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INTEGRAL CLOSURE AND OTHER OPERATIONS ON MONOMIAL IDEALS

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ABSTRACT. In this paper we give a characterization of integrally closed monomial ideals in two variables. The notion of x- and y-tight ideals is introduced. We also present a monomial version of a result of Watanabe on chains of integrally closed monomial ideals. Using the developed techniques, we prove results about the quadratic transforms and products in classes of monomial ideals, and describe multiplication in two classes of ideals in an integral domain.

1. Introduction and preliminaries. Let R be a polynomial (localized) ring or a power series ring. The maximal (irrelevant) ideal is denoted by $\mathfrak{m}$. An ideal I is called $\mathfrak{m}$-primary if its radical $\sqrt{I} = \mathfrak{m}$. An ideal is *simple*, if it is not a product of two proper ideals.

A power product is an element $x_1^{a_1} \cdots x_n^{a_n}$. If a monomial ideal is written as $I = \langle x^{a_i} y^{b_i} \rangle$, we usually assume that the generators are ordered in such a way that $a_i < a_{i+1}$ and $b_i > b_{i+1}$.

An element $r \in R$ is said to be *integral* over an ideal I in R, if r satisfies an *equation of integral independence*

$$r^l + a_1 r^{l-1} + \cdots + a_{l-1} r + a_l = 0 \text{ where } a_j \in I^j.$$

The *integral closure* of I is defined as the set of all elements in R which are integral over I. This closure is denoted by $\overline{I}$.

If $r \in R$, then $o(r) = \max(l | r \in \mathfrak{m}^l)$. The order of an ideal $I \subset R$ is defined as $o(I) = \min\{o(r) | r \in I\}$. The least number of generators of I is denoted by $\mu(I)$.

In Appendix 5 of [**14**], which is based on [**13**], it is proved that in a two-dimensional regular local ring the product of integrally closed ideals is integrally closed. This is not the case in a three-dimensional

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regular local ring. Example (3.1) in [**5**] shows: if $I, J \subset k[x,y]_{\langle x,y\rangle}$ are integrally closed primary ideals such that $I + J$ is not integrally closed, then the product $\langle I, z\rangle\langle J, z\rangle \subset k[x,y,z]_{\langle x,y,z\rangle}$ is not integrally closed although $\langle I, z\rangle, \langle J, z\rangle$ are both integrally closed. A monomial example in the three-dimensional case can thus be constructed taking $I = \langle y^4, xy^2, x^2\rangle$ and $J = \langle y^2, x^2y, x^4\rangle$.

Moreover, Zariski proves that any integrally closed ideal is in a unique way the product of simple such ideals. Since then the theory of integrally closed ideals has been frequently studied, see [**3, 4, 7, 9, 10**], among others. In Appendix 5 of [**14**] and in [**6**], where Huneke presents the main facts, the theory of contracted ideals and the quadratic transform plays an important role.

Let I be an ideal in R. If there is an $r \in \mathfrak{m}/\mathfrak{m}^2$ such that $I \cdot R\big[\mathfrak{m}/r\big] \cap R = I$, then I is contracted from the extension ring $R\big[\mathfrak{m}/r\big]$.

Proposition 1.1 [**6**]. *An ideal I is contracted if and only if $\mu(I) = o(I) + 1$.*

Let $\mathfrak{m} = \langle r, s\rangle \subset R$, and let I be an ideal of order l. If $a \in I$, then $a/r^l \in R[\mathfrak{m}/r]$. Thus we can write $I \cdot R[\mathfrak{m}/r] = r^l I'$ for some ideal $I' \subset R[\mathfrak{m}/r]$. The ideal I' is called the *quadratic transform* of I in $R[\mathfrak{m}/r]$.

Among the results about contracted and integrally closed ideals, and quadratic transforms we have Proposition 3.4 in [**6**], which states that if I is integrally closed then the transform I' is integrally closed. It is uncertain whether the converse holds. By Corollary 3.2 in [**6**] integrally closed ideals are contracted. The converse is not true, but it is interesting to examine under which conditions the converse holds.

Throughout Section 1, it will be tacitly understood that the ring R is either a polynomial ring $k[x_1,\dots,x_n]$, its localization $k[x_1,\dots x_n]_{\langle x_1,\dots,x_n\rangle}$ or a power series ring $k[[x_1,\dots,x_n]]$ over a field k.

There are two alternative descriptions of the integral closure of a monomial ideal, the algebraic and the one using a staircase diagram, which simplifies the study of the integral closure.

1.1. An algebraic description of the integral closure. The following lemma is a generalization of Lemma 7.3.1 [**11**]. For proof see [**2**, Lemma 2.6].

Lemma 1.2. *Let $I \subset R$ be an ideal, and $\{m_i\}_{i=1}^{q\geq 2}$ a set of different power products such that for integers $s_i > 0$ and $0 \leq r_i \leq s_i$:*

$$\begin{aligned} m_1^{s_1} &\in I^{s_1-r_1}\langle m_2, m_3, \dots, m_q\rangle^{r_1} \\ m_2^{s_2} &\in I^{s_2-r_2}\langle m_1, m_3, \dots, m_q\rangle^{r_2} \quad \dots \\ \dots \quad m_q^{s_q} &\in I^{s_q-r_q}\langle m_1, m_2, \dots, m_{q-1}\rangle^{r_q}. \end{aligned} \tag{1.1}$$

Then, for each $1 \leq i \leq q$, there is some $l_i > 0$ such that $m_i^{l_i} \in I^{l_i}$.

Proposition 1.3. *Let I be a monomial ideal in R. Then its integral closure $\overline{I}$ is also a monomial ideal. In fact,*

$$\overline{I} = \langle m \in R \mid m \text{ is a monomial and } m^l \in I^l \text{ for some } l > 0\rangle. \tag{1.2}$$

Proof. See proof of Proposition 2.7 [**2**]. Also, for somewhat different proofs, see Propositions 7.3.2 and 7.3.3 [**11**]. □

It is worth noting that, in general, if a monomial $m \in \overline{I}$, then we cannot claim that $m^l \in I^l$ for all l.

Example 1.4. Let $I = \langle y^2, x^2\rangle \subset k[x,y]$. Then $xy \in \overline{I}$, and $(xy)^l \in I^l$ for even $l \geq 2$ only.

Also, if a polynomial $p \in \bar{I}$, then we cannot deduce that $p^l \in I^l$ for any $l \geq 2$.

Example 1.5. Let k be a field of characteristic distinct from 2 and 3. Consider the ideal $I = \langle y^4, x^3\rangle \subset k[x,y]$ and its integral closure $\overline{I} = \langle y^4, xy^3, x^2y^2, x^3\rangle$. We have $(x^2y^2)^2 \in I^2$ and $(xy^3)^3 \in I^3$.

Let $p_1 = x^2y^2 + xy^3 \in \overline{I} \setminus I$. For which l does $p_1^l \in I^l$? In other words, we want to determine an l such that for all $0 \leq i \leq l$ there is some j such that

$$(x^3)^j(y^4)^{l-j} \mid (x^2y^2)^i(xy^3)^{l-i} = x^{l+i}y^{3l-i}. \tag{1.3}$$

The equation is equivalent to the inequalities $3j - i \leq l \leq 4j - i$. For $i = 0$ they are satisfied by $l \in \{3,4\} \cup \{6,7,8\} \cup \{9,\dots\} = \{3,4\} \cup \{6,\dots\}$.

For $i = 1$ they are satisfied by $l \in \{2,3\} \cup \{5,\dots\}$. For each following i the intervals move one step to the left. The intersection of all allowed l's gives the result: $p_1^l \in I^l$ for $l \geq 6$.

Now, consider $p_2 = x^2y^2 + x^3 \in \bar{I} \setminus I$. Is there an l such that

$$(x^3)^j(y^4)^{l-j} \mid (x^2y^2)^i(x^3)^{l-i} = x^{3l-i}y^{2i} \tag{1.4}$$

for all $0 \leq i \leq l$ and some j? The corresponding inequalities in this case are $j + (i/3) \leq l \leq j + (i/2)$. Evidently no integer l satisfies them for $i = 1$. There is no l such that $(x^2y^2 + x^3)^l \in I^l$ because $(x^3)^{l-1}(x^2y) \notin I^l$ for all l.

Let us look more thoroughly at the difference between p_1 and p_2. Each term in the binomial expansion $p_1^6 = \sum_{i=0}^{l} \binom{i}{6}(x^2y^2)^i(xy^3)^{6-i}$ belongs to I^6. Clearly the outer terms belong to I^6. The first mixed power product is $(x^2y^2)(xy^3)^5 = (x^2y^2)(xy^3)^2(xy^3)^3$; its last factor belongs to I^3 initially. The intermediate factor $(xy^3)^2$ needs to be multiplied by xy^2 (and not necessarily by xy^3) in order to belong to I^3, which explains why this product belongs to I^6.

A monomial $m \in \overline{I} \setminus I$ may satisfy an even stronger condition $m^l \in I^{l'}$ where $l < l'$ or, roughly speaking, $m \in I^{l'/l}$ where $1 < (l'/l) \in \mathbf{Q}$. In fact, we have $(x^2y^2)^6 = (x^3)^4(y^4)^3 \in I^7$ and $(xy^3)^{12} = (x^3)^4(y^4)^9 \in I^{13}$.

1.2. The integral closure through staircase diagrams. Any monomial ideal I in n variables can be depicted by letting the set of the exponents of the power products in I be integral points in $\mathbf{R}^n$. Such a representation is essential for all the results we are going to present.

