Ratliff-Rush Monomial Ideals

Veronica Crispin Quiñonez

ABSTRACT. Let I be a regular $\mathfrak{m}$-primary ideal in $(R, \mathfrak{m}, k)$. Then its Ratliff-Rush associated ideal $\bar{I}$ is the largest ideal containing I with the same Hilbert polynomial as I. In this paper we present a method to compute Ratliff-Rush ideals for certain classes of monomial ideals in the rings $k[x, y]$ and $k[[x, y]]$. We find an upper bound for Ratliff-Rush reductions number for these ideals. Moreover, we establish some new characterizations of when all powers of I are Ratliff-Rush.

1. Introduction

Let R be a Noetherian ring and let an ideal I in it be regular, that is, let I contain a nonzerodivisor. Then the ideals $(I^{l+1} : I^l)$, $l \geq 1$, increase with l. The union $\tilde{I} = \bigcup_{l\geq 1}^{\infty}(I^{l+1} : I^l)$ was first studied by Ratliff and Rush in [**RR**]. They show that $(\tilde{I})^l = I^l$ for sufficiently large l and that $\tilde{I}$ is the largest ideal with this property. Hence, $\tilde{\tilde{I}} = \tilde{I}$. Moreover, they show that $\widetilde{I^l} = I^l$ for sufficiently large l. We call $\tilde{I}$ the Ratliff-Rush ideal associated with I, and an ideal such that $\tilde{I} = I$ a Ratliff-Rush ideal. The Ratliff-Rush reduction number of I is defined as $r(I) = \min\{l \in \mathbb{Z}_{\geq 0} \mid \tilde{I} = (I^{l+1} : I^l)\}$.

The operation $\tilde{}$ cannot be considered as a closure operation in the usual sense, since $J \subseteq I$ does not generally imply $\tilde{J} \subseteq \tilde{I}$. An example from [**RS**] shows this: let $J = \langle y^4, xy^3, x^3y, x^4\rangle \subset I = \langle y^3, x^3\rangle \subset k[x, y]$, then I is Ratliff-Rush but $x^2y^2 \in \tilde{J}\backslash\tilde{I}$.

Several results about Ratliff-Rush ideals are given in [**HJLS**], [**HLS**] and [**RR**]. In addition to general results, one can find many examples and counterexamples with respect to different properties [**RS**]. In [**E**] the author presents an algorithm for computing Ratliff-Rush associated ideals by computing the Poincaré series and choosing a tame superficial sequence of I.

One of the reasons to study Ratliff-Rush ideals is the following. Let I be a regular $\mathfrak{m}$-primary ideal in a local ring $(R, \mathfrak{m}, k)$. We know that the Hilbert function $H_I(l) = \dim_k(R/I^l)$ is a polynomial $P_I(l)$ called the Hilbert polynomial of I for all large l. Then $\tilde{I}$ can be defined as the unique largest ideal containing I and having the same Hilbert polynomial as I.

2000 *Mathematics Subject Classification.* Primary 13C05, 13D40; Secondary 13A30, 20M14.
Key words and phrases. Ratliff-Rush ideals, powers of ideals, (Ratliff-Rush) reduction number, numerical semigroups.

Ratliff-Rush ideals associated to monomial ideals are monomial by definition, which makes the computations easier. There is always a positive integer L such that $\tilde{I} = I^{L+1} : I^L$, but it is not clear how big that L is (see p. 2 in [**RS**]). If I is a monomial ideal and m is some monomial, then for all $l \geq 0$ we have

$$(mI)^{l+1} : (mI)^l = (m^{l+1}I^{l+1}) : (m^l I^l) = m(I^{l+1} : I^l). \tag{1.1}$$

Principal ideals are trivially Ratliff-Rush. Any non-principal monomial ideal J in the rings $k[x, y]$ and $k[[x, y]]$ can be written as $J = mI$, where m is a monomial and I is an $\langle x, y\rangle$-primary ideal; hence it suffices to consider $\langle x, y\rangle$-primary monomial ideals. Moreover, (1.1) shows that the Ratliff-Rush reduction numbers of I and mI are the same.

In this paper we show how to compute Ratliff-Rush ideals associated to certain classes of monomial ideals in the rings $k[x, y]$ and $k[[x, y]]$ and find an upper bound for Ratliff-Rash reduction number for such ideals. Section 2 is devoted to some results about numerical semigroups that are crucial for our work in Section 3. In Section 4 we duscuss several useful examples.

