{SO}_8$ there exist $\varphi_1, \varphi_2 \in \mathrm{SO}_8$ such that for any $x, y \in \mathbb{O}$,

$$\varphi(xy) = \varphi_1(x)\varphi_2(x). \tag{2.3}$$

The maps φ_1 and φ_2 are called *triality components*, and the pair (φ_1, φ_2) is uniquely determined by φ up to an overall sign. Note that $\varphi \in \mathrm{SO}_8$ is an automorphism of $\mathbb{O}$ if and only if (φ, φ) is a pair of triality components of φ. We then say that φ has trivial triality components.

The Principle of Triality has in fact been shown to hold for any octonion algebra over any field, and appears in different guises. Given an octonion algebra C with quadratic form q, a *related triple* is a triple $(\varphi, \varphi_1, \varphi_2) \in \mathrm{O}^+(q)^3$

such that (2.3) holds with respect to the multiplication in C. The set $T(q)$ of all such triples is in fact a subgroup of $\mathrm{O}^+(q)^3$, which is isomorphic to the spin group $\mathrm{Spin}(q)$.

Triality induces an automorphism of order three of the group $\mathrm{Spin}(q)$, and thence also of $\mathrm{PGO}^+(q)$, as described in [28, §35]. This is perhaps most transparently shown using symmetric composition algebras rather than octonion algebras. A composition algebra is called *symmetric* if the bilinear form associated to its quadratic form is associative, i.e. satisfies

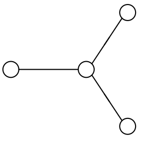
$$b_q(xy,z) = b_q(x,yz).$$

For any Hurwitz algebra H with standard involution κ, the *para-Hurwitz algebra* $H_{\kappa,\kappa}$ is an example of a symmetric composition algebra. As is shown in [28] and [9], to each symmetric composition algebra S of dimension eight with quadratic form q, and to each $\varphi \in \mathrm{GO}^+(q)$ there exist $\varphi_1, \varphi_2 \in \mathrm{GO}^+(q)$ such that (2.3) holds with respect to the multiplication in S. The pair (φ_1, φ_2) is unique up to certain scalar multiples, and the assignment $\varphi \mapsto \varphi_1$ induces an outer automorphism ρ^S of $\mathrm{PGO}^+(q)$ of order three: we indeed have

$$[\varphi] \xrightarrow{\rho^S} [\varphi_1] \xrightarrow{\rho^S} [\varphi_2] \xrightarrow{\rho^S} [\varphi], \tag{2.4}$$

where square brackets denote the quotient projection onto $\mathrm{PGO}^+(q)$. This also induces an automorphism of the corresponding affine group schemes.

The existence of an outer automorphism of order three is a property which algebraic groups in general do not have. Such automorphisms are induced by graph automorphisms of the corresponding Dynkin diagram. For $\mathrm{PGO}^+(q)$, this is the D_4-diagram



which admits graph automorphisms of order three, permuting the outer vertices. Among the finite Dynkin diagrams, D_4 is unique with this property, as all other diagrams have automorphism groups of order at most 2.

In fact, as we show in Paper V, triality can be defined with respect to any eight-dimensional composition algebra. The consequences of this are discussed in connection to that paper.

2.2.3 Lie Algebras of Derivations and their Representations

A classical approach to e.g. division algebras and composition algebras uses their derivation algebras. A *derivation* of an algebra A over a field k is a linear operator δ on A, satisfying

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for all $a,b \in A$. This relation, known as the Leibniz rule, generalizes the usual product rule for derivatives, which gives derivations their name. The set $\mathrm{Der}(A)$ of all derivations of A is a Lie algebra over k under the commutator bracket $[d,d'] = dd' - d'd$, known as the *derivation algebra of* A.

If A is a finite-dimensional real composition algebra, then $\mathrm{Aut}(A)$ is a real Lie group, and $\mathrm{Der}(A)$ is the Lie algebra of $\mathrm{Aut}(A)$. In fact (see [28, §21]) for an arbitrary finite-dimensional algebra over any field, the derivation algebra is the Lie algebra of the automorphism group scheme of A. In this sense, the derivation algebra encodes the symmetries of the algebra. As $\mathrm{Der}(A)$ acts on A in the obvious way, this action endows A with the structure of a $\mathrm{Der}(A)$-module, i.e. a representation of the Lie algebra $\mathrm{Der}(A)$.

The advantage of viewing algebras as modules over their derivation algebras is that one can then use tools from the well-developed representation theory of Lie algebras to study and classify them. Indeed, if $\varphi : A \to B$ is an isomorphism of finite-dimensional algebras, then the map

$$\mathrm{Der}(A) \to \mathrm{Der}(B), \quad \delta \mapsto \varphi\delta\varphi^{-1},$$

is an isomorphism. Thus the isomorphism type of the derivation algebra is an isomorphism invariant of the algebras, and isomorphisms of algebras from A to B map $\mathrm{Der}(A)$-submodules to $\mathrm{Der}(B)$-submodules. Therefore, given a category of algebras, knowledge about the derivation algebras and the corresponding submodule structure of the algebras simplifies the classification problem: on the one hand, the category splits into blocks according to the type of the derivation algebra, and on the other hand, in each block the isomorphisms are a priori known to respect the submodule structure. The more non-trivial the derivation algebra is, the more useful this approach becomes.

Derivation algebras in general and in connection to octonion algebras were already studied by Jacobson in [24] and [25]. More recently, in [3] and [4] the authors study the derivation algebras of finite-dimensional real division algebras in detail, while in [19] and [32], the derivation approach is applied to finite-dimensional composition algebras over general fields. Among composition algebras with large derivation algebras, one finds the octonion and para-octonion algebras, and certain algebras known as Okubo algebras, which were discovered by Okubo in connection to $\mathrm{SU}(3)$ particle physics, and which have been further studied in e.g. [18].

2.3 Group Action Categories

The use of a category theoretic language offers a conceptual framework from which we will benefit throughout the thesis. For each field k, we denote by $\mathscr{C}(k)$ the category in which the objects are the finite-dimensional composition algebras over k, and the morphisms are the algebra homomorphisms between them which are isometries. (Recall that algebra isomorphisms between objects in $\mathscr{C}(k)$ are automatically isometries.) We also write $\mathscr{D}(k)$ for the category of finite-dimensional division algebras over k, with non-zero algebra homomorphisms as morphisms. Over the real numbers we write $\mathscr{A}$ for the category of all finite-dimensional absolute valued algebras with non-zero algebra homomorphisms as morphisms. The morphisms in $\mathscr{A}$ are known to be isometries. Thus $\mathscr{A} = \mathscr{C}(\mathbb{R}) \cap \mathscr{D}(\mathbb{R})$ and is a full subcategory of $\mathscr{D}(\mathbb{R})$. For each $d \in \{1,2,4,8\}$ we further write $\mathscr{A}_d$ for the full subcategory consisting of all algebras in $\mathscr{A}$ of dimension d.

The morphisms in these categories are injective, and thus the full subcategories consisting of all objects of a fixed dimension are *groupoids*, where a groupoid is a (not necessarily small) category in which all morphisms are isomorphisms. In contrast to module categories over associative algebras, and to other categories arising in representation theory, the categories we consider here are not abelian, nor even additive, and therefore most methods used for these types of categories fail. We thus need a different approach.

An important class of groupoids consists of those arising from group actions. Let G be a group acting from the left on a set X. The corresponding *group action category* is then defined as the category where the objects are the elements of X, and the morphisms are given by the action of G in the sense that for each $x,y \in X$, the set of all morphisms from x to y is

$$_GX(x,y) = \{(x,y,g) | g \in G, gx = y\}.$$

The inclusion of x and y in the notation is done in order to distinguish morphisms between different pairs of objects. Thus a morphism is essentially an element $g \in G$ mapping x to y under the action of G. This category is clearly a groupoid. Note that different actions of G on X give rise to different group action categories, whence $_GX$ also depends on the choice of an action. We will however always subsume this in the notation, as in practice it will always be clear which action is in question. Following [15], we define a *description (in the sense of Dieterich)* of a groupoid $\mathscr{C}$ as a quadruple $(G,X,\alpha,\mathscr{F})$, where G is a group, X is a set, $\alpha : G \times X \to X$ is a group action, and $\mathscr{F} : {}_GX \to \mathscr{C}$ is an equivalence of categories. We then say that $\mathscr{C}$ is *described by* $_GX$.

Descriptions were first systematically introduced in [15], where they were used as a tool to study subcategories of $\mathscr{D}(\mathbb{R})$. The idea is that once a description is explicitly found, the problem of classifying the groupoid $\mathscr{C}$ up to isomorphism is transferred to the normal form problem for the action at hand, i.e. the problem of finding a cross-section for its orbits.