Definition 1.6. Let $x_1^{a_1} \cdots x_n^{a_n} = X^{\mathrm{a}}$ be a power product in R. We set $\Gamma(X^{\mathrm{a}}) = \mathrm{a}$. Let I be a monomial ideal; then we define the semigroup ideal $\Gamma(I) = \{\Gamma(m) \mid m \in I,\ m \text{ a power product}\} \subset \mathbf{R}^n_{\geq 0}$. Moreover, we define $\Gamma^*(I)$ as the set of integral points in the convex hull of $\Gamma(I) + \mathbf{R}^n_{\geq 0}$.

Remark 1.7. In this setting Proposition 1.3 states that $\Gamma(\overline{I}) = \{\mathrm{a} \in \mathbf{Z}^n_{\geq 0} \mid \mathrm{a}l = b_1 + \cdots + b_l$ for some l and some $b_i \in \Gamma(I)\}$.

The following result is a nice description of the integral closure. Here we consider the whole set $\Gamma(I)$, instead of the Γ of the minimal generating set for I only, which is done in Proposition 7.3.4 in [**11**]. See also [**8**] for reference.

Proposition 1.8. *Let $I \subset R$ be a monomial ideal. Then the integral closure $\overline{I}$ is generated by such powers products m that $\Gamma(m) \in \Gamma^*(I)$. That is,*

$$\Gamma(\overline{I}) = \Gamma^*(I).$$

Proof. Let $\mathrm{a} \in \Gamma^*(I)$. Then $\mathrm{a} = \sum_{i=1}^{q} \lambda_i \mathrm{a}_i$, where $\mathrm{a}_i \in \Gamma(I)$, $\lambda_i \in \mathbf{Q}_{\geq 0}$ and $\sum_{i=1}^{q} \lambda_i = 1$. Since there is an integer $l > 0$ such that $l\lambda_i \in \mathbf{Z}_{\geq 0}$ for all i, we obtain $(X^{\mathrm{a}})^l = (X^{\sum \lambda_i \mathrm{a}_i})^l = (X^{\mathrm{a}_1})^{l\lambda_1} \cdots (X^{\mathrm{a}_q})^{l\lambda_q} \in I^l$. Thus $X^{\mathrm{a}} \in \overline{I}$, that is, $\mathrm{a} \in \Gamma(\overline{I})$.

On the other hand, if $\mathrm{b} \in \Gamma(\overline{I})$, then there is an integer l such that $\mathrm{b}l = \mathrm{b}_1 + \cdots + \mathrm{b}_l$ where all the b_i's (not necessarily different) belong to $\Gamma(I)$ (compare with Remark 1.7). Thus, $\mathrm{b} = \sum_{i=1}^{l} \frac{1}{l}\mathrm{b}_i$, and it follows that $\mathrm{b} \in \Gamma^*(I)$. □

It follows directly from Proposition 1.8 that principal monomial ideals are integrally closed.

Lemma 1.9. *Let J be a monomial ideal and assume that $J = mI$ where m is a power product and I is a monomial ideal. Then J is integrally closed if and only if I is integrally closed.*

Proof. It is clear that $\Gamma^*(I) = \Gamma(m) + \Gamma^*(J)$. The result follows. □

1.3. Chains of integrally closed monomial ideals. We will show that, given two integrally closed $\mathfrak{m}$-primary monomial ideals $J \supset I$, there is a composition series between J and I, consisting of integrally closed monomial ideals only. The result is a monomial ideal version of Watanabe's result in [**12**], but it is also a stronger result, since all the ideals in the composition series are monomial.

We recall that a composition series of I is a chain $I = I_0 \supset I_1 \cdots \supset I_l = 0$ such that I_i/I_{i+1} has no nontrivial subideals. The *length* of a composition series of M is denoted by $l(M)$.

Lemma 1.10. *Let $S \subseteq \mathbf{R}^n_{\geq 0}$ be a convex set and $p \in \mathbf{R}^n_{\geq 0} \setminus S$ a point. Then, for every $x \in \operatorname{conv}(S \cup \{p\})$, $x \neq p$, we have $\operatorname{conv}(S \cup \{x\}) \subsetneq \operatorname{conv}(S \cup \{p\})$.*

Proof. Since $x \in \operatorname{conv}(S \cup \{p\})$ there is some $s \in S$ and $0 \leq \lambda < 1$ such that

$$x = \lambda p + (1 - \lambda)s. \tag{1.5}$$

Assume $p \in \operatorname{conv}(S \cup \{x\})$. Then there is an $s' \in S$, such that for some $0 \leq \mu \leq 1$, we have $p = \mu x + (1 - \mu)s'$. Hence, by (1.5): $(1 - \mu\lambda)p = \mu(1 - \lambda)s + (1 - \mu)s'$. We have $1 \neq \mu\lambda$ since $\lambda < 1$, but then we deduce that $p \in S$ which is a contradiction. Hence, the assumption was false and the lemma is true. □

Proposition 1.11. *Let $J \supset I$ be $\mathfrak{m}$-primary integrally closed monomial ideals in R. Then there is an integrally closed monomial ideal I' such that $J \supseteq I' \supsetneq I$ and $l_R(I'/I) = 1$.*

Proof. We have $l_R(J/I) = |\Gamma^*(J) \setminus \Gamma^*(I)|$. Pick an integrally closed monomial ideal I' such that $J \supseteq I' \supset J$ and with minimal (positive) length over I. We claim that $l_R(I'/I) = 1$.

Assume the contrary. Then there are power products m_1 and m_2 such that $\{\Gamma(m_1),\ \Gamma(m_2)\} \in \Gamma^*(I') \setminus \Gamma^*(I)$. Consider the integral closure of $I + \langle m_1 \rangle$. Then $I' = \overline{I + \langle m_1 \rangle}$ by minimality of I' and, thus, $m_2 \in \overline{I + \langle m_1 \rangle}$. That is, $\Gamma(m_2) \in \Gamma^*(I + \langle m_1 \rangle)$. Then, by Lemma 1.10, we have $\Gamma^*(I + \langle m_2 \rangle) \subset \Gamma^*(I + \langle m_1 \rangle) = \Gamma^*(I')$, which violates minimality of I'. □

Corollary 1.12. *Let $J \supset I$ be $\mathfrak{m}$-primary integrally closed monomial ideals in R. Then there is a chain of $\mathfrak{m}$-primary integrally closed monomial ideals $J = I_0 \supset I_1 \supset \cdots \supset I_l = I$ such that $l_R(I_i/I_{i+1}) = 1$ for every i.*

Given two $\mathfrak{m}$-primary integrally closed monomial ideals $J \subset I$, we consider the set of integrally closed monomial ideals between J and I. For any two ideals I_1 and I_2 in this set we define their join as

$I_1 \vee I_2 = \overline{I_1 + I_2}$ and their meet as $I_1 \wedge I_2 = I_1 \cap I_2$. It is an easy exercise to show that the intersection of two arbitrary integrally closed ideals is also integrally closed. Hence, this set is a lattice – a partially ordered set $\widehat{\overline{J/I}}$ with $\hat{1} = J$ and $\hat{0} = I$.

The only interesting property that this lattice possesses is lower semimodularity. A finite lattice is called upper semimodular if it satisfies the following condition: if x and y cover $x \wedge y$, then $x \vee y$ covers x and y. A finite lattice that satisfies the dual condition is called lower semimodular. Lattices of integrally closed monomial ideals are not upper semimodular as the following shows. Consider the ideals $\langle y^2, x^2y, x^3 \rangle$ and $\langle y^3, xy^2, x^2 \rangle$. They both cover their meet $\mathfrak{m}^3$, but their join $\mathfrak{m}^2$ covers neither of them.

Let A and B be ideals. Then the following statements are true.

(1) If A and B contain some ideal C such that $l(A/C) = l(B/C) = 1$, then $l((A+B)/A) = l((A + B/B)) = 1$.

(2) If A and B are contained in some ideal C such that $l(C/A) = l(C/B) = 1$, then $l(A/(A \cap B) = l(B/(A \cap B)) = 1$.

The first property is equivalent to upper semimodularity. The problem that may arise in the case when all the ideals are integrally closed is that the sum of two integrally closed ideal is not necessarily integrally closed. Thus, the lengths in the second part of (1) may be larger after taking the closure of that sum.

The second property, equivalent to lower semimodularity, is valid also when we restrict the subject of our interest to integrally closed ideals only. Hence, for arbitrary integrally closed $J \supset I$ the lattice $\widehat{\overline{J/I}}$ is lower semimodular.

We conclude by presenting the main results of the paper.

In Section 2 we give a full explicit description of integrally closed monomial ideals and the unique factorization into simple such ideals in a two-dimensional polynomial (localized) ring or a power series ring. All this is done using staircase diagrams from subsection 1.2. In subsection 2.1 we introduce x-tight and y-tight ideals. Then in Section 3 we describe the quadratic transform of x- and y-tight ideals and state the conditions when the converse to Proposition 3.4 and Corollary 3.2 in [**6**] holds. Powers and products of two classes of ideals

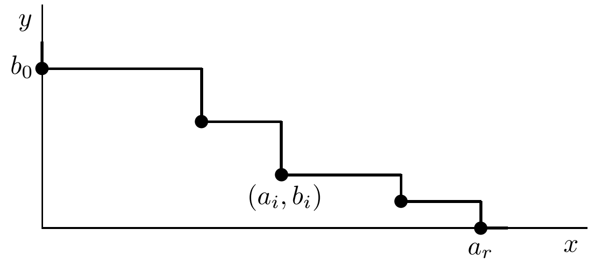

FIGURE 1. The semigroup ideal $\Gamma(I) = \{(0, b_0), \dots, (a_i, b_i), \dots, (a_r, 0)\}$.

in an integral domain are described in Section 4. Many of the results have been previously presented.