2. Some results on numerical semigroups

A numerical semigroup S is a set of linear combinations $\lambda_1 a_1 + \cdots + \lambda_r a_r$, where $a_i \in \mathbb{Z}_{\geq 0}$ are the generators and $\lambda_i \in \mathbb{Z}_{\geq 0}$ are the coefficients. There is a partial ordering $\leq_S$ where for any pair s, s' in S, if there is $s'' \in S$ such that $s' = s + s''$ then $s \leq s'$. The set of minimal elements in $S \backslash \{0\}$ in this ordering is called a *minimal set of generators* for S. If a semigroup is generated by a set $\{a_i\}_{i=1}^r$, then we denote it by $\langle a_1, \ldots, a_r\rangle$.

Definition 2.1. Let $S = \langle a_i\rangle$ be a numerical semigroup and $\gcd(a_i) = h$. The greatest multiple of h that does not belong to S is called the *Frobenius number* of S and is denoted by $g(S)$. If $\gcd(a_i) = 1$, then the Frobenius number is the greatest integer that does not belong to S. A list of references to the papers written about this subject can be found in [**FGH**], pp. 1-2.

We notice that for any $h \in \mathbb{Z}_+$ the numerical semigroups $\langle a_i\rangle$ and $\langle ha_i\rangle$ are isomorphic.

Definition 2.2. Let $S = \langle a_1, \ldots, a_r\rangle$, where $a_1 < \cdots < a_r$, be a numerical semigroup. For $s \in S$ the coefficients in a linear combination $s = \sum \lambda_i a_i$ are not necessarily unique. We define the function $\lambda : S \to \mathbb{Z}_{\geq 0}$ by $\lambda(s) = \min\{\sum \lambda_i \mid s = \sum \lambda_i a_i\}$. Then we define the following positive number:

$$\Lambda = \Lambda(S) = \max\{\lambda(s) \mid s \leq g(S) + a_r\}. \tag{2.1}$$

Corollary 2.3. *Let $S = \langle a_1, \ldots, a_r\rangle$ with $a_1 < \cdots < a_r$. Then for $s \in S$ we have* $\lim_{s\to\infty} \frac{s}{\lambda(s)} = a_r$.

Proof. For each $s > g(S)$ there is $n \in \mathbb{Z}_{\geq 0}$ such that $g(S) + a_r n + 1 \leq s \leq g(S) + a_r(n + 1)$. Then, obviously, $\lambda(s) \geq n$ and $\lambda(s) \leq \Lambda + n$ by Definition 2.2. Hence, $\frac{g(S)+a_r n+1}{\Lambda+n} \leq \frac{s}{\lambda(s)} \leq \frac{g(S)+a_r(n+1)}{n}$. The limits of both the right hand side and the left hand side are a_r as $s \to \infty$. □

Proposition 2.4. *Let $S = \langle a_1, \ldots, a_r\rangle$ be a numerical semigroup generated by nonnegative integers $a_1 < \cdots < a_r$. Let $\alpha < 1$ and β be real nonnegative numbers. Then there is a number L such that for every integer $l \geq L$ the following is true:*

if $s \in S$ and $s \leq a_r \cdot \alpha l + \beta$, then $\lambda(s) \leq l$.

PROOF. For each $s > g(S)$ there is $n \in \mathbb{Z}_{\geq 0}$ such that $g(S) + a_r n + 1 \leq s \leq g(S) + a_r(n+1)$. Thus, $\lambda(s) \leq \Lambda + n \leq \Lambda + \frac{s-g(S)-1}{a_r}$. Hence, if $s \leq a_r \alpha l + \beta$ we get $\lambda(s) \leq \Lambda + \alpha l + \frac{\beta - g(S)-1}{a_r}$. We want to find an L such that $\lambda(s) \leq l$ for all $l \geq L$. This occures if

$$l \geq \frac{a_r \Lambda + \beta - g(S) - 1}{a_r(1-\alpha)}, \tag{2.2}$$

which is an upper bound for the number L. □

REMARK 2.5. It is easy to see the necessity of the condition $\alpha < 1$. Consider the numerical semigoup $S = \langle 2, 5 \rangle$. Let $l \in \mathbb{Z}_{\geq 0}$ and $\beta = 4$. Then for any l there is no $\lambda_1 \in \mathbb{Z}_{\geq 0}$ such that $s = 5l + 4 = \lambda_1 \cdot 5 + (l - \lambda_1) \cdot 2$.

COROLLARY 2.6. *Let $S = \langle a_i \rangle_{i=0}^r$ and $T = \langle b_i \rangle_{i=0}^r$, where $a_0 = b_r = 0$ and $a_i + b_i = d$ for all i, be numerical semigroups. Then there is a number L such that for every integer $l \geq L$ and some fixed β the following is true:*
if $s \in S$ and $s \leq d \cdot \alpha l + \beta \leq dl$, then there are $\lambda_0, \ldots, \lambda_r$ such that $s = \sum_{i=0}^r \lambda_i a_i$ and $\sum_{i=0}^r \lambda_i = l$; moreover, $dl - s = \sum_{i=0}^r \lambda_i b_i \in T$.