In what follows we will construct descriptions of various categories, prove structural results for general descriptions, and generalize the concept by introducing *quasi-descriptions*, as motivated by the problems we shall encounter.

3. Summary of Results

Felix qui potuit rerum cognoscere causas.
Publius Vergilius Maro

Our study of finite-dimensional composition algebras is to a large part concerned with real composition algebras which are division algebras, i.e. with absolute valued algebras. We begin by determining the morphisms of such algebras of dimension at most 4, and studying their properties, which we do in Paper I. With the eight-dimensional case in mind, we develop in Paper II a general framework for constructing full subcategories of group action categories, and apply it to $\mathscr{A}_8$ to obtain the first instances of algebras determined by hyperplane reflections. This is extended in Paper III, where we find a decomposition of the category $\mathscr{A}$ using invariants defined through reflections, and thus obtain in dimension eight a number of subcategories whose classification problem we simplify and express in geometric terms. In Paper IV we use representation theory of Lie algebras and obtain a classification of all algebras in $\mathscr{A}$ having a non-abelian derivation algebra. Finally, in Paper V, we work over general fields and establish a correspondence between eight-dimensional composition algebras and certain pairs of automorphisms of affine group schemes. In this chapter we shall outline the main ideas and results of the papers.

3.1 Morphisms and Paper I

Among all algebras which we have mentioned above, some of the most completely understood algebras are the absolute-valued algebras of dimension at most four. For these algebras, an explicit classification up to isomorphism exists. The case of dimension 1 and 2 has already been mentioned; the category of four-dimensional absolute valued algebras has been shown to consist of four double-sign blocks, each equivalent to

$$ {}_{\mathrm{SO}_3}(\mathrm{SO}_3 \times \mathrm{SO}_3) $$

where the action is by simultaneous conjugation, and this has led to a classification. (See [20] and [15].) The automorphism groups of the algebras in the classification are known as well. In order to arrive at a full understanding of the category $\mathscr{A}_{\leq 4}$ of all absolute valued algebras of dimension at most four, it remains to describe all morphisms of this category. This is the topic of Paper I. As the morphisms are injective, we need only consider, on the one hand, morphisms from the one-dimensional algebras to those of dimension 2 and 4, and morphisms from two-dimensional to four-dimensional algebras.

3.1.1 Idempotents and Subalgebras

We first consider morphisms from the unique (up to isomorphism) one-dimensional algebra $\mathbb{R}$ in $\mathscr{A}$ to algebras of higher dimensions. For each $A \in \mathscr{A}$, the assignment $\varphi \mapsto \varphi(1)$ gives a bijection between morphisms $\mathbb{R} \to A$ and non-zero idempotents in A. Determining the morphisms from $\mathbb{R}$ to A thus amounts to describing the set $\mathrm{Ip}(A)$ of all non-zero idempotents of A. By a result of Segre from [36], $\mathrm{Ip}(A)$ is nonempty whenever A is a finite-dimensional real division algebra. For the four two-dimensional algebras in (2.1), it is easy to see that all algebras except the para-complex algebra $\mathbb{C}_{\kappa,\kappa}$ have the complex number 1 as their unique non-zero idempotent, while for the para-complex numbers, the non-zero idempotents are the third roots of unity. In Paper I we determine the idempotents of all four-dimensional absolute valued algebras, using the aforementioned classification. For some algebras, the idempotents are determined explicitly. In the general case, the idempotents of an algebra A are given explicitly up to finding the real roots of an explicitly constructed rational polynomial p_A of degree five. As it turns out, whenever A does not have double sign $(-,-)$, the quintic p_A factors into a quadratic and a cubic polynomial, and is therefore solvable by radicals. In the $(-,-)$-case, however, we prove that there exist algebras whose corresponding polynomial is unsolvable. A new feature in dimension four is the existence of algebras with infinitely many idempotents.

The general picture is captured by the following result, which sums up the behaviour of the idempotents.

Theorem 3.1.1. *Let A be an absolute valued algebra of dimension at most four. Then A satisfies one of the following conditions.*

(i) $\mathrm{Ip}A$ *is finite and* $|\mathrm{Ip}A|$ *is 1, 3 or 5.*

(ii) The double sign of A is not $(-,-)$, *and* $\mathrm{Ip}A = S \cup \{p\}$ *with S a 1-sphere and p equidistant to S.*

(iii) The double sign of A is $(-,-)$, *and* $\mathrm{Ip}A = S \cup \{p\}$ *with S a 2-sphere and p equidistant to S.*

All possible cases occur.

To shed some further light on these results, we recall that it was proved in [6] that for any finite-dimensional absolute valued algebra A, the set $\mathrm{Ip}A$ is a union of a finite number (possibly zero) of smooth manifolds and an odd number of isolated points. Moreover, some open problems were formulated as to the number of isolated points and the types of manifolds that can occur. Our above result answers this problem in the case of dimension at most four.

The next step is to determine the morphisms from the two-dimensional algebras to the four-dimensional ones. As the morphisms are injective, this precisely amounts to describing all ways to embed a two-dimensional absolute valued algebra as a subalgebra of a four-dimensional one. The question as to which four-dimensional algebras admit a subalgebra isomorphic to one of the four isotopes of $\mathbb{C}$ from (2.1) was answered by Ramirez in [34]. It however remains to describe the morphisms themselves, which is done explicitly in Paper I. As a consequence, we obtain the following general picture.

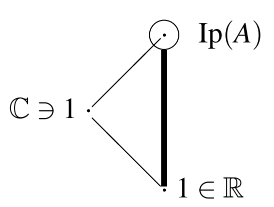
Theorem 3.1.2. *Let $C \in \mathscr{A}_2$ and $A \in \mathscr{A}_4$. If the morphism set $\mathscr{A}(C,A)$ is non-empty, then it consists of m disjoint copies of the sphere $\mathbb{S}^n$, where $m \in \{1,3\}$ is the cardinality of* $\mathrm{Ip}(C)$*, and $n \in \{0,1,2\}$ depends only on A.*

A key part of the proof consists of explicitly determining the idempotents of the algebras in $\mathscr{A}_4$ which admit a two-dimensional subalgebra.

3.1.2 Irreducibility and Actions of Automorphism Groups

A morphism is called *irreducible* if it is not an isomorphism and cannot be written as the composition of two non-isomorphisms. In view of injectivity, a morphism from an absolute valued algebra C to another one A is irreducible whenever there is no chain of proper subalgebras $C \subset B \subset A$. When such a chain exists, the question of irreducibility becomes non-trivial. In the present setting, this occurs precisely when $\dim C = 1$ and $\dim A = 4$, and A contains a two-dimensional subalgebra. For such A we must therefore investigate which $e \in \mathrm{Ip}A$ correspond to morphisms which factor over two-dimensional subalgebras. Recalling from the above that we have an explicit description of the idempotents of A whenever it contains a two-dimensional subalgebra, we can carry this out, and as a result we obtain the number of idempotents in A which correspond to irreducible and reducible morphisms. The following example illustrates the situation more closely in one case.

Example 3.1.3. Let $a = \cos\theta + i\sin\theta \in \mathbb{H}$ with $\pi/3 < \theta < \pi/2$ and, noting that the norm of a is 1, let $A = \mathbb{H}_{L_a,R_a}$. This algebra satisfies item (ii) of Theorem 3.1.1. The graph below has as vertices all non-zero idempotents in all subalgebras of A, and as arrows all irreducible morphisms between them, understood to be directed upwards. Thus the isolated idempotent of A corresponds to a reducible morphism which factors through $\mathbb{C}$, while the other idempotents all correspond to irreducible morphisms. (The thickened line is interpreted as one arrow to each point of the circle.)



We then turn to the behaviour of the morphisms under the actions of the automorphism groups of the algebras. Given two algebras C and A, the automorphism groups $\mathrm{Aut}(A)$ and $\mathrm{Aut}(C)$ act on $\mathscr{A}(C,A)$ by composition from the left and right, respectively. The number of orbits of each action measures, in some sense, the rigidity of the set of morphisms. The sets $\mathscr{A}(C,A)$ being explicitly described for each $C \in \mathscr{A}_2$ and $A \in \mathscr{A}_4$, and the automorphisms of all such C and A being known, we can compute these numbers for each possible action. Denoting the number of orbits with respect to the $\mathrm{Aut}(A)$-action by n_A and with respect to the $\mathrm{Aut}(C)$-action by n_C, we find the following.