2. Integrally closed monomial ideals in two variables. Let R be $k[x, y]$, $k[x, y]_{\langle x,y\rangle}$ or $k[[x, y]]$ over a field k. Here we present a full classification of integrally closed monomial ideals in R. Moreover, we show how the unique factorization of integrally closed monomial ideals into simple ones works and give a full characterization of the simple integrally closed monomial ideals.

If $mI = J$ is a monomial ideal, where m is the greatest common divisor of the generators of J, then I is $\mathfrak{m}$-primary or the ring itself. The last case is trivial. Thus, by Lemma 1.9 it suffices to consider $\mathfrak{m}$-primary monomial ideals in our study.

In the sequel, by $I = \langle y^{b_0}, \dots, x^{a_i}y^{b_i}, \dots, x^{a_r}\rangle$ we mean that the generators are minimal and ordered in such a way that $a_i < a_{i+1}$ (and $b_i > b_{i+1}$). The semigroup ideal $\Gamma(I) = \{(0, b_0), \dots, (a_i, b_i), \dots, (a_r, 0)\} + \mathbf{Z}^2_{\geq 0}$ can be interpreted as the lattice points on and above the thick lines in Figure 1. Such a representation is called a *staircase diagram.*

2.1. On products of monomial ideals. We start subsection 2.1 by introducing a special class of monomial ideals. This class is closely related to lexsegment ideals. Later on we go through the graphical conditions on integral closedness and describe them algebraically.

Definition 2.1. An $\mathfrak{m}$-primary monomial ideal I is called *x-tight* if the power of x in every generator is exactly by one greater than of

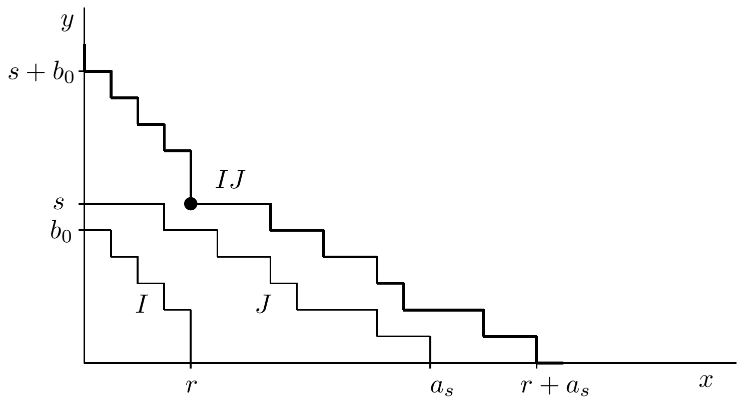

FIGURE 2. $IJ = y^s I + x^r J$.

the preceding generator. That is, I is x-tight of order r if and only if $I = \langle x^i y^{b_i} \rangle_{i=0}^{r}$ with $b_0 > \cdots > b_r = 0$. If $J = \langle x^{a_j} y^{s-j} \rangle_{j=0}^{s}$ is an ideal where $0 = a_0 < \cdots < a_s$, then J is called y-tight of order s.

Proposition 2.2. *Let I be x-tight of order r and J be y-tight of order s. Then $IJ = y^s I + x^r J$.*

The staircase diagram of the proposition statement is in Figure 2.

Proof. It is clear that $y^s I + x^r J \subseteq IJ$.

There are two cases in showing that any power product in the generating set for IJ, that is, any $x^{i+a_j} y^{b_i+s-j}$, belongs to $y^s I$ or $x^r J$.

If $i + j \geq r$, then we have:

$$\begin{cases} a_j - (r-i) & \geq a_{j-(r-i)} \\ b_i & \geq r - i \end{cases}$$

or

$$\begin{cases} i + a_j & \geq r + a_{i+j-r} & = r + a_{j'} \\ b_i + s - j & \geq s - (i+j-r) & = s - j' \end{cases} .$$

Thus, $x^{i+a_j} y^{b_i+s-j} \in x^r J$.

Similarly, if $i + j \leq r$ we get $x^{i+a_j} y^{b_i+s-j} \in \langle x^{i+j} y^{s+b_{i+j}} \rangle \subset y^s I$. ◻

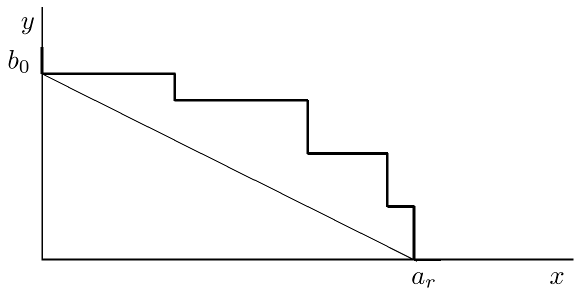

FIGURE 3. A simple monomial ideal.

Remark 2.3. In fact, if an ideal can be graphically depicted as in Figure 2, that is, an x-tight followed by an y-tight, then it is a product of the corresponding x- and y-tight ideals.

Proposition 2.4. *Let* $I = \langle x^{a_i} y^{b_i} \rangle_{i=0}^{r}$ *be an* $\mathfrak{m}$*-primary ideal, and assume that* $(a_i/(b_0 - b_i)) > (a_r/b_0)$ *for all* $1 \leq i \leq r-1$. *Then* I *is a simple monomial ideal, that is, not a product of proper monomial ideals.*

In Figure 3 we show a staircase diagram of such an ideal. Notice that none of the intermediate generators crosses the line from $(0, b_0)$ to $(a_r, 0)$.

Proof. Assume that I is a product of two ideals, say, $J_1 = \langle y^{b'}, \ldots, x^{a'} \rangle$ and $J_2 = \langle y^{b''}, \ldots, x^{a''} \rangle$, where $a' + a'' = a_r$ and $b' + b'' = b_0$.

First we consider the element $x^{a'} y^{b''} \in J_1 J_2$. By assumption it belongs to the ideal I, and then the inequality $(a'/(b_0 - b'')) > (a_r/b_0)$ must be valid. Now consider $x^{a''} y^{b'} \in J_1 J_2$. This element cannot belong to I, since $(a''/(b_0 - b')) = ((a_r - a')/b'') < [a_r - a_r((b_0 - b'')/b_0)]/b'' = (a_r/b_0)$. Hence, the assumption was false and I is not a product of two monomial ideals. □

2.2. Necessary and sufficient conditions for integral closedness. Let $I = \langle x^{a_i} y^{b_i} \rangle_{i=0}^{r}$ with $a_i > a_{i+1}$. We know that the integral points in the convex hull of $\{(a_i, b_i)_{i=0}^{r}\}$ generate the integrally closed ideal $\overline{I}$. In order to find a necessary condition on a monomial ideal to be integrally closed, we look at conv $(\{(a_i, b_i), (a_{i+1}, b_{i+1})\} + \mathbf{Z}^2_{\geq 0})$, the convex hull of two consecutive exponents. Particularly, this area contains the triangle defined by the vertices (a_i, b_i), (a_{i+1}, b_i) and (a_{i+1}, b_{i+1}). If the ideal is integrally closed, then there must not be any integral points in this triangle. Obviously, this is the case if and only if either $a_{i+1} - a_i = 1$ or $b_i - b_{i+1} = 1$.

Remark 2.5. Thus, a necessary condition on a monomial ideal to be integrally closed is that every generator is followed by such a generator that either its power of x is increased by one, or its power of y is decreased by one.

Next, we will improve this condition. Assume that in an ideal the condition in Remark 2.5 is fulfilled for each pair of consecutive generators. Assume further that $a_{i+1} - a_i \geq 2$ and $b_j - b_{j+1} \geq 2$ for some $i < j$, where i and j are the greatest, respectively the smallest, index such that the situation occurs. The integral points in the area above and to the right of the diagonal line from $(a_{i+1} - 2, b_i)$ to $(a_{j+1}, b_j - 2)$ belong to the integral closure, especially the point $(a_{j+1} - 1, b_j - 1)$. Thus, the power product $x^{a_{j+1}-1} y^{b_j - 1}$ will always appear in the integral closure, and an ideal, fulfilling the assumption, cannot be integrally closed.

Thus, a necessary condition on a monomial ideal to be integrally closed is that its staircase diagram consists of two parts: an x-tight ideal followed by a y-tight ideal.

Proposition 2.6. *Let I be an integrally closed $\mathfrak{m}$-primary monomial ideal. Then the ideal is a product of an x-tight ideal and a y-tight ideal.*

Moreover, let J_1 be x-tight and J_2 be y-tight. Then their product $J_1 J_2$ is integrally closed if and only if both J_1 and J_2 are integrally closed.

Proof. The first statement is a direct consequence of Remark 2.3 and the following discussion. The second statement follows from Figure 2. □

According to Proposition 2.6 it suffices to determine which x-tight and y-tight ideals are integrally closed, in order to give a full description of integrally closed monomial ideals.

We will restrict ourselves to y-tight ideals. To begin with we will consider a special class of these ideals. In the next section we will show that every integrally closed y-tight ideal is a product of ideals from that class.

Let $a_r > r$ be positive integers, and consider the monomial ideal I, such that $\Gamma(I)$ consists of the integral points in the area, that is limited by the x-axis, y-axis and the line from $(0, r)$ to $(a_r, 0)$. Then I is y-tight and integrally closed.

In general, the values of a_i in an integrally closed ideal $I = \langle x^{a_i} y^{r-i} \rangle_{i=0}^{r}$ are obtained using the following:

$$i\frac{a_r}{r} \le a_i < i\frac{a_r}{r} + 1, \quad \text{that is, } a_i = \left\lceil i\frac{a_r}{r} \right\rceil. \tag{2.1}$$

If $\gcd(r, a_r) = d$ then, except for $(0, b_0)$ and $(r, 0)$, there are $d-1$ points in $\Gamma(I)$ lying *on* the diagonal line. These points divide the staircase diagram of I into d copies of some ideal I'. As we see later on, this means $I = (I')^d$.