PROOF. By Proposition 2.4 there is some L such that for all $l \geq L$ if $S \ni s \leq d\alpha l + \beta$ then $s = \sum_{i=1}^r \lambda_i a_i$, where $\sum_{i=1}^r \lambda_i \leq l$. Letting $\lambda_0 = l - \sum_{i=1}^r \lambda_i$ we can write $s = \sum_{i=0}^r \lambda_i a_i$ where $\sum_{i=0}^r \lambda_i = l$. Then, clearly, $dl - s = d\sum_{i=0}^r \lambda_i - \sum_{i=0}^r \lambda_i a_i = \sum_{i=0}^r \lambda_i b_i \in T$. □

If S and T are as in Corollary 2.6, then for any $s \in S$ and $t \in T$ such that $s + t = dl$ we have either $s \leq \frac{dl}{2} + \beta$ or $t \leq \frac{dl}{2} + \beta$ for some $\beta \leq \frac{dl}{2}$. This estimation will be used frequently in the next two sections when we apply our results on calculating powers and Ratliff-Rush associated ideal of some monomial ideals.

EXAMPLE 2.7. In Proposition 2.4 let $\alpha = \frac{1}{2}$ and $\beta = 0$. Thus, for every $l \geq 2\Lambda - \frac{2g(S)+2}{a_r}$, if $s \in S$ and $s \leq \frac{a_r l}{2}$, then $\lambda(s) \leq l$.

EXAMPLE 2.8. For any $l \in \mathbb{Z}_{\geq 0}$ every $s \in S$ belongs to the interval $\frac{a_r(l+j)}{2} \leq s \leq \frac{a_r(l+j+1)-1}{2}$ for some $j \geq -l$. That is, the assumptions in Proposition 2.4 are fulfilled for $\alpha = \frac{1}{2}$ and $\beta = \frac{a_r(j+1)-1}{2}$. Hence, for all $l \geq 2\Lambda + (j+1) - \frac{3+2g(S)}{a_r}$ if $s \leq \frac{a_r l}{2} + \frac{a_r(j+1)-1}{2}$ then $\lambda(s) \leq l$.

3. Ratliff-Rush ideals associated to certain monomial ideals

Now we will apply the results from the previous section in order to compute Ratliff-Rush ideals for some monomial cases. We start with the case where all the minimal generators for the ideal have the same degree.

3.1. Ideals generated by monomials of the same degree. Let $I = \langle x^{a_i} y^{b_i} \rangle_{i=0}^r$ be an $\mathfrak{m}$-primary ideal generated by the monomials of the same degree d ordered in such a way that $a_i < a_{i+1}$ and $b_i > b_{i+1}$; in other words, $a_0 = b_r = 0$ and $b_i = d - a_i$ for all i. To this ideal we associate the numerical semigroups $S = \langle a_i \rangle_{i=0}^r$ and $T = \langle b_i \rangle_{i=0}^r$.

The ideal I^l is generated by monomials of degree dl, namely by

$$\{\prod_{\sum l_i = l} (x^{a_i} y^{b_i})^{l_i} = x^{\sum l_i a_i} y^{\sum l_i b_i}\}. \tag{3.1}$$

Here $\sum l_i a_i \in S,\ \sum l_i b_i \in T$ and $\sum l_i a_i + \sum l_i b_i = \sum l_i(a_i + b_i) = dl$.

Theorem 3.1. *Let an ideal $I = \langle x^{a_i} y^{b_i}\rangle_{i=0}^{r} \subset R$ and the corresponding numerical semigroups $S = \langle a_i\rangle_{i=0}^{r}$ and $T = \langle b_i\rangle_{i=0}^{r}$. Then there is an integer L such that for any $l \geq L$ the following is true:*

$$I^l = \langle x^s y^t \mid s \in S \text{ and } t \in T \text{ such that } s + t = dl\rangle. \tag{3.2}$$

Moreover, for l sufficiently large:

(1) *if $s \in S,\ s \leq u$ and $s + u \geq dl$, where $u \in \mathbb{Z}_{\geq 0}$, then $x^s y^u \in I^l$;*
(2) *if $t \in T,\ t \leq v$ and $t + v \geq dl$, where $v \in \mathbb{Z}_{\geq 0}$, then $x^v y^t \in I^l$.*

Proof. The inclusion $I^l \subseteq \langle x^s y^t \mid s \in S,\ t \in T \text{ and } s + t = dl\rangle$ is true for all l, which is clear from the text preceeding the theorem.