Proposition 3.1.4. *Let $C \in \mathscr{A}_2$ and $A \in \mathscr{A}_4$. Then the pair (n_C, n_A) attains one of*

$$(1,1),(1,2),(1,3),(1,6),(\infty,1),(\infty,3).$$

All of these pairs do occur for suitable $C \in \mathscr{A}_2$ and $A \in \mathscr{A}_4$. Moreover, the action of $\mathrm{Aut}(A) \times \mathrm{Aut}(C)$ *on $\mathscr{A}(C,A)$, defined by $(\tau,\sigma)\cdot\varphi = \tau\varphi\sigma^{-1}$, is transitive.*

This essentially means that all morphisms are equal up to composition by automorphisms of C and A, but not up to composition by automorphisms of only one of the algebras.

3.2 Stabilizers of Group Actions and Paper II

The study of a category for which a description in the sense of Dieterich exists is transferred, by means of this description, to the study of a group action category. For some such categories, such as $\mathscr{A}_8$, the full classification problem is beyond reach, and one may ask for a method which uses the description to construct suitable subcategories. In Paper II, we propose a systematic framework for constructing subcategories of any category for which a description exists, which we apply to $\mathscr{A}_8$. Using the isomorphism condition (2.2) from [6], we first give a description of the category: the group SO_8 acts on

$$(\mathrm{O}_8 \times \mathrm{O}_8)\,/\,\{\pm(1,1)\} \tag{3.1}$$

by triality, i.e. for all $f,g \in O_8$,

$$\varphi\cdot[f,g] = \left[\varphi_1 f\varphi^{-1}, \varphi_2 g\varphi^{-1}\right],$$

where (φ_1,φ_2) are triality components with respect to $\mathbb{O}$, and square brackets denote the quotient projection. The equivalence from the group action category thus arising to $\mathscr{A}_8$ is then defined on objects by mapping $[f,g]$ to $\mathbb{O}_{f,g}$, and on morphisms by mapping each $\varphi \in \mathrm{SO}_8$ to the algebra homomorphism φ. The fact that triality components are difficult to compute, together with the fact that the dimension of $\mathrm{O}_8 \times \mathrm{O}_8$ is 56, render the classification problem difficult, which serves as a motivation for our approach.

3.2.1 Full Subsets

When constructing subcategories of a category $\mathscr{C}$ for which a description is given, certain non-trivial constraints need to be imposed. To begin with, we require that the subcategory be full. With the classification problem in mind, this implies that a classification of the subcategory up to isomorphism gives a classification of the objects up to isomorphism in $\mathscr{C}$. Secondly, in order to be able to take advantage of the description, we require that if $\mathscr{C}$ is described by the action of a group G on a set X, then the subcategory be described by the induced action of a subgroup $H \leq G$ on a subset $Y \subseteq X$.

In Paper II, this is done in full generality. Let X be a set and let G be a group acting on X. Given any subset $Y \subseteq X$, we consider its stabilizer

$$\mathrm{St}(Y) = \{g \in G | g \cdot Y \subseteq Y\}.$$

This subset is not in general a group, as it does not necessarily contain the inverses of its elements, and one can easily find examples where it actually fails to be a group. On the other hand the action of G on X induces an action of H on Y for a subgroup H of G if and only if H is contained in $\mathrm{St}(Y)$. Since we are interested in full subcategories, we need to use the largest subgroup contained in $\mathrm{St}(Y)$. This is $\mathrm{St}(Y) \cap \mathrm{St}(Y)^{-1}$, which we denote by $\mathrm{St}^*(Y)$. We can now formulate our problem in precise terms.

Question. What are sufficient and necessary condition on a subset $Y \subseteq X$ in order for ${}_{\mathrm{St}^*(Y)}Y$ to be a full subcategory of ${}_GX$?

This problem is solved in Paper II as follows, where the destabilizer $\mathrm{Dest}(Y)$ of $Y \subseteq X$ is the set of all $g \in G$ such that $g \cdot Y \cap Y = \emptyset$.

Theorem 3.2.1. *Let G be a group acting on a set X, and let $\emptyset \neq Y \subseteq X$. Then the following conditions are equivalent.*
(i) The subcategory ${}_{\mathrm{St}^*(Y)}Y$ of ${}_GX$ is full.
(ii) $G = \mathrm{St}(Y) \sqcup \mathrm{Dest}(Y)$.
(iii) The collection $\pi = \{g \cdot Y | g \in G\}$ is a partition of $G \cdot Y \subseteq X$.
If any, hence all, of the above conditions hold, then $\mathrm{St}^*(Y) = \mathrm{St}(Y)$, and moreover, there is a bijection $\rho : G/\mathrm{St}^*(Y) \to \pi$ between the left cosets of $\mathrm{St}^*(Y)$ and the classes of π, given by $g\,\mathrm{St}^*(Y) \mapsto g \cdot Y$.

We call the set Y *full* if it satisfies the equivalent conditions of the theorem.

3.2.2 Application

When the conditions of the theorem are fulfilled, we are able to derive structural results for the category ${}_GX$. Thus equipped, we return to $\mathscr{A}_8$ and its above quoted description, and look for full subsets Y of (3.1) which moreover satisfy

$$\mathrm{St}^*(Y) \leq \mathrm{Aut}(\mathbb{O})$$

in order for the triality components to be trivial.

Our inspiration comes from the full subcategory of $\mathscr{A}_8$ whose objects have a one-sided unity, which was classified in [10]. Let us consider the left unital algebras. These are exhausted by all $\mathbb{O}_{f,g}$ with $f(1) = 1$ and $g = \mathrm{Id}$. The stabilizer of the corresponding collection of objects in (3.1) is $\mathrm{Aut}(\mathbb{O})$ and, as is proved in [10], classifying the algebras amounts to solving the normal form problem for the action of this group on the set of all $f \in \mathrm{O}_8$ which fix $1 \in \mathbb{O}$, which the is identified with O_7, by conjugation.

In Paper II we consider what is in a sense the simplest possible extension, namely the collection of all $[f,g]$ in (3.1) with $f(1) = 1$ and g being the reflection in a hyperplane containing 1. The isotopes $\mathbb{O}_{f,g}$ corresponding to these form a full subcategory of $\mathscr{A}_8$, which was observed by Dieterich to be dense in the full subcategory $\mathscr{A}_8^S$ of $\mathscr{A}_8$ consisting of all algebras in which left multiplication by some idempotent is a hyperplane reflection. The category $\mathscr{A}_8^S$ is closed under isomorphisms, and we obtain the following result.

Proposition 3.2.2. *Fix an element $u \in \mathbb{O}$ with norm one, orthogonal to the unity. The set*

$$Y = \Big\{ [f,g] \in (\mathrm{O}_8 \times \mathrm{O}_8)\,/\,\{\pm(1,1)\} \,|\, f(1) = 1 \text{ and } g \text{ is the reflection in } u^\perp \Big\}$$

is full in $(\mathrm{O}_8 \times \mathrm{O}_8)/\{\pm(1,1)\}$, and $\mathrm{St}^*(Y)$ is a subgroup of $\mathrm{Aut}(\mathbb{O})$ isomorphic to the semidirect product $\mathrm{SU}_3 \rtimes C_2$. Moreover, ${}_{\mathrm{St}^*(Y)}Y$ is equivalent to $\mathscr{A}_8^S$, and to the category arising from the action of $\mathrm{St}^*(Y)$ on O_7 by conjugation.

The normal form problem for the action of $\mathrm{St}^*(Y)$ on O_7 by conjugation strictly refines the normal form problem solved in [10]. Using and generalizing an argument by Dieterich, we reformulate the normal form problem at hand modulo the one solved. Thence we reduce the problem to the study of the action of certain subgroups of $\mathrm{Aut}(\mathbb{O})$ on the projective space of octonions orthogonal to the unity, which we solve in some cases.

3.3 The Reflection Approach and Paper III

The above application in Paper II deals with eight-dimensional absolute valued algebras in which left multiplication is, in some sense, determined by a hyperplane reflection. In general, left multiplication by elements of norm 1 in such algebras is given by an orthogonal operator on eight-dimensional Euclidean space. By the Cartan–Dieudonné theorem, such operators are generated by reflections.

Theorem 3.3.1. *(Cartan–Dieudonné) Every orthogonal operator f with respect to a non-degenerate quadratic form on a vector space of finite dimension d over a field of characteristic not two is the product of at most d hyperplane reflections.*

The least such number is called the *length* $\lambda(f)$ of f. A proof is found in [16], where the characteristic two case is also discussed. This provides a means to generalize our approach. If A is an absolute valued algebra of finite dimension d and $e \in A$ is a non-zero idempotent, then multiplicativity of the norm implies that left and right multiplication by e are orthogonal. The above theorem and the fact that multiplication by idempotents has non-trivial fixed points imply that the pair $(\lambda(L_e), \lambda(R_e))$ belongs to $\{0, \ldots, d-1\}^2$. Since any finite-dimensional absolute valued algebra has non-zero idempotents in view of the previously mentioned result by Segre, we may define the *left reflection type* of A as the minimum of $(\lambda(L_e), \lambda(R_e))$ with respect to the lexicographic order, as e ranges over all non-zero idempotents of A. In the same way we define the *right reflection type* of A as the minimum of $(\lambda(R_e), \lambda(L_e))$, and the *minimal reflection type* as the minimum of the left and right reflection type. This assigns to A three pairs of natural numbers between 0 and $d-1$.