Correspondingly, if $I = \langle x^i y^{b_i} \rangle_{i=0}^{r}$ with

$$(r-i)\frac{b_0}{r} \le b_i < (r-i)\frac{b_0}{r} + 1, \quad \text{that is, } b_i = \left\lceil (r-i)\frac{b_0}{r} \right\rceil, \tag{2.2}$$

then I is an integrally closed x-tight ideal.

2.3. Simple integrally closed monomial ideals and factorization. Let r and a_r be relatively prime, and let $I = \langle x^{a_i} y^{r-i} \rangle_{i=0}^{r}$, where the a_i's are determined by (2.1). Since $\gcd(r, a_r) = 1$ we have $i(a_r/r) < a_i$ for all $1 \le i \le r-1$. By Proposition 2.4 such an ideal is simple. Hence, to a given rational number (a_r/r) corresponds exactly one simple integrally closed $\mathfrak{m}$-primary monomial ideal.

Definition 2.7. Let a and b be positive integers with $\gcd(a, b) = 1$. Then there is a unique simple integrally closed monomial ideal containing x^a and y^b in its minimal generating set, $\overline{\langle y^b, x^a \rangle}$. We call such an ideal an (a, b)-*block* or a *block ideal*. Moreover, the ideal is the least integrally closed ideal possessing x^a and y^b.

If $a > b$, then the (a, b)-block is equal to $\langle x^{a_i} y^{b-i} \rangle_{i=0}^{b}$ where $i(a/b) \le a_i < i(a/b) + 1$ with equality only if $i = 0, b$.

If $a < b$, then the block ideal is equal to $\langle x^i y^{b_i} \rangle_{i=0}^{a}$ where $(a-i)(b/a) \le b_i - 1 < (a-i)(b/a) + 1$ with equality only for $i = 0, a$.

Proposition 2.8. *Let I be an (a,b)-block and J a (c,d)-block. Assume further that $a/b \leq c/d$. Then $IJ = y^d I + x^a J$.*

The proposition states that the staircase diagram of IJ is the ideal I translated to $(0,d)$ and the ideal J translated to $(a,0)$. Note also that IJ is the least integrally closed monomial ideal, containing the power products $y^{b+d}, x^a y^d$ and x^{a+c}.

Proof. We prove the proposition for all three kinds of ratios a/b and c/d.

If $a/b < 1$ and $c/d > 1$, then Proposition 2.8 is a special case of Proposition 2.2.

Suppose $a/b > 1$. Let $I = \langle x^{a_i} y^{r-i} \rangle_{0 \leq i \leq r}$ with $a_i = \lceil i(a_r/r) \rceil$, and let $J = \langle x^{c_j} y^{s-j} \rangle_{0 \leq j \leq s}$, where $c_j = \lceil j(c_s/s) \rceil$ and $(a_r/r) \leq (c_s/s)$. We will show that $IJ = y^s I + x^{a_r} J$.

Clearly $y^s I + x^{a_r} J \subseteq IJ$.

On the other hand, the ideal IJ is generated by power products on the form

$$x^{a_i+c_j} y^{r+s-i-j} = \begin{cases} x^{a_i+c_j} y^{r-i-j} \cdot y^s & \text{if } i+j \leq r \\ x^{a_r} \cdot x^{a_i+c_j-a_r} y^{r+s-i-j} & \text{if } i+j \geq r. \end{cases} \tag{2.3}$$

To prove $x^{a_i+c_j} y^{r+s-i-j} \in y^s I + x^{a_r} J$ we will use:

$$\lceil q_1 \rceil + \lceil q_2 \rceil \geq \lceil q_1 + q_2 \rceil \geq \lceil q_1 \rceil + \lceil q_2 \rceil - 1 \tag{2.4}$$

$$\lceil q_1 + q_2 \rceil = \lceil q_1 \rceil + \lceil q_2 \rceil - 1, \quad \text{if } q_1 + q_2 \in \mathbf{Z} \text{ but } q_1, q_2 \notin \mathbf{Z}. \tag{2.5}$$

If $i+j \geq r$, we get

$$\begin{aligned} a_i + c_j - a_r = a_i - a_r + \lceil j \frac{c_s}{s} \rceil &\overset{(2.4)}{\geq} a_i - a_r + \lceil (r-i) \frac{c_s}{s} \rceil \\ + \lceil (j-r+i) \frac{c_s}{s} \rceil - 1 &\geq \left(\lceil i \frac{a_r}{r} \rceil + \lceil (r-i) \frac{a_r}{r} \rceil - 1 \right) \\ - a_r + \lceil (i+j-r) \frac{c_s}{s} \rceil &\overset{(2.5)}{=} c_{i+j-r}, \end{aligned}$$

whence $x^{a_i+c_j} y^{r+s-i-j} = x^{a_r+n'} x^{c_{i+j-r}} y^{s-(i+j-r)}$ for some n'.

Similarly, if $i + j \leq r$, then $x^{a_i + c_j} y^{r+s-i-j} = x^n \cdot x^{a_{i+j}} y^{r-(i+j)} \cdot y^s$ for some n.

If $c/d < 1$, then the proof is a modification of the case $a/b > 1$, and we are done. □

We use the result to show the main result of this section, which will be a generalization of Proposition 2.8. By assigning to any positive rational number a_k/b_k with $\gcd(a_k, b_k) = 1$ an (a_k, b_k)-block, we get a one-to-one correspondence between ascending chains of positive rational numbers and integrally closed $\langle x, y\rangle$-primary monomial ideals.

Theorem 2.9. *Let $(I_k)_{1 \leq k \leq n}$ be a sequence of (a_k, b_k)-blocks such that $a_k/b_k \leq a_{k+1}/b_{k+1}$. Then the product is the integrally closed ideal*

$$\prod_{k=1}^{n} I_k = \sum_{k=1}^{n} x^{A_k} y^{B_{k,n}} I_k, \tag{2.6}$$

where

$$A_k = \sum_{k'=1}^{k-1} a_{k'}, \quad B_{k,n} = \sum_{k'=k+1}^{n} b_{k'} \quad \text{and} \quad A_1 = B_{n,n} = 0. \tag{2.7}$$

Conversely, any integrally closed monomial ideal can be written uniquely as a product of block ideals and some monomial.

Proof. To prove (2.6) we use induction on n. For $n = 2$ the theorem is equivalent to Proposition 2.8.

Assume the theorem is true for some $n = p \geq 2$. Then we have

(2.8)

$$\begin{aligned}
\prod_{k=1}^{p+1} I_k = I_{p+1} \prod_{k=1}^{p} I_k &\overset{\{\text{ind. hyp.}\}}{=} \sum_{k=1}^{p} x^{A_k} y^{B_{k,p}} (I_k I_{p+1}) \\
&\overset{\{\text{Proposition 2.8}\}}{=} \sum_{k=1}^{p} x^{A_k} y^{B_{k,p}} (y^{b_{p+1}} I_k + x^{a_k} I_{p+1}) \\
&= \sum_{k=1}^{p} (x^{A_k} y^{B_{k,p} + b_{p+1}} I_k + x^{A_k + a_k} y^{B_{k,p}} I_{p+1})
\end{aligned}$$

$$= \left(\sum_{k=1}^{p} x^{A_k} y^{B_{k,p}+b_{p+1}} I_k \right) + x^{A_p+a_p} y^{B_{p,p}} I_{p+1}$$
$$+ \sum_{k=1}^{p-1} x^{A_k+a_k} y^{B_{k,p}} I_{p+1} \overset{\{B_{p,p}=B_{p+1,p+1}=0\}}{=} \sum_{k=1}^{p+1} x^{A_k} y^{B_{k,p+1}} I_k$$
$$+ \sum_{k=1}^{p-1} x^{A_k+a_k} y^{B_{k,p}} I_{p+1}.$$

We need to show that the second sum in the last row in (2.8) is contained in the first sum, which is our claim (2.6) for $n = p + 1$.

Pick some $1 \leq k \leq p - 1$, and assume that $b_{p+1} \leq b_{k+1}$. Then we compare the kth term in the second sum, that is, the ideal $x^{A_k+a_k} y^{B_{k,p}} I_{p+1} = x^{A_{k+1}} y^{b_{k+1}+B_{k+1,p}} I_{p+1}$, with the $(k+1)$th term in the first sum, that is, the ideal $x^{A_{k+1}} y^{B_{k+1,p+1}} I_{k+1} = x^{A_{k+1}} y^{B_{k+1,p}+b_{p+1}} \cdot$ $\cdot I_{k+1}$. Since $(a_{k+1}/b_{k+1}) \leq (a_{p+1}/b_{p+1})$ and $b_{p+1} \leq b_{k+1}$, it is easy to see that $y^{b_{k+1}} \overline{\langle y^{b_{p+1}}, x^{a_{p+1}} \rangle} \subseteq y^{b_{p+1}} \overline{\langle y^{b_{k+1}}, x^{a_{k+1}} \rangle}$ is valid.

If $b_{k+1} < b_{p+1}$ for some $1 \leq k \leq p - 1$, then we look at the sequence $\{k + 1, \dots, k + j\}$ of the minimal length, such that $b_{p+1} \leq b_{k+1} + \cdots + b_{k+j}$, and consider the ideals $x^{A_k+a_k} y^{B_{k,p}} I_{p+1}$ and $x^{A_{k+1}} y^{B_{k+1,p+1}} I_{k+1} + \cdots + x^{A_{k+j}} y^{B_{k+j,p+1}} I_{k+j}$.

Look at the staircase diagram in Figure 4. The ideal depicted with thick lines is fully contained in the sum of the ideals depicted by thin lines. Thus, each term in the second sum of the last row in (2.8) is contained in the first sum, that is, $\prod_{k=1}^{p+1} I_k = \sum_{k=1}^{p+1} x^{A_k} y^{B_{k,p+1}} I_k$ and we have proved the first part of the theorem.