The other inclusion needs to be proved since $s+t = dl$ does not generally imply that $s = \sum_{i=0}^{r} \lambda_i a_i$ with $\sum_{i=0}^{r} \lambda_i = l$ or $t = \sum \mu_i b_i$ with $\sum \mu_i = l$. However, this is asserted by Corollary 2.6 as we will see below. Thus, we will prove the second part of the theorem, because this other inclusion is a special case of it, since if $s+t = dl$ then either $s \leq t$ or $t \leq s$.

(1) If $s+u \geq dl$ then there is some j such that $d(l+j) \leq s+u \leq d(l+j+1)-1$. We will show that for any such s, u and j we have $x^s y^u \in I^{l+j} \subset I^l$. Clearly, it is sufficient to consider the case $j = 0$, that is, suppose $dl \leq s + u = dl + b \leq d(l + 1) - 1$. If $s \leq u$ then $s \leq \frac{dl}{2} + \frac{d-1}{2}$. Then, by Corollary 2.6, for sufficiently large l we can write $s = \sum \lambda_i a_i$ and $u = b + \sum \lambda_i b_i$, where $\sum \lambda_i = l$. Hence, $x^s y^u = y^b \prod_i (x^{a_i} y^{b_i})^{\lambda_i} \in I^l$.

Part (2) is proved similarly. □

Remark 3.2. By Example 2.8 an upper bound for the least integer L in Theorem 3.1 is $\lceil \max\left(2\Lambda(S) + 1 - \frac{3+2g(S)}{d}, 2\Lambda(T) + 1 - \frac{3+2g(T)}{d}\right)\rceil$, where $\lceil c\rceil$ denotes the least integer which is greater or equal to c.

Definition 3.3. Let the assumptions be as in Theorem 3.1. We introduce the following ideals: $I_S = \langle x^s y^{d-s} | s \in S \text{ and } s \leq d\rangle$ and $I_T = \langle x^{d-t} y^t | t \in T \text{ and } t \leq d\rangle$.

Proposition 3.4. *Let the assumptions be as in Theorem 3.1. Then for every l sufficiently large*

$$I^l = \left(y^{dl-d}\right) I_S + \left(x^{dl-d}\right) I_T + \left(x^d y^d\right) I_{M,l} \tag{3.3}$$

for some ideal $I_{M,l}$.

Proof. Let $l \geq \max(\{\lambda(s) \mid S \ni s \leq d\}, \{\lambda(t) \mid T \ni t \leq d\})$.

Then $y^{dl-d}(x^s y^{d-s}) \in I^l$ if and only if $s = \sum \lambda_i a_i \leq d$ where $\sum \lambda_i a_i = l$. Equivalently, $x^{dl-d}(x^{d-t} y^t) \in I^l$ if and only if $t = \sum \lambda_i b_i \leq d$. Finally, the generators for I^l such that both the power of x and y is equal to or greater than d can be written as the third term in (3.3) where $I_{M,l} = I : (x^d y^d)$. □

Example 3.5. Let $I = \langle y^7, x^2y^5, x^5y^2, x^7\rangle$. Then $I_S = \langle y^7, x^2y^5, x^4y^3, x^5y^2, x^6y, x^7\rangle$ and $I_T = \langle y^7, xy^6, x^2y^5, x^3y^4, x^5y^2, x^7\rangle$. For $l \geq 3$ we can write $I^l = y^{7l-7} I_S + x^{7l-7} I_T + x^7 y^7 I_{M,l}$ for some $I_{M,l}$. For $l \geq 4$ the ideal $I_{M,l} = \mathfrak{m}^{7l-14}$.

Remark 3.6. Generally, if $g(S)$ and $g(T)$ are less or equal to $d - 1$, then $I_{M,l} = \mathfrak{m}^{d(l-2)}$ for all sufficiently large l.

PROPOSITION 3.7. *Let $I = \langle x^{a_i}y^{b_i}\rangle_{i=0}^r \subset R$, $S = \langle a_i\rangle_{i=0}^r$ and $T = \langle b_i\rangle_{i=0}^r$. Then the Ratliff-Rush ideal associated to I is*

$$\tilde{I} = I_S \cap I_T.$$

PROOF. We will show that $I^{l+1} : I^l = I_S \cap I_T$ for all sufficiently large l. Since I is monomial, a polynomial p belongs to I^{l+1} if and only if every power product in p belongs to I^{l+1}. Hence, it suffices to consider monomial ring elements.

