MINIMAL REDUCTIONS OF MONOMIAL IDEALS IN DIMENSION TWO

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Abstract. The concept of reduction is tightly connected with the integral closure, since given two ideals $J \subseteq I$ we know that J is a reduction of I if and only if I is contained in the integral closure of J. It is well known that minimal reductions exist in local rings. We present a process of determining a minimal reduction of monomial ideals in two variables generalized to certain ideals in a two-dimensional local ring. The method is then generalized to some classes of ideals in integral domains and applied to monomial subrings.

1. Introduction

Let R be a Noetherian local ring. A reduction of an ideal $I \subset R$ is defined as an ideal $J \subseteq I$ such that $JI^l = I^{l+1}$ for some integer l. A *minimal reduction of* I is a reduction which is minimal with respect to inclusion.

The concept of reduction was introduced by Northcott and Rees in [10]. It is closely related to integral closure and, hence, multiplicity. An ideal $J \subseteq I$ is a reduction of I if and only if I is integral over J, that is $I \subseteq \bar{J}$, and $\bar{J}$ is the unique largest ideal for which J is a reduction. A step by step proof of this fact is given in [11]. An overview of the relation between these subjects is found in [7].

Let $(R, \mathfrak{m})$ be a local ring. The analytic spread of an ideal I, denoted by $\ell(I)$, is defined as the Krull dimension of the fibre cone $\mathcal{F}(I) = R/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \cdots = R[It]/\mathfrak{m}R[It]$, where $R[It] = \oplus I^j t^j$ is the Rees ring of I. It is known that $\mathrm{ht} I \leq \ell(I) \leq \min(\dim R, \mu(I))$, where $\mu(I)$ is the least number of generators of I (see [1], [7], [13]).

In a local Noetherian ring a minimal reduction of an ideal always exists. Moreover, if $(R, \mathfrak{m})$ is a local ring with infinite residue field, then for every minimal reduction J of I we have $\mu(J) = \ell(I)$, where $\mu(I)$ is the minimal number of generators of I ([10]).

If R is non-local the situation becomes much more complicated. However, in [9] Lyubeznik shows that every ideal in a polynomial ring in n variables over an infinite field has an n-generated reduction. Later Katz generalized the result to every ideal in a commutative Noetherian ring of dimension n in [8].

In a local ring with infinite residue field one can construct a minimal reduction of an ideal by taking a *sufficiently generic* sequence of the ideal elements. In the two-dimensional case there are two explicit results: the proposition in [2] for a special case in the polynomial localized ring and Theorem 5.5 in [4] for a special case in

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the polynomial ring. Then Chan and Liu presented an algorithm for computing a minimal reduction of a monomial ideal in $k[x,y]_{(x,y)}$ [3].

Here we present a rather convenient method of explicitly finding a minimal reduction for a class of ideals in local rings with infinite residue fields. We start with an algorithm for finding two-generated reductions in some cases in the two-dimensional polynomial ring over any field k.

In Section 3 we consider monomial ideals in the polynomial ring $k[x,y]$ with the maximal ideal $\mathfrak{m} = \langle x, y\rangle$. If the *minimal monomial reduction* I_{mmr} of a monomial ideal $I \subset k[x,y]$ is three-generated, we determine a two-generated minimal reduction of any ideal between I_{mmr} and I by using the relation between reduction and integral closure. A special case of this result coincides with the results from [2] and [4] mentioned above.

Section 4 is devoted to computing a minimal reduction for a certain class of $\mathfrak{m}$-primary ideals in a local integral domain. The application of the result gives a minimal reduction of any $\mathfrak{m}$-primary ideal in $k[x,y]_{\langle x,y\rangle}$ or $k[[x,y]]$. Hence, a special case of our theorem is Corollary 3.7 in [3]. Our approach is completely different. Also, our theorem can be used to find other minimal reductions of monomial ideals in a local ring.