Now let $I = \langle y^b, \dots, x^a \rangle$ be an integrally closed monomial ideal. Define $(a_1, b - b_1) \in \Gamma I$ as such a point, that there are no other points that belong to ΓI below or on the line between $(0, b)$ and $(a_1, b - b_1)$. Then for $k \geq 1$, let $(a_1 + \cdots + a_{k+1}, b - b_1 - \cdots - b_{k+1}) \in \Gamma I$ be the point that satisfies the same condition but for the line between $(a_1 + \cdots + a_k, b - b_1 - \cdots - b_k)$ and $(a_1 + \cdots + a_{k+1}, b - b_1 - \cdots - b_{k+1})$. Determined in such a way and considering that I is integrally closed, we have GCD $(a_k, b_k) = 1$. The corresponding (a_k, b_k)-blocks are unique and their product is equal to I by the first statement of the theorem. □

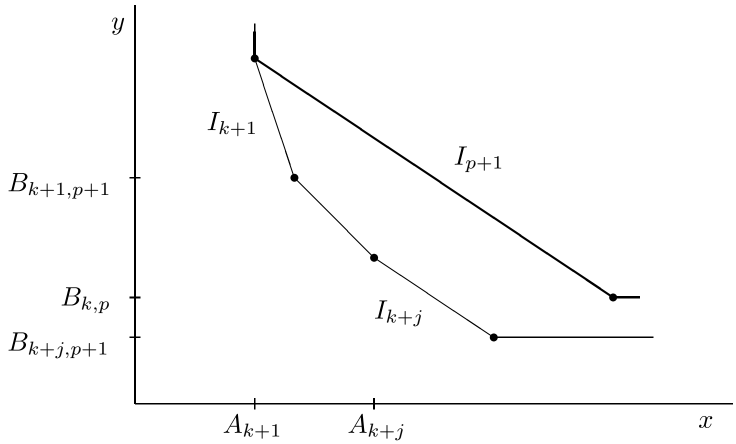

FIGURE 4.

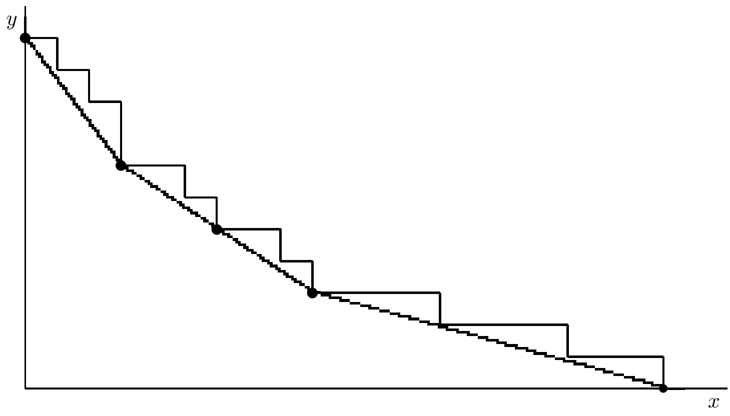

FIGURE 5.

Example 2.10. Consider the integrally closed ideal $I = \langle y^{11}, xy^{10}, x^2y^9, x^3y^7, x^5y^6, x^6y^5, x^8y^4, x^9y^3, x^{13}y^2, x^{17}y, x^{20}\rangle$. The staircase of this ideal is in Figure 5.

The points constructed by the procedure described in the second part of the proof of Theorem 2.9 are: $(0, 11)$, $(3, 7)$, $(6, 5)$, $(9, 3)$ and $(20, 0)$.

Thus, $I = I_1 I_2 I_3 I_4$ where $I_1 = \langle y^4, xy^3, x^2y^2, x^3\rangle$, $I_2 = \langle y^2, x^2y, x^3\rangle = I_3$ and $I_4 = \langle y^3, x^4y^2, x^8y, x^{11}\rangle$.

3. The quadratic transform and contracted monomial ideals. Let $(R, \mathfrak{m})$ be equal to $k[x,y]_{\langle x,y\rangle}$ or $k[[x,y]]$ over a field k. We give an explicit description of the quadratic transform of x-tight and y-tight ideals and show that if the quadratic transform is integrally closed then the ideal itself is integrally closed.

3.1. Transforms of x-tight and y-tight ideals. Let $\mathfrak{m} = \langle a, b\rangle$. Then there is an isomorphism $R[\mathfrak{m}/b] = R[a/b] \cong R[z]/\langle bz - a\rangle$ since $\{a, b\}$ is a regular sequence.

Let $I_x = \langle x^i y^{b_i}\rangle_{i=0}^r$ be an ideal of order r, not equal to any power of $\mathfrak{m}$. If $p \in I$, then $p/y^r \in R[x/y]$, and we may write $IR[x/y] = y^r\langle (x/y)^i y^{b_i - r + i}\rangle_{i=0}^r = y^r(I_x)'$, where $(I_x)'$ is an ideal in the ring $R[x/y]$. We will call $(I_x)'$ the *transform* of I_x in $R[x/y]$. Note that if $i + b_i = (i+1) + b_{i+1}$ for some i, then the generator $x^{i+1}y^{b_{i+1}}$ is superfluous in the minimal generating set of I'. The only maximal ideal in $R[x/y]$ containing $(I_x)'$ is $\mathfrak{n}_x = \langle x/y, y\rangle$.

Thus, the transforms of the ideals I_x, I_y in the respective localized rings are:

$$
\begin{aligned}
(I_x)'_{\langle (x/y), y\rangle} &= \left\langle \left(\frac{x}{y}\right)^i y^{b_i - r + i}\right\rangle \text{ in } R\left[\frac{x}{y}\right]_{\langle \frac{x}{y}, y\rangle} \\
(I_y)'_{\langle x, (y/x)\rangle} &= \left\langle x^{a_j - j}\left(\frac{y}{x}\right)^{s-j}\right\rangle \text{ in } R\left[\frac{y}{x}\right]_{\langle x, (y/x)\rangle}.
\end{aligned} \tag{3.1}
$$

By Proposition 2.2 the product of an x-tight and a y-tight ideal is equal to $I = I_x I_y = \langle x^i y^{s+b_i},\ x^{r+a_j} y^{s-j}\rangle$, and its order is $r+s$. For the computation of its transform we may consider the ring $R\big[\mathfrak{m}/(x+y)\big]$. Then we can write I as the product of $(x+y)^{r+s}$ and the ideal

$$
\begin{aligned}
I' = &\left\langle \left(\frac{x}{x+y}\right)^i (x+y)^{b_i - r + i}\left(\frac{y}{x+y}\right)^{s+b_i}\right\rangle \\
&+ \left\langle \left(\frac{x}{x+y}\right)^{r+a_j} (x+y)^{a_j - j}\left(\frac{y}{x+y}\right)^{s-j}\right\rangle.
\end{aligned}
$$

The two maximal ideals containing I' are $\mathfrak{n}_1 = \langle x/(x+y), x+y\rangle$ and $\mathfrak{n}_2 = \langle x+y, y/(x+y)\rangle$.

(3.2)

$$(I')_{\mathfrak{n}_1} = \left\langle \left((\frac{x}{x+y})^i (x+y\right)^{b_i - r + i} \right\rangle \text{ in } R\left[\frac{\mathfrak{m}}{x+y}\right]_{\mathfrak{n}_1} = R\left[\frac{x}{x+y}\right]_{\mathfrak{n}_1}$$

$$(I')_{\mathfrak{n}_2} = \left\langle (x+y)^{a_j - j} \left(\frac{y}{x+y}\right)^{s-j} \right\rangle \text{ in } R\left[\frac{\mathfrak{m}}{x+y}\right]_{\mathfrak{n}_2} = R\left[\frac{y}{x+y}\right]_{\mathfrak{n}_2}.$$

Obviously, the two localized rings, as well as the transforms in them in (3.1), are isomorphic to the ones in (3.2).

Remark 3.1. The quadratic transforms of the ideals we have described can be taken in a ring $R\big[\mathfrak{m}/(cx+dy)\big]$ for almost any $c, d \in k$. The resulting transforms and the maximal ideals containing them will also be the same up to isomorphism.

For example,

$$(I_x)' = \left\langle \left(\frac{x}{cx+dy}\right)^i (cx+dy)^{b_i - r + i} \left(\frac{y}{cx+dy}\right)^{b_i} \right\rangle \quad \text{in } R\left[\frac{\mathfrak{m}}{cx+dy}\right]$$

for any $d \neq 0$ and any c. The unique associated maximal ideal will be

$$\mathfrak{n}_x = \left\langle \frac{x}{cx+dy}, cx+dy \right\rangle \text{ and } (I_x)'_{\mathfrak{n}_x} = \left\langle \left(\frac{x}{cx+dy}\right)^i (cx+dy)^{b_i - r + i} \right\rangle.$$

If the given ideal is some power of $\mathfrak{m}$, then the transform must be done in the ring $R\big[\mathfrak{m}/(cx+dy)\big]$ with both c and d being nonzero.

We have proved the following proposition.

Proposition 3.2. *Let I_x be an x-tight ideal in a two-dimensional ring R. Then the quadratic transform of I_x in the ring $R\big[\mathfrak{m}/p_1\big]$, where $p_1 \neq x$, has one unique associated maximal ideal $\mathfrak{n}_x = \langle (x/p_1), p_1\rangle$. The quadratic transform of a y-tight ideal I_y in $R\big[\mathfrak{m}/p_2\big]$, where $p_2 \neq x$, has the unique associated maximal ideal $\mathfrak{n}_y = \langle p_2, (y/p_2)\rangle$.*

Further, let $I = I_x I_y$. The quadratic transform of I in $R\big[\mathfrak{m}/p_3\big]$, where $p_3 \notin \{x, y\}$, has the associated ideals $\mathfrak{n}_1 = \langle (x/p_3), p_3\rangle$ and $\mathfrak{n}_2 = \langle p_3, (y/p_3)\rangle$. Moreover, $(I_x)'_{\mathfrak{n}_x} \cong (I')_{\mathfrak{n}_1}$ and $(I_y)'_{\mathfrak{n}_y} \cong (I')_{\mathfrak{n}_2}$.