Let $m \in I_S \cap I_T$. Then $m = m'x^{s'}y^{d-s'} = m''x^{d-t''}y^{t''}$. We know that for all sufficiently large l the generators for I^l are on the form x^sy^t where $s \in S,\ t \in T$ and $s + t = dl$, that is either $s \leq \frac{dl}{2}$ or $t \leq \frac{dl}{2}$. Assume $s \leq \frac{dl}{2}$. Then, using the first equality for m, we get $m \cdot x^sy^t = m'x^{s+s'}y^{dl+d-(s+s')}$. Since $s + s' \leq \frac{dl}{2} + d = \frac{d(l+1)}{2} + \frac{d}{2}$, then by Corollary 2.6 there is some integer L_S use we can write such that for all $l \geq L_S$ we can write $s + s' = \sum \lambda_i a_i$ and $d(l + 1) - (s + s') = \sum \lambda_i b_i$ where $\sum \lambda_i = l + 1$. Hence, $mx^sy^t = \prod_i (x^{a_i}y^{b_i})^{\lambda_i} \in I^{l+1}$.

Using the equality $m = m''x^{d-t''}y^{t''}$ and Corollary 2.6 we show in the same way that there is some L_T such that for all $l + 1 \geq L_T$ if $t \leq \frac{dl}{2}$ then $mx^sy^t = \prod_i (x^{a_i}y^{b_i})^{\mu_i} \in I^{l+1}$.

On the other hand, assume $m \notin I_S$. Then $my^{dl} \notin y^{dl}I_S$ and, hence, $my^{dl} \notin I^{l+1}$ by Proposition 3.4. Analogously, if $m \notin I_T$ then $mx^{dl} \notin I^{l+1}$, which finishes the proof. □

COROLLARY 3.8. *Let $I = \langle x^{a_i}y^{b_i}\rangle_{i=0}^r \subset R$, $S = \langle a_i\rangle_{i=0}^r$ and $T = \langle b_i\rangle_{i=0}^r$ its corresponding numerical semigroups. If for every pair a_i and a_j we have either $a_i + a_j = a_k$ for some k or $a_i + a_j \geq d$, then I is Ratliff-Rush.*

PROOF. Clearly, the set $\{s \in S \mid s \leq d\} = \{a_i\}_{i=0}^r$ and then $I_S = I$. Since the inclusion $I \subseteq I_T$ is always valid, we conclude that $\tilde{I} = I_S \cap I_T = I \cap I_T = I$. □

PROPOSITION 3.9. *Let I, S and T be as in Theorem 3.1. Then there is an upper bound for the reduction number of I:*

$$r(I) \leq \lceil \max\left(2\Lambda(S) + 2 - \frac{g(S)+1}{d}, 2\Lambda(T) + 2 - \frac{g(T)+1}{d}\right)\rceil - 1. \tag{3.4}$$

PROOF. The proof of Proposition 3.7 asserts that the upper bound is, using the notations from there, equal to $\lceil \max(L_S, L_T)\rceil$. The result follows from the formula (2.2) in Proposition 2.4 with $\alpha = \frac{1}{2}$ and $\beta = d$. □

EXAMPLE 3.10. Let I be the ideal in Example 3.5. Then $\tilde{I} = I_S \cap I_T = \langle y^7, x^2y^5, x^4y^4, x^5y^2, x^7\rangle$. It is interesting to note that I^l satisfies (3.1) for all $l \geq 5$ by Remark 3.2, but actually for all $l \geq 4$. Further, $r(I) = 1$ while the upper bound suggested by 3.9 is five.

EXAMPLE 3.11. Let $I = \langle y^{18}, x^3y^{15}, x^{13}y^5, x^{18}\rangle$. Then $\tilde{I} = I_S \cap I_T = \langle y^{18}, x^3y^{15}, x^8y^{12}, x^9y^{10}, x^{13}y^5, x^{18}\rangle$ and $r(I) = 4$. Thus, the minimal generators for $\tilde{I}$ do not need to be of the same degree.

3.2. Ideals generated by $x^{a_i}y^{b_i}$ such that $\frac{a_i}{a_r}+\frac{b_i}{b_0}=1$. Here we discuss slight generalizations of the subject in Section 3.1 to $\langle x,y\rangle$-primary monomial ideals $\langle x^{a_i}y^{b_i}\rangle_{i=0}^r$ such that $\frac{a_i}{a_r}+\frac{b_i}{b_0}=1$ where $\gcd(b_0,a_r)=d$. We can, of course, apply the results directly using the numerical semigroups $S'=\frac{d}{a_r}\cdot S$ and $T'=\frac{d}{b_0}\cdot T$. However, it might be useful to devote some space to formulate the material differently in order to make it possible to widen the results.