The reflection types thus defined behave well with respect to isomorphisms. Denoting the full subcategory of $\mathscr{A}_d$ consisting of all algebras with left (resp. right, minimal) reflection type (m,n) by $\mathscr{L}_d^{m,n}$ (resp. $\mathscr{R}_d^{m,n}$, $\mathscr{M}_d^{m,n}$), we get the following block decompositions of $\mathscr{A}_d$.

Proposition 3.3.2. *For each $d \in \{1,2,4,8\}$ and each $(m,n) \in [d-1]^2$, the subcategories $\mathscr{L}_d^{m,n}$, $\mathscr{R}_d^{m,n}$ and $\mathscr{M}_d^{m,n}$ of $\mathscr{A}_d$ are closed under isomorphisms. Moreover,*

$$\mathscr{A}_d = \coprod_{0 \leq m,n \leq d-1} \mathscr{L}_d^{m,n} = \coprod_{0 \leq m,n \leq d-1} \mathscr{R}_d^{m,n} = \coprod_{0 \leq m \leq n \leq d-1} \mathscr{M}_d^{m,n}.$$

The decompositions by left and right reflection type are equivalent (in fact, isomorphic), while the minimal type decomposition essentially combines the two. The choice of which one to use depends largely on which kinds of algebras one is interested in studying in detail, since as we will see, with respect to each reflection type, the blocks which are the easiest to study will be those for which the particular reflection type is small with respect to the lexicographic order. This is exemplified by the results in [10] and our findings in Paper II. Using the left reflection type, this occurs for algebras which possess idempotents the left multiplication by which is the product of few reflections, and vice versa for the right reflection type. The minimal reflection type is left-right symmetric, as an algebra has low minimal reflection type if it contains an idempotent by which either left or right multiplication has simple structure. The first part of the paper establishes structural results for the different blocks.

3.3.1 Descriptions in Dimension Eight

The next step is to turn to the classification of these blocks in dimension eight, toward which the first step is to provide a description. As in Paper II, we are particularly interested in blocks for which the descriptions are given by actions of $\mathrm{Aut}(\mathbb{O})$ and its subgroups, thus avoiding triality. To do so, we note that conjugating an orthogonal map on $\mathbb{O}$ by elements from $\mathrm{Aut}(\mathbb{O})$ preserves its length, and that automorphisms of $\mathbb{O}$ fix $1 \in \mathbb{O}$. Thus for each $0 \le m, n \le 7$, $\mathrm{Aut}(\mathbb{O})$ acts on $\mathrm{O}^n \times \mathrm{O}^m$ by simultaneous conjugation, where O^k is the set of all isometries of $\mathbb{O}$ of length k which fix 1. This gives rise to the group action category

$$\mathscr{O}^{m,n} = {}_{\mathrm{Aut}(\mathbb{O})}\left(\mathrm{O}^n \times \mathrm{O}^m\right).$$

We then have the following results.

Proposition 3.3.3. *Let* $0 \le m, n \le 7$.

(i) If $(m,n) \preceq (4,3)$ *or* $n = 0$*, then* $\mathscr{L}_8^{m,n}$ *and* $\mathscr{R}_8^{m,n}$ *are equivalent to* $\mathscr{O}^{m,n}$.

(ii) If $m, n \le 4$*, then* $\mathscr{L}_8^{m,n}$*,* $\mathscr{L}_8^{n,m}$*,* $\mathscr{R}_8^{m,n}$ *and* $\mathscr{R}_8^{n,m}$ *are pairwise equivalent.*

(iii) If $m + n < 8$*, then* $\mathscr{M}^{m,n}$ *is the coproduct of one or two blocks, each equivalent to* $\mathscr{O}_8^{m,n}$.

The equivalences are given by means of descriptions, mapping each pair $(f,g) \in \mathrm{O}^n \times \mathrm{O}^m$ to $\mathbb{O}_{f,g}$, and acting as the identity on morphisms.

From here we deduce that the blocks for which we have an explicit description are described by the action of $\mathrm{Aut}(\mathbb{O})$ on $\mathrm{O}^n \times \mathrm{O}^m$ with $m \le 3$, which we therefore consider in detail. Our strategy is to find a transversal for the orbits of the action of $\mathrm{Aut}(\mathbb{O})$ on O^m by conjugation for each $m \le 3$, and then reduce the problem above to the study of the actions of the stabilizers of the elements in this transversal on O^n for $0 \le n \le 7$. We find that eight subgroups and a one-parameter family of subgroups of $\mathrm{Aut}(\mathbb{O})$ occur as stabilizers, as detailed in Table 1.

What remains is to find a normal form for the action of the subgroups H appearing in Table 3.1 on each O^n by conjugation. Notice that this was done in [10] for the case where H is the full automorphism group of $\mathbb{O}$, and generalizing the argument by Dieterich mentioned in connection to Paper II, we see that we can build our approach on this classification, provided that we have a good understanding of the coset spaces $\mathrm{Aut}(\mathbb{O})/H$ for all H appearing in the table, which are G-spaces for $G = \mathrm{Aut}(\mathbb{O})$. While solving the normal form is now feasible, it requires much technical work, and remains beyond the scope of the paper. Nevertheless, we give an equivariant geometric interpretation of these spaces. We refer to the paper for the full result, and content ourselves with one of the easier examples.

Example 3.3.4. Fix an orthonormal pair u, v of octonions orthogonal to 1. The group $G_2^{[v,uv]}$ is then defined as the subgroup of $\mathrm{Aut}(\mathbb{O})$ fixing the (v,uv)-plane

Block	Orbits	Transversals	Stabilizer	Parameters
O^0	O^0_0	$\{\mathrm{Id}\}$	$\mathrm{Aut}(\mathbb{O})$	
O^1	O^1_1	$\{\sigma_u\}$	$G_2^{[u]} \simeq \mathrm{SU}_3 \rtimes C_2$	
O^2	$\mathrm{O}^2(\pi/2)$	$\rho(\pi/2)$	$G_2^{[v,uv]} \simeq \mathbb{S}^3 \rtimes \mathrm{O}_2$	
	$\mathrm{O}^2(\theta)$	$\rho(\theta)$	$SG_2^{[v,uv]} \simeq \mathbb{S}^3 \rtimes \mathrm{SO}_2$	$\theta \in (0,\pi/2)$
O^3	$\mathrm{O}^3(\theta,0)$	$\rho(\theta,0)$	$SG_2^{[v,uv]} \simeq \mathbb{S}^3 \rtimes \mathrm{SO}_2$	$\theta \in (0,\pi/2)$
	$\mathrm{O}^3(\theta,\pi/2)$	$\rho(\theta,\pi/2)$	$SG_2^{[v,uv],[z]} \simeq C_2 \times \mathrm{SO}_2$	$\theta \in (0,\pi/2)$
	$\mathrm{O}^3(\theta,\eta)$	$\rho(\theta,\eta)$	$SG_2^{[v,uv],z} \simeq \mathrm{SO}_2$	$\theta,\eta \in (0,\pi/2)$
	$\mathrm{O}^3(\pi/2,0)$	$\rho(\pi/2,0)$	$G_2^{\mathbb{H}} \simeq \mathrm{SO}_4$	
	$\mathrm{O}^3(\pi/2,\eta)$	$\rho(\pi/2,\eta)$	$\widehat{\eta}\Delta(H_u)\widehat{\eta}^{-1} \simeq \mathrm{SO}_3$	$\eta \in (0,\pi/2)$
	$\mathrm{O}^3(\pi/2,\pi/2)$	$\rho(\pi/2,\pi/2)$	$G_2^{[v,uv,z]} \simeq \mathrm{O}_3$	

Table 3.1. *The sets* O^m *with* $m \leq 3$ *under the action of* $\mathrm{Aut}(\mathbb{O})$ *by conjugation. We refer to Paper III for the precise definitions of the orbits, transversals and stabilizers in the table.*

as a set. Consider the Grassmannian $\mathrm{Gr}(2,1^\perp)$ of all planes in $\mathbb{O}$ orthogonal to 1, with the action of $\mathrm{Aut}(\mathbb{O})$ induced by its action on $\mathbb{O}$ by automorphisms. Then there is an (equivariant) isomorphism of $\mathrm{Aut}(\mathbb{O})$-spaces from $G_2^{[v,uv]}$ to $\mathrm{Gr}(2,1^\perp)$, mapping the coset of $\varphi \in G_2$ to the span of $\varphi(v)$ and $\varphi(uv)$.