3.2. Contracted monomial ideals. We have from Huneke [**6**] that an ideal I is contracted if and only if $\mu(I) = o(I) + 1$ (Proposition 2.3), and that an integrally closed ideal is contracted (Corollary 3.2). Proposition 3.4 states that if I is integrally closed, then its quadratic transform is integrally closed. Finally, if I in Proposition 3.4 is simple and $\mathfrak{m}$-primary, then $I'R[\mathfrak{m}/a]$ is simple (Proposition 3.5).

We can easily see that any x-tight or y-tight ideal is contracted.

It is not known whether the converse to Proposition 3.4 above is valid. We show that this is the case if the transform I' is monomial. We prove the result for the case when I' is a block and I is y-tight. If I is x-tight the proof is similar.

Proposition 3.3. *Let $I \subset R$ be a y-tight ideal, and let I' be its quadratic transform in the ring $R[\mathfrak{m}/p]$, where $p \neq y$; assume I' is a simple integrally closed ideal. Then I is a product of some power of the maximal ideal $\mathfrak{m}$ and a simple integrally closed monomial ideal.*

Proof. Without loss of generality we consider the extension ring $R[\mathfrak{m}/x] = R[y/x] \cong R[z]/\langle xz - y\rangle$. Let $I = \langle x^{a_i} y^{r-i}\rangle_{i=0}^{r}$; then the transform is $I' = \langle x^{a_i - i} z^{r-i}\rangle_{i=0}^{r}$. The assumption yields that I' is either x- or z-tight. Our goal is to show that

$$I = \mathfrak{m}^{r-l}\overline{\langle x^{a_r - l}, y^{r-l}\rangle} \tag{3.3}$$

with $\gcd(a_r - l, r - l) = 1$ for some l, that is, $a_i = i$ for all $i \leq l$ and $a_i - 1 < (i-l)(a_r - l)/(r - l) + l \leq a_i$ for all $i \geq l + 1$ and with equality only if $i = r$.

I' is x-tight: Let I' be minimally generated by $\langle x^j z^{d_j}\rangle_{j=0}^{s}$. Then the following hold for the a_i's:

$$\begin{aligned} a_i - i &= 0 \quad \text{for all } 0 \leq i \leq l = r + d_0; \\ a_i - i &= j \quad \text{for all } i \quad \text{such that} \quad d_j \leq r - i \leq d_{j-1} - 1. \end{aligned} \tag{3.4}$$

I' is assumed to be integrally closed and simple, so by Definition 2.7 we have $d_j - 1 < (s - j)(d_0/s) < d_j$ for all $1 \leq j \leq s - 1$, which combined with (3.4) gives

$$(s - j)\frac{d_0}{s} < r - i < (s - (j - 1))\frac{d_0}{s}, \tag{3.5}$$

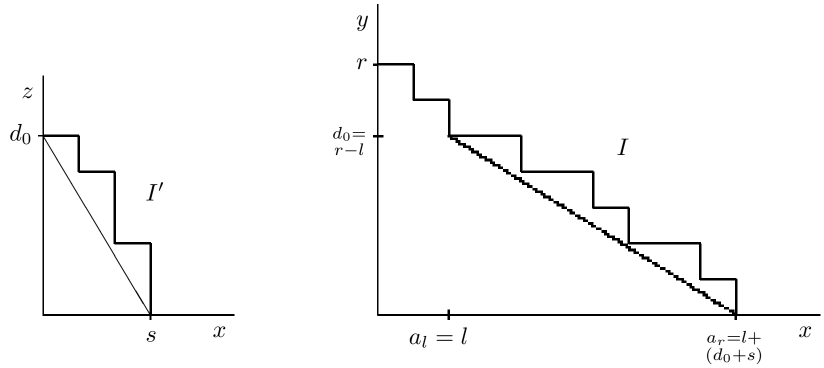

FIGURE 6. I' and I if I' is x-tight.

valid for all corresponding $l+1 \leq i \leq r-d_{s-1}$. Moreover, (3.5) is valid for $j=s$ and $i \leq r-1$, since $0 = d_s < r-i \leq d_{s-1}-1$, and is equivalent to

$$j-1 < s-(r-i)\frac{s}{d_0} < j. \tag{3.6}$$

Applying (3.4) on (3.6) and using $s = a_r - r$ yields

$$a_i - 1 < (i-l)\frac{a_r - l}{r-l} + l < a_i \quad \text{for all} \quad l+1 \leq i \leq r-1. \tag{3.7}$$

Thus, the ideal I precisely a product of $\mathfrak{m}^l$ and an $(a_r - l, r-l)$-block.

I' is z-tight: Similarly, we show that $a_i - 1 < (i-l)(a_r-l)/(r-l) + l < a_i$ where $l = r-s$, and $I = \mathfrak{m}^l\overline{\langle a_r - l, r-l\rangle}$. □

Corollary 3.4. *Let I be a y-tight ideal of order r and its transform an (a,b)-block, $\overline{\langle z^b, x^a\rangle}$. Then $I = \mathfrak{m}^{r-b}\overline{\langle y^b, x^{a+b}\rangle}$.*

If I is x-tight of order r and $I' = \overline{\langle z^b, x^a\rangle}$. Then $I = \mathfrak{m}^{r-a}\overline{\langle y^{a+b}, x^a\rangle}$.

Proof. If I is y-tight, then the formula (3.3) together with (3.4) and its analogue for z-tight I', yields the result. The statement for an x-tight ideal follows if one proves Proposition 3.3 for an x-tight ideal in the same way we did. □

Proposition 3.3 shows that if the transform I' is a simple integrally closed monomial ideal and the order of I is given, then it is possible to determine the y-tight ideal I. Moreover, this I is unique. It is also possible to determine I if its quadratic transform is *any* integrally closed monomial ideal.

Example 3.5. Let $I = \langle x^{a_i} y^{6-i} \rangle_{i=0}^{6}$ be a y-tight ideal of order 6, and consider its transform in the ring $R[y/x] \cong R[w]/\langle xz - y \rangle$ to be $I' = \langle z^5, xz^3, x^2z^2, x^4z, x^5 \rangle = \langle z^2, x \rangle \langle z, x \rangle \langle z^2, x^2z, x^3 \rangle$–a product of the blocks (1,2), (1,1) and (3,2).

Since $I' = \langle x^{a_i - i} z^{6-i} \rangle$ we need to solve a number of linear equations, keeping in mind that $a_i - i \le a_{i+1} - (i+1)$. For example, $x^{a_1 - 1} z^5$ is a generator and then $a_1 = 1$; since $x^{a_2 - 2} z^4 \in I'$ we must have $a_2 - 2 = 1$, and so on.

Thus, the unique solution is $I = \langle y^6, xy^5, x^3y^4, x^4y^3, x^6y^2, x^9y, x^{11} \rangle = \mathfrak{m} \cdot \langle y^2, x^2y, x^3 \rangle \langle y, x^2 \rangle \langle y^2, x^3y, x^5 \rangle$–the product of the maximal ideal and the blocks (3,2), (2,1) and (5,2).

The following result is the converse of Proposition 3.4 [**6**].

Proposition 3.6. *Let I be y-tight, and assume that the quadratic transform I' is integrally closed. Then I is integrally closed.*

Moreover, let the order of I be equal to r and $I' = \prod_{i=0}^{n} \overline{\langle z^{d_k}, x^{c_k} \rangle}$. Then

$$I = \mathfrak{m}^{r - \sum_{k=1}^{n} d_k} \prod_{i=1}^{n} \overline{\langle y^{d_k}, x^{c_k + d_k} \rangle}. \tag{3.8}$$

Correspondingly, if I is an x-tight ideal of order r and its quadratic transform $I' = \prod_{i=0}^{n} \overline{\langle y^{d_k}, z^{c_k} \rangle}$ is integrally closed. Then I is integrally closed and equal to

$$I = \mathfrak{m}^{r - \sum_{k=1}^{n} c_k} \prod_{i=1}^{n} \overline{\langle y^{c_k + d_k}, x^{c_k} \rangle}. \tag{3.9}$$

Proof. Let $I = \langle x^{a_i} y^{r-i} \rangle_{i=0}^{r}$ and its transform $I' = \langle x^{a_i - i} z^{r-i} \rangle \subset R[y/x]$. The transform is $I' = \prod_{k=1}^{n} \overline{\langle z^{d_k}, x^{c_k} \rangle} = \sum_{k=1}^{n} x^{C_k} z^{D_{k,n}} \overline{\langle z^{d_k}, x^{c_k} \rangle}$

with $C_k = \sum_{k'=1}^{k-1} c_{k'}$, $D_{k,n} = \sum_{k'=k+1}^{n} d_{k'}$ and $C_1 = D_{n,n} = 0$, by assumption and Theorem 2.9.

For all i such that $r - i \geq D_{0,n}$, we have $a_i = i$. So one factor in I is $\mathfrak{m}^{r-D_{0,n}}$.