Corollary 3.12. *Let $S=\langle a_i\rangle_{i=0}^r$ and $T=\langle b_i\rangle_{i=0}^r$, where $a_0=b_r=0$ and $\frac{a_i}{a_r}+\frac{b_i}{b_0}=1$ for all i, be numerical semigroups. Then there is a number L such that for every integer $l\geq L$ and some fixed β the following is true:*

if $s\in S$ and $s\leq a_r\cdot\alpha l+\beta\leq dl$ then there are $\lambda_0,\dots,\lambda_r$ such that $s=\sum_{i=0}^r\lambda_i a_i$ and $\sum_{i=0}^r\lambda_i=l$; moreover, $l-\frac{s}{a_r}=\frac{1}{b_0}\sum_{i=0}^r\lambda_i b_i\in\frac{1}{b_0}\cdot T$.

Proof. The proof differs from the one of Corollary 2.6 by the last sentence, which here should be:
$b_0(l-\frac{s}{a_r})=b_0(\sum\lambda_i-\frac{1}{a_r}\sum\lambda_i a_i)=b_0\big(\sum\lambda_i(1-\frac{a_i}{a_r})\big)=\sum\lambda_i b_i\in T.$ □

Theorem 3.13. *Let $I=\langle x^{a_i}y^{b_i}\rangle_{i=0}^r\subset R$ be an $\mathfrak{m}$-primary ideal such that $a_0=b_r=0$ and $\frac{a_i}{a_r}+\frac{b_i}{b_0}=1$. Let $S=\langle a_i\rangle_{i=0}^r$ and $T=\langle b_i\rangle_{i=0}^r$ be numerical semigroups. Then there is an integer L such that for any $l\geq L$ the following is true:*

$$I^l=\langle x^sy^t\mid s\in S\text{ and }t\in T\text{ such that }\frac{s}{a_r}+\frac{t}{b_0}=l\rangle. \tag{3.5}$$

Moreover, for l sufficiently large:

1. *if $s\in S$, $\frac{s}{a_r}\leq\frac{u}{b_0}$ and $\frac{s}{a_r}+\frac{u}{b_0}\geq l$ for some $u\in\mathbb{Z}_{\geq0}$, then $x^sy^u\in I^l$;*
2. *if $t\in T$, $\frac{t}{b_0}\leq\frac{v}{a_r}$ and $\frac{t}{b_0}+\frac{v}{a_r}\geq l$ for some $v\in\mathbb{Z}_{\geq0}$, then $x^vy^t\in I^l$.*

Proof. The ideal I^l is a subideal of the right hand side of (3.5) by the definition of S and T and the condition on the exponents.

To prove (1) it suffices to show that if $l\leq\frac{s}{a_r}+\frac{u}{b_0}=l+q\leq l+1$ for some rational q then $x^sy^u\in I^l$; compare to the proof of Theorem 3.1.

If $\frac{s}{a_r}\leq\frac{u}{b_0}$ and $\frac{s}{a_r}+\frac{u}{b_0}\leq l+1$, then $s\leq\frac{a_rl}{2}+\frac{a_r}{2}$. Thus, by Corollary 3.12, for sufficiently large l we can write $s=\sum\lambda_i a_i$ where $\sum\lambda_i=l$. Further, let some $t=\sum\lambda_i b_i$, then $\frac{s}{a_r}+\frac{t}{b_0}=\sum\lambda_i(\frac{a_i}{a_r}+\frac{b_i}{b_0})=l$. Hence, $u\geq t$ and we get $x^sy^u\in I^l$.

Part (2) is proved similarly. □

Proposition 3.14. *Let $I=\langle x^{a_i}y^{b_i}\rangle\subset R$, S and T be as in Theorem 3.13. Then the Ratliff-Rush ideal associated to I is*

$$\begin{aligned}\tilde{I}=&\langle x^sy^u\mid s\in S,\ s\leq a_r\text{ and }u\text{ such that }\frac{s}{a_r}+\frac{u}{b_0}=1\rangle\cap\\ &\langle x^vy^t\mid t\in T,\ t\leq b_0\text{ and }v\text{ such that }\frac{v}{a_r}+\frac{t}{b_0}=1\rangle.\end{aligned} \tag{3.6}$$

Example 3.15. Let $I=\langle y^{12},x^6y^8,x^9y^6,x^{15}y^2,x^{18}\rangle$. Then we have $S=\langle 6,9\rangle$, $T=\langle 2\rangle$ and $I_S=\langle y^{12},x^6y^8,x^9y^6,x^{12}y^4,x^{15}y^2,x^{18}\rangle$, $I_T=\langle y^{12},x^3y^{10},x^6y^8,x^9y^6,x^{12}y^4,x^{15}y^2,x^{18}\rangle$. Thus, the Ratliff-Rush associated ideal is $\tilde{I}=I_S\cap I_T=I+\langle x^{12}y^4\rangle$.

4. Examples

In the sequel we let $I = \langle x^{a_i}y^{b_i}\rangle_{i=0}^{r}$ be an $\langle x, y\rangle$-primary ideal such that $a_i+b_i = d$ for all i.