3.4 Derivations and Paper IV

Over the past recent decades much research in division and composition algebras has been devoted to trying to understand those algebras which exhibit a high degree of symmetry. As we have seen, one way to quantify the amount of symmetry is through the Lie algebra of derivations. One is thus led to study composition algebras with a non-abelian derivation algebra. For these algebras, no complete classification is known, even in the case where the algebras are finite-dimensional real division composition algebras, i.e. absolute valued algebras. In Paper IV we achieve a classification of these algebras.

As a basis of our work we use [32], where the finite-dimensional division composition algebras with a non-abelian derivation algebra over a field of characteristic not two or three are explicitly expressed as isotopes of Hurwitz algebras. The possible Lie algebras of derivations are listed, as well as the possible dimensions of the irreducible submodules into which the composition algebras decompose as modules over their derivation algebras. The fact that the algebras are completely reducible is derived in [32], using the properties of the quadratic form of the algebras.

In what follows, for each $d \in \{1,2,4,8\}$ and each partition π of d, we write $\mathscr{D}_\pi$ for the full subcategory of $\mathscr{A}_d$ consisting of all algebras whose derivation algebra is non-abelian, and where the partition π of d is given by the dimensions of the irreducible submodules into which the algebras decompose as modules over their derivation algebras.

Example 3.4.1. The derivation algebra of each $A \in \mathscr{A}_1 \cup \mathscr{A}_2$ is abelian. The category of all $A \in \mathscr{A}_4$ with a non-abelian derivation algebra is precisely $\mathscr{D}_{1,3}$, and is classified by

$$\left\{\mathbb{H}, \mathbb{H}_{\mathrm{Id},\kappa}, \mathbb{H}_{\kappa,\mathrm{Id}}, \mathbb{H}_{\kappa,\kappa}\right\},$$

where κ is the standard involution on $\mathbb{H}$. The derivation algebra is of type $\mathfrak{su}_2$ in all cases, and the decomposition into irreducible submodules corresponds to the decomposition into real and imaginary quaternions.

The category $\mathscr{D}$ of all eight-dimensional absolute valued algebras with a non-abelian derivation algebra is thus the main topic of the paper. From [32], and since the partition π is invariant under isomorphisms, we know that $\mathscr{D}$ decomposes into blocks as follows.

Proposition 3.4.2. *The category $\mathscr{D}$ decomposes as the coproduct*

$$\mathscr{D}_{1,7} \amalg \mathscr{D}_8 \amalg \mathscr{D}_{1,1,6} \amalg \mathscr{D}_{1,3,4} \amalg \mathscr{D}_{1,1,2,4} \amalg \mathscr{D}_{1,1,1,1,4} \amalg \mathscr{D}_{3,5} \amalg \mathscr{D}_{1,1,3,3}.$$

For each $A \in \mathscr{D}$, $\mathrm{Der}(A)$ *is of type $\mathfrak{g}_2$ if $A \in \mathscr{D}_{1,7}$, $\mathfrak{su}_3$ if $A \in \mathscr{D}_8 \amalg \mathscr{D}_{1,1,6}$, $\mathfrak{su}_2 \times \mathfrak{su}_2$ or $\mathfrak{su}_2 \times \mathfrak{a}$ if $A \in \mathscr{D}_{1,3,4}$, and $\mathfrak{su}_2 \times \mathfrak{a}$ otherwise, where $\mathfrak{a}$ is an abelian Lie algebra with* $\dim \mathfrak{a} \leq 1$.

Note that on the one-dimensional submodules, the action of the derivation algebra is trivial. The block $\mathscr{D}_8$ consists of all algebras which are irreducible over their derivation algebras. It is known that these are precisely the real Okubo algebras that are division algebras, and it is further known that they constitute a unique isomorphism class. Apart from this block, the only blocks for which a classification is known are $\mathscr{D}_{1,7}$, for which the classification is analogous to that of $\mathscr{D}_{1,3}$, with $\mathbb{H}$ replaced by $\mathbb{O}$, and $\mathscr{D}_{1,1,6}$, which was essentially classified by four 2-parameter families of algebras in [19].

In the first part of the paper, we use the description of $\mathscr{A}_8$ from Paper II to obtain a description of the above blocks, with the exception of $\mathscr{D}_8$, $\mathscr{D}_{3,5}$ and $\mathscr{D}_{1,1,3,3}$. The descriptions are all given by actions of subgroups of $\mathrm{Aut}(\mathbb{O})$, which avoids triality. Nevertheless, they are quite technical and it is difficult to obtain an overview of the structure of the category. This motivates the search for a simpler concept. By examining the descriptions at hand, we observe that if one does not want to keep track of all isomorphisms, but only of the property of being isomorphic, a simplification is possible.

3.4.1 Quasi-Descriptions

The simplification lies in the introduction of quasi-descriptions.

Definition 3.4.3. A functor $\mathscr{F} : \mathscr{B} \to \mathscr{C}$ between two categories $\mathscr{B}$ and $\mathscr{C}$ is said to *detect non-isomorphic objects* if $\mathscr{F}(B) \not\simeq \mathscr{F}(B')$ in $\mathscr{C}$ whenever $B \not\simeq B'$ in $\mathscr{B}$. A *quasi-description* of $\mathscr{C}$ is a quadruple $(G, X, \alpha, \mathscr{F})$ where G is a group, X is a set, α is a left action of G on X, and $\mathscr{F} : {}_G X \to \mathscr{C}$ is a dense functor which detects non-isomorphic objects.

This can be compared to the definition of descriptions, which form a special case of quasi-descriptions. Dense functors detecting non-isomorphic objects map classifications of $\mathscr{B}$ to classifications of $\mathscr{C}$. Thus, as in the case of descriptions, classifying a category for which a quasi-description is given amounts to solving the normal form problem of the corresponding group action. The advantage is that the action can be taken with respect to a smaller group than would have been needed to obtain a description. In some cases such as the one at hand, this simplifies the problem.

Thus equipped, we return to the category $\mathscr{D}$. We set

$$\mathscr{L}_0 = \mathscr{D}_8 \amalg \mathscr{D}_{1,7} \amalg \mathscr{D}_{1,3,4} \quad \text{and} \quad \mathscr{H}_0 = \mathscr{D}_{1,1,6} \amalg \mathscr{D}_{1,1,2,4} \amalg \mathscr{D}_{1,1,1,1,4}.$$

Then we have $\mathscr{D} = \mathscr{L} \amalg \mathscr{H}$, where

$$\mathscr{L} = \mathscr{L}_0 \amalg \mathscr{D}_{3,5}$$

is the full subcategory of $\mathscr{D}$ consisting of all algebras with trivial submodule of dimension at most one, and

$$\mathscr{H} = \mathscr{H}_0 \amalg \mathscr{D}_{1,1,3,3}$$

is the full subcategory of $\mathscr{D}$ consisting of all algebras with trivial submodule of dimension at least two. For the categories $\mathscr{L}_0$ and $\mathscr{H}_0$, we obtain quasi-descriptions. The group actions are induced by the action of $\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{SO}_3$ on the sphere $\mathbb{S}^3$, viewed as the unit sphere in $\mathbb{H}$. We call this action *the action of* SO_3 *on* $\mathbb{S}^3$.

Theorem 3.4.4. *For each subcategory $\mathscr{C}$ of $\mathscr{D}$, let $\mathscr{C}^{ij}$ be the full subcategory consisting of the algebras having double sign (i, j).*

(i) For each $(i, j) \in \{+,-\}^2$, the category $\mathscr{L}_0^{ij}$ is quasi-described by the action of SO_3 *on $(\mathbb{S}^3 \times \mathbb{S}^3)$ induced by the action of* SO_3 *on $\mathbb{S}^3$.*

(ii) For each $(i, j) \in \{+,-\}^2$, the category $\mathscr{H}_0^{ij}$ is quasi-described by the action of SO_3 *on S induced by the action of* SO_3 *on $\mathbb{S}^3$, where S is obtained from*

$$\left((\mathbb{S}^3 \times \mathbb{S}^3)/\{\pm(1,1)\}\right)^2$$

by removing the four elements $([1, \pm 1], [1, \pm 1])$.

The functors in each quasi-description are explicitly given in the paper. The normal form problem for these group actions is now manageable, and is completely solved in the paper.