Any term $x^{C_k} z^{D_{k,n}} \overline{\langle z^{d_k}, x^{c_k}\rangle}$ in I' is equal to $x^{C_k} z^{D_{k,n}} \langle x^{a_i - i - C_k} z^{r-i-D_{k,n}}\rangle$ $= \langle x^{a_i - i} z^{r-i}\rangle$ for all i such that $r - D_{k-1,n} \leq i \leq r - D_{k,n}$. Let $j = i - (r - D_{k-1,n})$; then for all $0 \leq j \leq d_k$ we get the equality

(3.10)
$$\overline{\langle z^{d_k}, x^{c_k}\rangle} = \langle x^{(a_{(j+r-D_{k-1,n})} - r + D_{k-1,n} - C_k) - j} z^{d_k - j}\rangle = \langle x^{a'_{j'} - j} z^{d_k - j}.\rangle$$

By (3.7), we know that for all $r - D_{k-1,n} < i < r - D_{k,n}$, we have $a_i - 1 < (i - (r - D_{k-1,n}))(c_k + d_k)/d_k + (r - D_{0,n}) + C_k + \left(\sum_{k'=1}^{k-1} d_{k'}\right) < a_i$. The messy expression becomes clearer, if we look at Figure 7.

Since
$$\frac{c_k}{d_k} \leq \frac{c_{k+1}}{d_{k+1}} \iff \frac{c_k}{c_k + d_k} \leq \frac{c_{k+1}}{c_{k+1} + d_{k+1}},$$
with equality on one of side if and only if there is equality on the other, we have determined the shape of I:

$$I = \mathfrak{m}^{r - D_{0,n}} \cdot \prod_{k=1}^{n} \overline{\langle y^{d_k}, x^{c_k + d_k}\rangle}.$$

Hence, the ideal I is a product of blocks and is therefore integrally closed.

If I is x-tight, then using the same methods we can deduce (3.9). □

Let I be a contracted monomial ideal of order r. We may assume that I is $\mathfrak{m}$-primary. Then $I = \langle x^{a_i} y^{b_i}\rangle_{0 \leq i \leq r}$ and there is an l such that $o(x^{a_l} y^{b_l}) = a_l + b_l = r$. Since the given set of generators is minimal, we have $a_l \geq a_{l-1} + 1 \geq \cdots \geq l$ and $r - a_l = b_l \geq b_{l+1} + 1 \geq \cdots \geq r - l$. Hence, $a_l = l$ and I is the product of an x-tight ideal of order l and a y-tight ideal of order $r - l$.

Proposition 3.7. *Let I be a contracted monomial ideal and assume that its quadratic transform I' is integrally closed. Then I is integrally closed.*

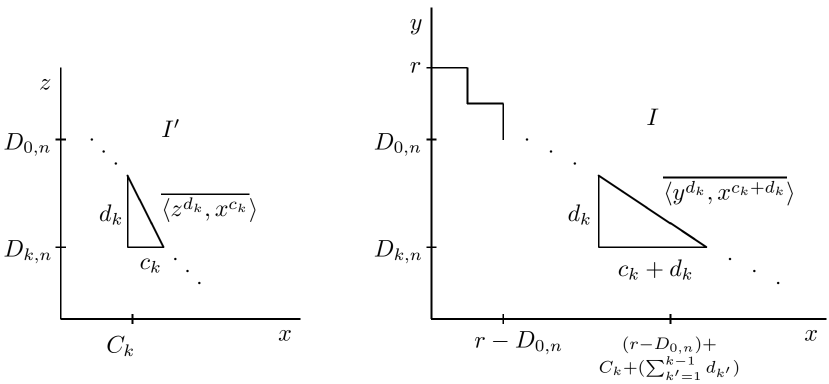

FIGURE 7.

Proof. If I is x- or y-tight, we apply Proposition 3.6.

If I is the product of an x-tight and a y-tight ideal, then we consider $I' \subset R[\mathfrak{m}/(x+y)]$. The transform I' is assumed to be integrally closed, then so are all its localizations. Hence, according to Proposition 3.2 and using the notations from it, the ideals $(I'_x)_{\mathfrak{n}_x}$ and $(I'_y)_{\mathfrak{n}_y}$ are integrally closed monomial ideals. Moreover, $\mathfrak{n}_x \subset R[\mathfrak{m}/y]$ and $\mathfrak{n}_y \subset R[\mathfrak{m}/x]$ are the unique maximal ideals containing I'_x and I'_y respectively. Thus, both I'_x and I'_y must be integrally closed. Then both I_x and I_y are integrally closed by the previous proposition, and the result follows. □

4. Powers and products of ideals. In subsection 2.3 we saw that the lth power of a simple integrally closed monomial ideal I can be depicted as l copies of the ideal I. Moreover, if ordered in the right way, the product of a family of integrally closed monomial ideal can be depicted as the ideals one after the other. In a way this property can be extended to certain kinds of ideals in an integral domain.

4.1. Dividing generators, I. Let R be an integral domain and $F(R)$ its field of fractions. Let $\alpha, \beta \in F(R)$; then we say that α divides β, denoted by $\alpha|\beta$, if there is a $p \in R$ such that $\alpha \cdot p = \beta$.

Proposition 4.1. *Let R be an integral domain and $I = \langle r_0, \dots, r_n\rangle \subset R$ an ideal where*

$$r_i = r_{i-1}\alpha_i = r_0(\alpha_1 \cdots \alpha_i) \quad \text{with} \quad \alpha_i \in F(R) \quad \text{and} \quad \alpha_i|\alpha_{i+1}. \tag{4.1}$$

Then for any non-negative integer l, we have

$$I^l = \langle r_0^l, r_0^{l-1} r_1, \dots, r_0 r_1^{l-1}, r_1^l, r_1^{l-1} r_2, \dots, r_1 r_2^{l-1}, \dots \dots, r_{n-1}^l, r_{n-1}^{l-1} r_n, \dots, r_{n-1} r_n^{l-1}, r_n^l \rangle = \left\langle r_i^{l-t} r_{i+1}^t \mid 0 \leq i \leq n-1 \text{ and } 0 \leq t \leq l \right\rangle . \tag{4.2}$$

Remark 4.2. Pick some r_i and $r_{i'}$ where $i < i'$. We have $r_i r_{i'} = (r_{i+1}/\alpha_{i+1}) \cdot r_{i'-1}\alpha_{i'} = r_{i+1} r_{i'-1}(\alpha_{i'}/\alpha_{i+1})$. Then the condition $\alpha_{i+1} \mid \alpha_{i'}$ yields $r_{i+1} r_{i'-1} \mid r_i r_{i'}$. That is, $\langle r_i r_{i'} \rangle \subseteq \langle r_{i+1} r_{i'-1} \rangle$. □

Proof. The inclusion $\langle r_i^{l-t} r_{i+1}^t \rangle \subseteq I^l$ is valid naturally.

I^l is generated by the elements of the form $r_0^{l_0} \cdots r_n^{l_n}$ where $\sum_{i=0}^n l_i = l$. Assume $l_0 \leq l_n$ in some generator, then by Remark 4.2 I^l is contained in the ideal, where this generator is replaced by the product $r_1^{l_0+l_1} \cdots r_{n-1}^{l_{n-1}+l_0} r_n^{l_n - l_0}$. (If $l_0 \geq l_n$, then we replace the generator by a product where r_n is eliminated.) Now, depending on $\min(l_0+l_1, l_n-l_0)$, we replace this product with another, in which either r_1 or r_n is eliminated. Continuing in such a way, we see that I^l is contained in the ideal, in which the generator $r_0^{l_0} \cdots r_n^{l_n}$ is replaced by r_i^l or $r_i^{l-t} r_{i+1}^t$ for some i and t. Repeating the procedure for all generators of I^l, we see that $I^l \subseteq \langle r_i^{l-t} r_{i+1}^t \rangle$, where $0 \leq i \leq n-1$ and $0 \leq t \leq l$. □

Note that I^l itself satisfies the condition (4.1) on I in Proposition 4.1.

Let R now be equal to $k[x,y]$, $k[x,y]_{\langle x,y \rangle}$ or $k[[x,y]]$. We will prove that a monomial ideal in R satisfying the condition (4.1) factors quite simply.

Suppose $I = \langle m_i \rangle_{i=0}^n = \langle x^{A_i} y^{B_i} \rangle_{i=0}^n$ satisfies (4.1). We may assume that I is $\mathfrak{m}$-primary, that is, $A_0 = B_n = 0$. Then $\alpha_i = x^{A_i - A_{i-1}} / y^{B_{i-1} - B_i} = x^{a_i}/y^{b_i}$. Since $\alpha_i | \alpha_{i+1}$ we have $a_i \leq a_{i+1}$ and $b_i \geq b_{i+1}$ for all $1 \leq i \leq n-1$.

Proposition 4.3. *Let $I = \langle x^{A_i} y^{B_i} \rangle_{i=0}^n \subset R$, and let $a_i = A_i - A_{i-1}$ and $b_i = B_{i-1} - B_i$. Assume $a_i \leq a_{i+1}$ and $b_i \geq b_{i+1}$. Then*

$$I = \prod_{i=1}^n \langle y^{b_i}, x^{a_i} \rangle. \tag{4.3}$$

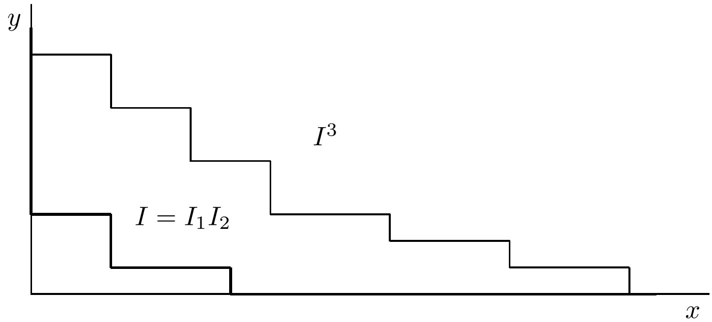

FIGURE 8.