Example 4.1. Assume $a_1 \geq \frac{d}{2}$, then $a_i+a_j \geq d$ for all pairs of a_i and a_j. Hence, the condition in Corollary 3.8 is fulfilled and I is Ratliff-Rush. This generalizes the example of a non integrally closed Ratliff-Rush ideal $\langle y^4, x^2y^2, x^3y, x^4\rangle$ in [**RS**], p. 2.

The only integrally closed monomial ideals such that the generators have the same degree are $\langle x, y\rangle^d$.

By the total $N = 2^{d-1}$ of $\langle x, y\rangle$-primary ideals generated by degree d monomials there are $2\cdot 2^{\lceil \frac{d}{2}\rceil}$ such that $\frac{d}{2} \leq a_1$ or $\frac{d}{2} \leq b_1$. Hence, such monomial ideals generated by the same degree there are $2\sqrt{N}$ Ratliff-Rush ideals if d is odd and $2\sqrt{2N}$ if d is even.

4.1. Ideals such that all their powers are Ratliff-Rush. It is shown in [**HLS**], (1.2), that all the powers of a regular ideal in a Noetherian ring are Ratliff-Rush if and only if the depth of the associated graded ring $gr_I(R)$ is positive.

Example 4.2. In [**HJLS**], (6.3), the authors conjecture that for any d the ideal $I_d = \langle y^d, x^{d-1}y, x^d\rangle$ and all its powers are Ratliff-Rush. The conjecture was later proved in [**RS**] by actual computation of the depth. An alternative way to show this uses Corollary 3.8.

The numerical semigroups associated with the ideal I_d are $S_d = \bigcup_{l=0}^{\infty}\{ld - i\}_{0\leq i\leq l}$ and $T_d = \mathbb{Z}_{\geq 0}$. Obviously, if $s \in S_d$ and $s \leq dl$, then $\lambda(s) \leq l$. Let $S_{d,l}$ be the numerical semigroup associated with the ideal I_d^l. Then $\{s \in S_{d,l} \mid s \leq dl\} = \{$exponents of x in the minimal generating set for $I_d^l\}$. Hence, I_d^l is Ratliff-Rush for all l by Corollary 3.8.

This family of ideals is part of a larger one such that all their powers are Ratliff-Rush.

Let $I_{d,k} = \langle y^d, x^{d-k}y^k, x^{d-k+1}y^{k-1}, \ldots, x^{d-1}y, x^d\rangle$. For example, the family $I_{d,1}$ are the ideals we have discussed previously. The corresponding numerical semigroups are $S_{(d,k)} = \bigcup_{l=0}^{\infty}\{ld - i\}_{0\leq i\leq lk}$ and $T_{(d,k)} = \mathbb{Z}_{\geq 0}$. If $s \in S_{(d,k)}$ and $s \leq ld$ then $\lambda(s) \leq l$. Then the exponents of x among the generators for $I_{d,k}^l$ fulfil the assumption in Corollary 3.8, which finishes the proof.

Example 4.3. In [**HJLS**], (E3), the authors examine the ideal $I = \langle y^8, x^3y^5, x^5y^3, x^8\rangle$ using MACAULAY. Among other things they show that I is Ratliff-Rush but I^3 is not. We will look at all the powers I^l.

Using Proposition 3.7 we see that $\tilde{I} = \langle y^8, x^3y^5, x^5y^3, x^6y^2, x^8\rangle \cap \langle y^8, x^2y^6, x^3y^5, x^5y^3, x^8\rangle = I$.

Further, $g(S) = g(\langle 0, 3, 5, 8\rangle) = 7$ and $\lambda(s) \leq 4$ if $s \leq 24$. Thus, for every $s \leq 8k$ we have $\lambda(s) \leq k + 1$, that is, for all $l \geq 4$ if $s \leq 8 \cdot \frac{l}{2}$ then $\lambda(s) \leq \lceil \frac{l}{2}\rceil + 1 \leq l$. Exactly the same is valid for the numerical semigroup T. Hence,

$$I^l = \langle x^s y^{8l-s} \mid s \in S \text{ and } s \leq 4l\rangle + \langle x^{8l-t}y^t \mid t \in T \text{ and } t \leq 4l\rangle \tag{4.1}$$

for all $l \geq 4$. (Compare to Remark 3.6.) Moreover, I^2 is on that form too, but not I^3 since $\lambda(12) = 4$.

Now we will show that if I^l is on the form (4.1), then I^l is Ratliff-Rush. Let S_l and T_l be the numerical semigroups defined by I^l, then $I_{S_l} = \langle x^s y^{8l-s} \mid s \leq$

$4l\rangle + x^{4l}\mathfrak{m}^{4l}$ and $I_{T_l} = \langle x^{8l-t}y^t \mid t \leq 4l\rangle + y^{4l}\mathfrak{m}^{4l}$. It is easy to see that $I_{S_l} \cap I_{T_l} = \langle x^s y^{8l-s} \mid s \leq 4l\rangle + \langle x^{8l-t}y^t \mid t \leq 4l\rangle = I^l$.