We are thus left with the blocks $\mathscr{D}_{3,5}$ and $\mathscr{D}_{1,1,3,3}$ which fall outside the quasi-descriptions, and which we treat directly. The block $\mathscr{D}_{3,5}$ is then seen to consist of precisely three isomorphism classes, and by fixing an Okubo algebra in $\mathscr{D}_8$, we obtain a classification of $\mathscr{D}_{3,5}$ consisting of three isotopes of the Okubo algebra. The block $\mathscr{D}_{1,1,3,3}$ is more complicated. With the help of an argument by Elduque and several computations we were able to obtain a classification consisting of twelve 2-parameter families of algebras, appearing as isotopes of algebras in $\mathscr{D}_{1,1,6}$.

3.5 Triality, Algebraic Groups and Paper V

As the reader has perhaps now inferred, when dealing with eight-dimensional composition algebras, one inevitably has to deal with triality. In Papers II-IV, our approach has been to avoid triality in the sense of considering or systematically constructing subcategories of algebras for the classification problem of which the considerations associated with triality disappear. In Paper V we take a different approach, where the aim is to understand these considerations conceptually and on a higher level.

The inspiration for this paper comes from the recent publications [9] and [8], where the authors establish a correspondence between eight-dimensional symmetric composition algebras with quadratic form q and trialitarian automorphisms of the affine group scheme $\mathbf{PGO}^+(q)$, i.e. outer automorphisms of order three. More specifically, the authors assign to each symmetric composition algebra S with quadratic form q the automorphism ρ^S of $\mathrm{PGO}^+(q)$ from (2.4). This induces an automorphism of the affine group scheme $\mathbf{PGO}^+(q)$, and it is proved that isomorphisms of composition algebras correspond to conjugation in $\mathrm{Aut}(\mathbf{PGO}^+(q))$. The authors further classify the objects on either side using this correspondence and a classification of the objects on the other.

In Paper V, we generalize this approach to arbitrary eight-dimensional composition algebras. To begin with, we establish triality for general composition algebras over any field of characteristic not two.

Proposition 3.5.1. *Let C be an eight-dimensional composition algebra with quadratic form q over a field k of characteristic different from two. Then for each $\varphi \in \mathrm{GO}^+(q)$ there exist $\varphi_1, \varphi_2 \in \mathrm{GO}^+(q)$ such that for each $x, y \in C$,*

$$\varphi(x) = \varphi_1^C(x)\varphi_2^C(x).$$

The pair $\left(\left[\varphi_1^C\right], \left[\varphi_2^C\right]\right) \in \mathrm{PGO}^+(q)^2$ is moreover uniquely determined by φ, and the assignments

$$\rho_i^C : [\varphi] \mapsto \left[\varphi_i^C\right]$$

define outer automorphisms of $\mathrm{PGO}^+(q)$, which induce automorphisms of the affine group scheme $\mathbf{PGO}^+(q)$.

We notice that for a general composition algebra C, the map ρ_2^C is not the inverse of ρ_1^C, as is the case for symmetric composition algebras. In fact, for each composition algebra C with quadratic form q there exist $f,g \in \mathrm{O}(q)$ such that $S = C_{f,g}$ is symmetric, and then

$$\rho_1^C = \kappa_{[f]}\rho^S \quad \text{and} \quad \rho_2^C = \kappa_{[g]}\left(\rho^S\right)^2$$

where $\kappa_{[f]}$ is conjugation by $[f]$ in $\mathrm{PGO}(q)$, which induces an automorphism of $\mathbf{PGO}^+(q)$, and likewise for g.

The quotient of the automorphism group of $\mathbf{PGO}^+(q)$ by the group of inner automorphisms is isomorphic to the symmetric group S_3, viewed as the automorphism group of the Dynkin diagram D_4. For a trialitarian automorphism τ, the quotient projection of $\{\tau,\tau^2\}$ consists of the two different order three elements in S_3. Generalizing this, we assign to each eight-dimensional composition algebra with quadratic form q what we call a *trialitarian pair* of automorphisms of $\mathbf{PGO}^+(q)$. This is a pair (τ_1,τ_2) of (necessarily outer) automorphisms such that the quotient projection of $\{\tau_1,\tau_2\}$ consists of the two order three elements of S_3. The group $\mathrm{PGO}(q)$ acts on pairs of automorphisms of $\mathbf{PGO}^+(q)$ by simultaneous conjugation by $\mathbf{PGO}(q)$-inner automorphisms, which gives rise to a group action category in the usual way. Denoting by $\mathfrak{Tri}(q)$ the full subcategory of this consisting of all trialitarian pairs, we arrive at the main result of the paper, where $\mathfrak{Comp}(q)$ is the category of all composition algebras with quadratic form q, where the morphisms are all algebra isomorphisms.

Theorem 3.5.2. *For each 3-Pfister form q over a field of characteristic different from two, the map $C \mapsto \left(\rho_1^C,\rho_2^C\right)$ defines an equivalence of categories $\mathfrak{Comp}(q) \to \mathfrak{Tri}(q)$, acting on morphisms by $\varphi \mapsto [\varphi]$.*

From this result we are able to deduce the isomorphism condition (2.2) for eight-dimensional absolute valued algebras due to [6], and, over fields of characteristic not two, a generalization of it to eight-dimensional composition algebras over arbitrary fields due to [12]. We can moreover express the double sign of a composition algebra C in terms of the order of the quotient projection of ρ_1^C and ρ_2^C onto S_3.

4. Epilogue

The papers summarized above have dealt with composition algebras, division algebras, absolute valued algebras, algebraic groups, representation theory of Lie algebras, and category theory. Most results have been obtained due to fruitful interactions between these areas, by combining tools inherent in them with those developed or refined along the way. In this manner, many questions about the structure and classification of composition algebras have been answered, and new questions have arisen. The aim of this chapter is to sketch a work in progress and a few paths for future research.

4.1 Trace Invariant Maps

In the recent paper [14], Dieterich introduced the concept of trace invariant maps. The idea is, roughly speaking, that non-trivial information about an algebra A over a field k is contained in various linear and bilinear forms given by traces of multiplication operators. Using this information, one aim is to obtain decompositions of categories of algebras in meaningful ways. This is illustrated by the following classical example.

Example 4.1.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic zero. Consider the bilinear form κ on $\mathfrak{g}$, defined by

$$\kappa(x,y) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)).$$

This is the Killing form of $\mathfrak{g}$, and it is a classical result from the theory of Lie algebras that $\mathfrak{g}$ is semisimple if and only if κ is non-degenerate.

Along the lines of thought in [14], this can be phrased as follows. For each $d \in \mathbb{N}$, the category $\mathscr{L}_d(k)$ of all d-dimensional Lie algebras over k, with isomorphisms as morphisms, decomposes as the coproduct of two blocks, one of which consisting of all semisimple objects, and the other of all object that are not semisimple.

The framework of trace invariant maps, of which this example becomes a special case, is set up in [14] as follows. Fix a field k and a natural number d. Let $\mathscr{I}\mathscr{V}_d(k)$ be the category where the objects are all 9-tuples

$$(V, \lambda_1, \lambda_2, \beta_1, \dots, \beta_6)$$

where V is a k-vector space of dimension d, λ_i is a linear form on V for each i, and β_j is a bilinear form on V for each j, and the morphisms are all linear isomorphisms which are compatible with the linear and bilinear forms. For each d-dimensional k-algebra A one associates the object

$$\mathscr{F}(A) = (A, \tau_L, \tau_R, \beta_L, \beta_R, \beta_{LL}, \beta_{RR}, \beta_{LR}, \beta_{RL}) \in \mathscr{I}\mathscr{V}_d(k),$$

where the linear forms are defined by $\tau_L(x) = \operatorname{tr} L_x^A$, $\tau_R(x) = \operatorname{tr} R_x^A$, and the bilinear forms are given by $\beta_M(x,y) = \operatorname{tr} M_{xy}^A$ and $\beta_{MN}(x,y) = \operatorname{tr} M_x^A N_y^A$ for each $M,N \in \{L,R\}$. This defines a functor from the category $\mathscr{I}\mathscr{A}_d(k)$ of all d-dimensional k-algebras, with algebra isomorphisms as morphisms, to $\mathscr{I}\mathscr{V}_d(k)$, acting on morphisms by $\mathscr{F}(\varphi) = \varphi$. A *trace invariant map* on a full subcategory $\mathscr{C}$ of $\mathscr{I}\mathscr{A}_d(k)$ is then any map $\hat{h}$ from the object class $\operatorname{Ob}(\mathscr{C})$ to any set I, satisfying

$$\hat{h} = h \circ \mathscr{F}$$

for some function $h : \mathscr{I}\mathscr{V}_d(k) \to I$ which is constant on isomorphism classes.