Proof. For all $0 \leq k \leq n$ we have $A_k = \sum_{i=1}^{k} a_i$ and $B_k = \sum_{i=k+1}^{n} b_i$. Clearly y^{B_0}, x^{A_n} and $x^{A_k} y^{B_k}$ belong to $\prod_{i=1}^{n} \langle y^{b_i}, x^{a_i} \rangle$.

On the other hand, the ideal $\prod_{i=1}^{n} \langle x^{a_i}, y^{b_i} \rangle$ is generated by the monomials $x^{\sum_{i \in R} a_i} y^{\sum_{j \in S} b_j}$, where $R \cup S$ is a partition of $\{1, \ldots, n\}$. But $\sum_{i \in R} a_i \geq \sum_{i=1}^{|R|} a_i$, and $\sum_{j \in S} b_j \geq \sum_{|R|+1}^{n} b_j$. Hence, this generating monomial belongs to I. Since the chosen monomial generator was arbitrary, the proof is finished. □

Example 4.4. Let $I_1 = \langle y^3, x^2 \rangle$ and $I_2 = \langle y, x^4 \rangle$. Then $I = I_1 I_2 = \langle y^4, x^2 y, x^6 \rangle$. We have $I^l = \langle (y^4)^{l-t} (x^2 y)^t \rangle + \langle (x^2 y)^{l-t} (x^6)^t \rangle$ by Proposition 4.1, and $I^l = \langle x^{2t} y^{3(l-t)+l} \rangle + \langle x^{2l+4t} y^{l-t} \rangle$ by Proposition 4.3. Figure 8 shows the staircase diagram for $l = 3$.

Let $(I_i) = (\langle x^{a_i}, y^{b_i} \rangle)$, where $a_i \leq a_{i+1}$ and $b_i \geq b_{i+1}$, be a family of ideals. Proposition 4.3 states that in the product $\prod_i I_i = \langle x^{A_i} y^{B_i} \rangle$ the differences $A_{i+1} - A_i$ increase and the differences $B_i - B_{i+1}$ decrease. Moreover, we have proved the following result in two different ways. The lth power of $\prod_i I_i = \langle x^{A_i} y^{B_i} \rangle$ is such that the first l differences of two consecutive powers of x are equal to $A_1 - A_0$ and the first l differences of two consecutive powers of y are all equal to $B_0 - B_1$. The next l differences of two consecutive powers of x are equal to $A_2 - A_1$ while the corresponding l differences for y are equal to $B_1 - B_2$, and so on. See Figure 8.

4.2. Dividing generators, II. In this section we consider ideals with the condition (4.1) being reversed.

Proposition 4.5. *Let R be an integral domain and $I = \langle r_0, \dots, r_n \rangle \subset R$ an ideal with*

$$r_i = r_{i-1}\beta_i = r_0(\beta_1 \cdots \beta_i) \quad \text{with} \quad \beta_i \in F(R) \quad \text{and} \quad \beta_{i+1} \mid \beta_i. \tag{4.4}$$

Then for any non-negative integer l we have

$$\begin{aligned} I^l = \langle & r_0^l, r_0^{l-1}r_1, \dots, r_0^{l-1}r_{n-1}, \\ & r_0^{l-1}r_n, r_0^{l-2}r_1r_n, \dots, r_0^{l-2}r_{n-1}r_n, \dots \\ & \dots r_0r_n^{l-1}, r_1r_n^{l-1}, \dots, r_{n-1}r_n^{l-1}, r_n^l \rangle \\ = \langle & r_0^{l-t}r_ir_n^{t-1} \mid 0 \le i \le n \text{ and } 1 \le t \le l \rangle \\ = & \sum_{t=1}^{l} r_0^{l-t}r_n^{t-1}I. \end{aligned} \tag{4.5}$$

Remark 4.6. For any r_i and $r_{i'}$ with $1 \le i \le i' \le n-1$ we have

$$r_ir_{i'} = (r_{i-1}\beta_i) \cdot \left(\frac{r_{i'+1}}{\beta_{i'+1}}\right) = r_{i-1}r_{i'+1}\left(\frac{\beta_i}{\beta_{i'+1}}\right).$$

That is, $\langle r_ir_{i'} \rangle \subseteq \langle r_{i-1}r_{i'+1} \rangle$ since $\beta_{i'+1}|\beta_{i'} \mid \beta_i$.

Proof. Clearly, $\langle r_0^{l-t}r_ir_n^{t-1} \rangle \subseteq I^l$.

Assume $\sum_{i=1}^{n-1} l_i \ge 2$ for some generator $m = \prod_{i=0}^{n} r_i^{l_i}$. Then there are $j \le j'$ such that $l_j,\ l_{j'} \ne 0$. Assume that $l_j \le l_{l'}$. Then, by Remark 4.6, I^l is contained in the ideal, in which the product $r_0^{l_0} \cdots r_{j-1}^{l_{j-1}+l_j} r_{j+1}^{l_{j+1}} \cdots r_{j'}^{l_{j'}-l_j} r_{j'+1}^{l_{j'+1}+l_j} \cdots r_n^{l_n}$ replaces m. We then replace any pair of factors in $\prod_{i=0}^{n} r_i^{l_i}$ by a pair of outer factors (if such exist) until $\sum_{i=1}^{n-1} l_i \le 1$. Then $I^l \subseteq \langle r_0^{l-t}r_ir_n^{t-1} \rangle$. □

Now let R be the polynomial or regular local ring in two variables. Let $I = \langle m_i \rangle_{i=0}^{n} = \langle x^{A_i}y^{B_i} \rangle_{i=0}^{n} \subset R$ satisfy (4.4). We may assume that I is $\mathfrak{m}$-primary. Define $a_i = A_i - A_{i-1}$ and $b_i = B_{i-1} - B_i$ as previously. Then $\beta_i = x^{a_i}/y^{b_i}$ with $a_i \ge a_{i+1}$ and $b_i \le b_{i+1}$. The differences of

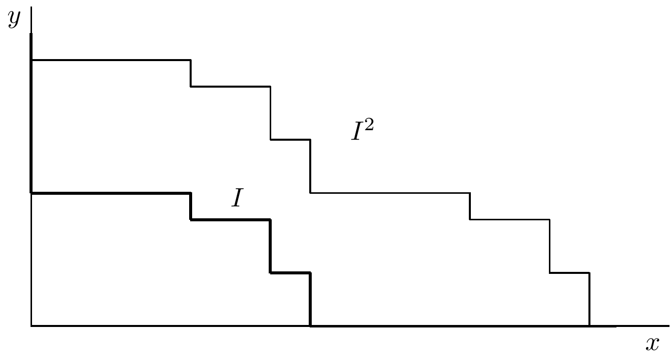

FIGURE 9.

two consecutive powers of x decrease, and the differences of powers of y increase. Moreover for any $1 \leq k \leq n$ we have

$$\frac{A_k}{B_0 - B_k} = \frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} \geq \frac{\sum_{i=1}^{k+1} a_i}{\sum_{i=1}^{k+1} b_i} \geq \cdots \geq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = \frac{A_n}{B_0}.$$

The equality occurs if and only if all the a_i's and the b_i's are constants. Then $I = \prod \langle y^{b_i}, x^{a_i} \rangle$ by Proposition 4.3. If $a_i > a_{i+1}$ or $b_i < b_{i+1}$ for some i, then $A_k/(B_0 - B_k) > (A_n/B_0)$ for all $1 \leq k \leq n-1$. Hence, I is simple as monomial ideal by Proposition 2.4; see Figure 3.

By (4.5) we have $I^l = x^{tA_n} y^{(l-t)B_0} I$. The staircase diagram is the ideal I repeated l times. In Figure 9 we can clearly see that generally I^l does *not* fulfill the condition on I.

REFERENCES

1. V. Crispin Quiñonez, *Integrally closed monomial ideals and powers of ideals*, Research Reports **7**, Department of Mathematics, Stockholm University, 2002. Available at http://www2.math.su.se/reports/2002/7/2002-7.pdf

2. ———, *Integral closure and related operations on monomial ideals*, Doctoral thesis, Stockholm University, 2006.

3. S.D. Cutkosky, *On unique and almost unique factorization of complete ideals*, Amer. J. Math. **111** (1989), 417–433.

4. ———, *On unique and almost unique factorization of complete ideals.* II, Invent. Math. **98** (1989), 59–74.

5. C. Huneke, *The primary components of and integral closures of ideals in* 3*-dimensional regular local rings*, Math. Ann. **275** (1986), 617–635.

6. C. Huneke, *Complete ideals in two-dimensional regular local rings*, *Commutative algebra*, (Berkeley, CA, 1987), 325–338, Math. Sci. Res. Inst. Publ. **15**, Springer, New York, 1989.

7. C. Huneke and J.D. Sally, *Birational extension in dimension two and integrally closed ideals*, L. Algebra **115** (1988), 481–500.

8. C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Math. Soc. Lect. Note Series **336**, Cambridge University Press, Cambridge, 2006.

9. M. Kim, *Product of distinct simple integrally closed ideals in two-dimensional regular local rings*, Proc. Amer. Math. Soc. **125** (1997), 315–321.

10. J. Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Ètudes Sci. Publ. Math. **36** (1969), 195–279.

11. R. Villarreal, *Monomial algebras*, Marcel Dekker, Inc., New York, 2001.

12. K. Watanabe, *Chains of integrally closed ideals*, *Commutative algebra* (Grenoble/Lyon, 2001), 353–358, Contemp. Math. **331**, American Mathematical Society, Providence, RI, 2003.

13. O. Zariski, *Polynomial ideals defined by infinitely near base points*, Amer. J. Math. **60** (1938), 151–204.

14. O. Zariski and P. Samuel, *Commutative algebra*, Vol. 2, D. Van Nostrand Co., Inc., Princeton, 1960.

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