Finally, we get $\widetilde{I^3} = I^3 + \langle x^{12}y^{12}\rangle$ using Proposition 3.7.

Example 4.4. Let $I_k = \langle y^{6k+1}\rangle + \left\langle x^{2(k+i)+1}y^{4k-2i}\right\rangle_{i=0}^{k-1} + \left\langle x^{4k+i+1}y^{2k-i}\right\rangle_{i=0}^{2k}$. For example, if $k = 2$ then $I_2 = \langle y^{13}, x^5y^8, x^7y^6, x^9y^4, x^{10}y^3, x^{11}y^2, x^{12}y, x^{13}\rangle$.

We will prove that all positive powers of I_k are Ratliff-Rush by showing that the numerical semigroup determined by I is $S = \{a_i\} \cup \{n \in \mathbb{Z} \mid n \geq 6k+1\}$ and if $s \in S$ is such that $s \leq l(6k+1)$ then $\lambda(s) \leq l$. Hence, the generators for I_k^l will fulfil the condition in Corollary 3.8.

We use induction on l.

If $l = 1$ we are done, since $\{s \in S \mid s \leq 6k+1\} = \{a_i\}$.

Let $l = 2$. We will show that all the elements in $\{6k+2, \ldots, 12k+2\}$ are linear combinations of at most two generators a_i and a_j. For all $0 \leq i \leq 2k$ we have $6k+2 \leq (2k+1)+(4k+i+1) \leq 8k+2$. Further, any integer $n \in [8k+2, \ldots, 12k+2]$ is a linear combination of two elements in $\{4k+i+1\}_{i=0}^{2k}$.

Assume our claim is true for all $l \leq p$. Let $l = p+1$. We need to show that if $p(6k+1)+1 \leq n \leq l(6k+1)+6k+1$ then $n = \sum \lambda_i a_i$ with $\sum \lambda_i \leq p+1$. By the induction hypothesis $\{p(6k+1) - 4k + i\}_{i=0}^{4k} \subset S$ and the values of the λ-function of these elements are always less or equal to p. Thus, $\big(p(6k+1)-4k+i\big)+4k+1 = \sum \lambda_i a_i$ with $\sum \lambda_i \leq p+1$ for all $0 \leq i \leq 4k$. Clearly, the same is valid for each sum $p(6k+1) + (4k+i+1)$ for all $0 \leq i \leq 2k$, and we are done.

This example of ideals can be varied in many different ways. Moreover, the induction proof that we used can be applied on other families of ideals. For example, $I_{n,k} = \left\langle x^{in}y^{n(k+1-i)-1}\right\rangle_{i=0}^{k} + \left\langle x^{kn+j}y^{n-j-1}\right\rangle_{j=0}^{n-1}$. If $n = 3$ we get the family $I_{3,k} = \langle y^{3k+2}, x^3y^{3k-1}, x^6y^{3k-4}, \ldots, x^{3k}y^2, x^{3k+1}y, x^{3k+2}\rangle$.

References

[C] V. Crispin Quiñonez, *Integrally Closed Monomial Ideals and Powers of Ideals*, Research Reports in Mathematics **7**, Department of Mathematics, Stockholms universitet, 2002.

[E] J. Elias, *On the Computation of the Ratliff-Rush Closure*, J. Symbolic Comput. **37** (2004), no. 6, 717-725.

[FGH] R. Fröberg, C. Gottlieb and R. Häggkvist, *On Numerical Semigroups*, Semigroup Forum **35** (1987), 63-83.

[HJLS] W. Heinzer, B. Johnston, D. Lantz and K. Shah, *Coefficient Ideals in and Blowups of a Commutative Noetherian Domain*, J. of Algebra **162** (1993), 355-391.

[HLS] W. Heinzer, D. Lantz and K. Shah, *The Ratliff-Rush Ideals in a Noetherian Ring*, Comm. in Algebra **20** (1992), 591-622.

[RR] L. J. Ratliff, Jr and D. E. Rush, *Two Notes on Reductions of Ideals*, Indiana Univ. Math. J. **27** (1978), no. 6, 929-934.

[RS] M. E. Rossi and I. Swanson, *Notes on the Behavior of the Ratliff-Rush Filtration*, Commutative Algebra (Grenoble/Lyon, 2001), 313-328, Contemp.Math., 331, Amer. Math. Soc., Providence, RI, 2003.

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
E-mail address: veronica@math.su.se