In the above example, we have $\mathscr{C} = \mathscr{L}_d(k)$, $I = \{0,1\}$, and

$$h : \mathscr{I}\mathscr{V}_d(k) \to I$$

defined by taking the value 1 if and only if β_3 is non-degenerate. The composed map $\hat{h}$ then gives the coproduct decomposition $\mathscr{C} = \mathscr{C}_0 \amalg \mathscr{C}_1$, where $\mathscr{C}_i$ is the full subcategory with object class $\hat{h}^{-1}(i)$, and $\mathscr{C}_1$ consists of all semisimple objects in $\mathscr{C}$.

In general, the requirement that h be constant on isomorphism classes guarantees that one obtains, in this manner, a coproduct decomposition of the full subcategory $\mathscr{C}$, i.e. the fibres of $\hat{h}$ are unions of isomorphism classes of algebras. Depending on what one wishes to achieve, by choosing h appropriately one obtains coarser or finer decompositions. For example, certain trace invariant maps give rise to decompositions into unions of *model equivalence classes* of algebras, where two algebras are called model equivalent if they become isomorphic after an extension of scalars. Such trace invariant maps are called *absolute*, and the Killing form above is, as proved in [14], one such example.

Returning to composition algebras, it is natural to ask what insight trace invariant maps can provide. In [14], Dieterich introduces an absolute trace invariant map on the category of all two-dimensional composition algebras over an arbitrary field, and proves that it is *model optimal*, i.e. that its fibres are precisely the model equivalence classes of such algebras. Using it, he thus achieves a classification of two-dimensional composition algebras up to model equivalence. The classification consists of four algebras and is analogous to (2.1). Thus for example, while a two-dimensional real composition algebra A which is not a division algebra is not isomorphic to an algebra in (2.1), it becomes so after an extension of scalars.

In a recently initiated joint work, Dieterich and the author aim to apply the framework of trace invariant maps to four-dimensional composition algebras over arbitrary fields. These algebras have been described in characteristic not two, but not classified. Questions of interest are which trace invariant maps give appropriate decompositions, and how these decompositions look. In addition, one may ask how close to a model optimal trace invariant map one can come. As this seems promising, one may further construct and study absolute trace invariant maps on eight-dimensional composition algebras, possibly guided by the results obtained in this thesis as to their classification problem.

In general, classifications up to model equivalence provide a systematization of the category at hand, and are also useful as an intermediate step toward a classification up to isomorphism. Indeed, *Galois cohomology* provides a means to classify model equivalent algebras up to isomorphism. The theory is described in [21] and briefly in [28], and an interesting line of research would be to investigate what this implies for composition algebras.

4.2 Automorphisms of Algebraic Groups

Paper V establishes a correspondence between eight-dimensional composition algebras with quadratic form q and pairs of automorphisms of the affine group scheme $\mathbf{PGO}^+(q)$. This provides a rich theory to apply to the study of composition algebras, as indicated by the results obtained in [8] for symmetric composition algebras. We will outline some potential lines of research.

As isomorphisms of algebras corresponds to simultaneous conjugation of automorphisms, it becomes possible to use properties of the automorphisms which behave well with respect to conjugation, in order to obtain isomorphism invariants of composition algebras. For example, given a trialitarian pair (τ_1, τ_2) of automorphisms of $\mathbf{PGO}^+(q)$ corresponding to a composition algebra C, one approach is to study the fixed points of the automorphisms. As mentioned in [9] and [8] in connection to symmetric composition algebras, the group of fixed points of a trialitarian automorphism of $\mathbf{PGO}^+(q)$ is either of type G_2 or of type A_2 (unless the characteristic of the field is three) and thus rather large. At the other end is the case of automorphisms without non-trivial fixed points. These can be studied in the light of a result by Auslander [2], which implies that certain powers of such isomorphisms are trivial, and thus gives a means to control these automorphisms.

One advantage of the fact that isomorphisms corresponds to simultaneous conjugation is that one can obtain partial results by studying the action of $\mathrm{PGO}(q)$ on automorphisms (rather than pairs of automorphisms) of $\mathbf{PGO}^+(q)$ by conjugation, and proceed by determining and using the stabilizers of this action in a manner reminiscent of what was done in Paper III for the action of $\mathrm{Aut}(\mathbb{O})$ on $\mathrm{O}_8 \times \mathrm{O}_8$ by simultaneous conjugation.

In the other direction, one may use certain known properties or partial classifications of composition algebras to get information about their trialitarian pairs. One such property is the type of the automorphism group or the derivation algebra of a composition algebra C, and the decomposition of C as a $\mathrm{Der}(C)$-module.

4.3 Extending the Results

In addition to the above, there are a number of natural ways to seek to extend the results of this thesis. Indeed, the results of Papers I-IV hold for finite-dimensional real division composition algebras, and it is natural to ask to what extent they can be generalized to arbitrary fields. For example, the framework of Paper II via corestricted group actions is not in itself dependent on the ground field. Moreover, the approach of Papers II and III through reflections uses the Cartan–Dieudonné theorem, which holds over arbitrary fields (with one exception in characteristic 2). However, it relies on the existence of idempotents, which is not guaranteed in composition algebras over arbitrary fields. The results thus do not generalize trivially, and working toward a generalization constitutes one possible line of research.

Paper IV deals with division composition algebras with non-abelian derivation algebras. The explicit description of such algebras over arbitrary fields of characteristic not 2 or 3, provided in [32], is rather similar to that over the real numbers. The success of the methods of Paper IV thus encourages an attempt to generalize these methods to such fields. If this succeeds, one can extend the approach further by lifting the requirement of being a division algebra, or investigating the case of characteristic two or three.

In conclusion, the rich structure of composition and division algebras provides several possible manners to investigate the problems dealt with above, or to use the methods introduced in connection to them, in greater generality. It is our hope that the results of this thesis will inspire further research in related areas.

5. Kompositionsalgebror via gruppverkningar och trialitet (Summary in Swedish)

5.1 Bakgrund

Studiet av kompositionsalgebror går tillbaka till 1800-talets mitt, då den irländske matematikern W. R. Hamilton sökte finna ett algebraiskt ramverk för det tredimensionella rummet, genom att generalisera operationerna på den endimensionella reella tallinjen och det tvådimensionella komplexa talplanet. Hamilton sökte en aritmetik där de fyra räknesätten kunde genomföras och där det fanns ett geometriskt avståndsbegrepp som var kompatibelt med dessa. Efter att förgäves ha sökt ett sådant system under lång tid, insåg Hamilton att det var nödvändigt att införa en fjärde dimension, och han upptäckte därmed *kvaternionerna*. Kort senare upptäcktes de åttadimensionella *oktonionerna* oberoende av T. Graves och A. Cayley. Båda dessa system medger de fyra räknesätten i någon form, och är utrustade med ett längdbegrepp, eller *absolutbelopp*, som är multiplikativt. För att möjliggöra detta var man tvungen att ge avkall på några andra egenskaper som multiplikationen av reella och komplexa tal åtnjuter: kvaternionernas multiplikation är inte kommutativ, och oktonionernas är varken kommutativ eller associativ.

Detta markerade ett steg mot studiet av abstrakta algebraiska objekt vars egenskaper är bortom vad vi är bekanta med genom vårt vardagliga räknande. De strukturer som vi här har att göra med kallas för *algebror*, där en algebra är ett vektorrum utrustat med en bilinjär multiplikation. I en algebra kan man således addera, subtrahera, multiplicera och skala elementen. Så långt kräver man varken att multiplikationen ska vara associativ eller kommutativ, eller att den ska medge division. Divisionsegenskapen hos reella tal kan karakteriseras av att ekvationerna $ax = b$ och $xa = b$ har en entydig lösning för varje nollskilt a och varje b. En nollskild algebra kallas således för en *divisionsalgebra* om motsvarande egenskaper gäller för algebrans element. En nollskild reell algebra där man har ett kompatibelt avståndsbegrepp i form av en multiplikativ norm kallas för en *absolutbeloppsalgebra*. De reella talen, de komplexa talen, kvaternionerna och oktonionerna utgör exempel på sådana algebror, och idag vet vi att ändligtdimensionella sådana algebror endast existerar i dimension 1, 2, 4 och 8. Detta förklarar således Hamiltons misslyckande med att finna en tredimensionell absolutbeloppsalgebra. Restriktionen på dimensionen gäller i allmänhet för alla ändligtdimensionella reella divisionsalgebror, men beviset för detta kräver metoder från algebraisk topologi.

En generalisering av ändligt-dimensionella absolutbeloppsalgebror fås om man betraktar *kompositionsalgebror*. Dessa är nollskilda algebror som har en multiplikativ, icke-degenererad kvadratisk form, och kan definieras över varje kropp. Den kvadratiska formen kan ses som en generalisering av absolutbeloppet, och faktum är att en ändligtdimensionell reell algebra är en absolutbeloppsalgebra om och endast om den är en divisionsalgebra såväl som en kompositionsalgebra. Över en godtycklig kropp gäller fortfarande att ändligtdimensionella kompositionsalgebror endast existerar i dimension 1, 2, 4 och 8. Deras struktur är fullständigt kartlagd i dimension 1 och 2, delvis utredd i dimension 4, men långt från förstådd i dimension 8. Ett sätt att uppnå en sådan förståelse är att klassificera algebrorna upp till isomorfi, vilket innebär att man avgör vilka algebror som är *isomorfa* eller algebraiskt ekvivalenta, och sedan konstruerar en lista med en representant från varje isomorfiklass. Detta har visat sig vara mycket svårt i dimension åtta. Ett annat problem är att förstå hur avbildningarna mellan algebror av olika dimensioner ser ut, och därmed hur algebrorna kan bäddas in i varandra.

Kompositionsalgebror har flera tillämpningar inom och utanför algebra. De är relaterade till algebraiska grupper och Liegrupper och används för att beskriva de så kallade exceptionella grupperna. I dimension åtta beskriver de trialitetsfenomenet, en viss typ av symmetri som endast förekommer i den dimensionen. Kvaternionerna och därtill besläktade algebror finner tillämpningar i kodningsteorin, medan vissa åtta-dimensionella kompositionsalgebror förekommer inom partikelfysiken. Detta tillsammans med algebrornas inneboende rikedom på struktur ger en motivation att studera och så gott det går förstå strukturen hos dem. Denna avhandling syftar till att vidga den systematiska förståelsen av (ändligtdimensionella) kompositionsalgebror i allmänhet och absolutbeloppsalgebror i synnerhet, genom att tillämpa av och utveckla nya metoder från kategoriteori, representationsteori, samt teorin för gruppverkningar och algebraiska grupper. Genom att använda sådana, ofta geometriska, metoder kan man få en konceptuell förståelse och i många fall en detaljerad insikt i strukturen hos algebrorna, vilket ömsom bäddar för, ömsom uppnår, en klassifikation upp till isomorfi.

5.2 Sammanfattning av avhandlingens resultat

Vår studie börjar med att betrakta absolutbeloppsalgebror av dimension högst fyra. För dessa existerar redan en klassifikation av algebrorna upp till isomorfi, medan ingen känd beskrivning av avbildningarna (morfismerna) mellan algebror av olika dimension existerar. Artikel I löser detta problem genom att beskriva alla morfismer från algebror av dimension 1 och 2 till sådana av dimension 4. Morfismerna från den upp till isomorfi entydigt bestämda endimensionella absolutbeloppsalgebran $\mathbb{R}$ till en absolutbeloppsalgebra A svarar mot nollskilda *idempotenter* i A. Dessa är element $a \in A$ som uppfyller $a \cdot a = a$ med avseende på multiplikationen i A. I Artikel I beräknar vi alla idempotenter i fyradimensionella absolutbeloppsalgebror, antingen explicit eller i term av rötter till reella polynom. Vi finner att antalet nollskilda idempotenter är 1, 3, 5, eller oändligt, och består i det oändliga fallet av en cirkel eller en sfär och en isolerad punkt. Detta besvarar några öppna frågor som ställts i [6]. Vi ger vidare en explicit och fullständig beskrivning av morfismerna från de tvådimensionella algebrorna till de fyradimensionella. Slutligen undersöker vi hur morfismerna sönderfaller i irreducibla faktorer, och hur de beter sig under verkan av de ingående algebrornas automorfigrupper.

I Artikel II och III betraktar vi kategorin av åttadimensionella absolutbeloppsalgebror, för vilka isomorfiproblemet inbegriper det så kallade trialitetsfenomenet. Detta tillsammans med kategorins storlek har medfört att endast relativt små underkategorier har kunnat klassificeras, såsom i [10]. Vi ger en beskrivning (i Dieterichs bemärkelse, [15]) av denna kategori, det vill säga en kategoriekvivalens till den från en gruppverkanskategori. I en sådan ges objekten av en mängd X och morfismerna av en grupp G som verkar på mängden X, och detta innebär att vi överför isomorfiproblemet till problemet att finna ett tvärsnitt för gruppverkningens banor. Då trialiteten medför tekniska svårigheter i hanteringen av detta problem, söker vi en metod för att systematiskt bryta ned problemet i mindre delar som är mer tillgängliga och där klassifikationsproblemet blir mer hanterligt. I tekniska termer söker vi konstruera fulla underkategorier av en gruppverkanskategori, som ges av enklare gruppverkningar. Vi utvecklar ett allmänt ramverk för att göra detta, och tillämpat på kategorin av åttadimensionella absolutbeloppsalgebror ger detta upphov till hittills ostuderade underkategorier i vars isomorfiproblem de trialitetsrelaterade svårigheterna försvinner. Vi avslutar Artikel II med att studera dessa i detalj.

I Artikel III utvecklar vi resultaten från Artikel II och uppnår en blockuppdelning av hela kategorin av ändligtdimensionella absolutbeloppsalgebror, där klassifikationsproblemet kan studeras i varje block för sig. Uppdelningen uppnås genom att införa geometriska invarianter som ges av antalet reflektioner som bygger upp vissa multiplikationer i algebrorna. I dimension åtta ger detta, utöver en systematisering av kategorin, upphov till ett väsentligt antal underkategorier där trialitetsproblemet försvinner. Bland dessa ingår såväl sådana underkategorier som studerats i [10] och i Artikel II, som hittills ostuderade sådana. Vi fortsätter med att reducera klassifikationsproblemet för dessa underkategorier genom att uttrycka algebraisomorfismerna i term av verkningar av undergrupper till oktonionernas automorfigrupp. Denna grupp är relativt välförstådd, och vi avslutar med att formulera problemet i geometriska termer.

Det finns en annan metod att angripa klassifikationsproblemet, som utretts för ändligtdimensionella kompositionsalgebror och reella divisionalgebror under de senaste årtiondena. Idén här är att söka klassificera de objekt som åtnjuter mest symmetri. Ett sätt att mäta symmetrin hos en algebra är genom deriveringar, det vill säga linjära operatorer δ på algebran som uppfyller den välbekanta Leibnizregeln

$$\delta(ab) = \delta(a)b + a\delta(b).$$

Mängden av alla deriveringar på en algebra A utgör en Liealgebra, och A får strukturen av en modul över sin deriveringsalgebra. Därmed kan verktyg från den välutvecklade representationsteorin för Liealgebror användas, och detta är mer användbart om deriveringsalgebran är väsentlig, exempelvis genom att vara icke-abelsk. Flera resultat i den riktningen har uppnåtts i bland andra [3], [19] och [32], emedan en klassifikation endast är känd i de mest symmetriska fallen. I artikel IV klassificerar vi kategorin av alla absolutbeloppsalgebror av ändlig dimension vars deriveringsalgebror är icke-abelska. Detta uppnås genom att införa begreppet *kvasibeskrivning*, vilket generaliserar beskrivningsbegreppet i Dieterichs mening. Kvasibeskrivningar innehåller mindre information än beskrivningar och blir därmed enklare att hantera, men bibehåller tillräcklig struktur för att vara användbara för lösningen av klassifikationsproblemet. Därigenom förenklas huvuddelen av problemet till studiet av verkningar av rotationsgrupper på tredimensionella sfärer.

En viktig komponent i arbetena ovan har varit hanteringen av trialitetsfenomenet, som uppstår i allmänhet i samband med oktonionalgebror över godtyckliga kroppar och som uppkommer på grund av extraordinära symmetrier hos vissa algebraiska grupper. På senare tid har man observerat att fenomenet bättre kan förstås via så kallade symmetriska kompositionsalgebror snarare än oktonionalgebror. Nyligen publicerades två artiklar, [9] och [8], där författarna konstruerar en korrespondens mellan symmetriska kompositionsalgebror och så kallade trialitära automorfismer av vissa algebraiska grupper. De relaterar vidare isomorfi hos algebror till konjugation av automorfismer, som är en bekant och konceptuellt välförstådd relation. Författarna använder sedan information från vardera sidan för att bättre förstå den andra. I Artikel V generaliserar vi detta till att omfatta alla åttadimensionella kompositionsalgebor över kroppar av karakteristik skild från två, och konstruerar en korrespondens mellan dessa och vissa par av automorfismer av algebraiska grupper. Denna korrespondens uttrycks som en ekvivalens av kategorier. Ur sambandet kan vi vidare ånyo härleda tidigare beskrivningar av kompositionsalgebror.

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... mathematics is nevertheless a sociable science.
Paul Halmos

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