2. Notation and preliminaries

Let I be a monomial ideal in n variables in the polynomial or a local ring. A graphical depiction of a monomial ideal in n variables is defined as follows. Let $\Gamma(cx_1^{a_1}\cdots x_n^{a_n}) = (a_1,\dots,a_n)$, then $\Gamma(I) = \{\Gamma(m) | m \in I,\ m \text{ monomial}\} \subset \mathbb{N}^n$. The integral closure of a monomial ideal $\bar{I} = \langle m \in R | m \text{ monomial},\ m^l \in I^l \text{ for some } l > 0\}$ is then interpreted as $\Gamma(\bar{I}) = (\text{conv}(\Gamma(I)) + \mathbb{R}^n_{\geq 0}) \cap \mathbb{N}^n$. For somewhat different proofs of this statement see Proposition 2.7 in [5] and Proposition 7.3.2 and 7.3.3 in [14].

Let R be $k[x,y]$, $k[x,y]_{\langle x,y\rangle}$ or $k[[x,y]]$ over an infinite field k. In [6] the normality of monomial ideals in R is studied. It is shown how integrally closed monomial ideals factors into simple ideals with the same porperty, and a criterion in terms of so called block ideals is established. We recall that an ideal is called simple if it is not a product of two proper ideals of R.

Let $I \subset R$ be a monomial ideal. By writing $I = \langle y^{b_0}, x^{a_1}y^{b_1}, \dots, x^{a_i}y^{b_i}, \dots, x^{a_{r-1}}y^{b_{r-1}}, x^{a_r}\rangle$ we mean that the generating set is minimal and the generators are ordered in such a way that $a_i < a_{i+1}$ (and $b_i > b_{i+1}$).

Definition 2.1 ([6], Definition 2.1)**.** An $\mathfrak{m}$-primary monomial ideal $I = \langle x^{a_i}y^{b_i}\rangle_{i=0}^r$ is called *x-tight* if $a_{i+1} - a_i = 1$ for all i, that is, $I = \langle x^i y^{b_i}\rangle_{i=0}^r$. An ideal of the form $I = \langle x^{a_i}y^{r-i}\rangle_{i=0}^r$ is called y-tight.

As usual, we denote the ceiling function of a real number r by $\lceil r\rceil$.

Definition 2.2 ([6], Definition 2.7)**.** Let a and b be positive integers with $\gcd(a,b) = 1$. Then there is a unique simple integrally closed monomial ideal containing x^a and y^b in its minimal generating set, $\overline{\langle y^b, x^a\rangle}$. We call such an ideal an (a,b)-*block* or a *block ideal*. Moreover, the ideal is the least integrally closed ideal possessing x^a and y^b.

If $a > b$ then the (a,b)-block is equal to $\langle x^{a_i}y^{b-i}\rangle_{i=0}^b$ where $a_i = \lceil i\frac{a}{b}\rceil$ with equality only if $i = 0, b$.

If $a < b$ then the block ideal is equal to $\langle x^i y^{b_i} \rangle_{i=0}^{a}$ where $b_i = \lceil (a-i)\frac{b}{a} \rceil$ with equality only for $i = 0, a$.

Theorem 2.3 ([6], Theorem 2.9)**.** *Let $(I_k)_{1 \le k \le n}$ be a sequence of (a_k, b_k)-blocks such that $\frac{a_k}{b_k} \le \frac{a_{k+1}}{b_{k+1}}$. Then the product is the integrally closed ideal*

$$\prod_{k=1}^{n} I_k = \sum_{k=1}^{n} x^{A_k} y^{B_{k,n}} I_k, \tag{2.1}$$

$$\text{where } A_k = \sum_{k'=1}^{k-1} a_{k'}, B_{k,n} = \sum_{k'=k+1}^{n} b_{k'} \text{ and } A_1 = B_{n,n} = 0. \tag{2.2}$$

Conversely, any integrally closed monomial ideal can be written uniquely as a product of block ideals and some monomial.

3. Reductions of monomial ideals of dimension two

Let $R = k[x, y]$ be the polynomial ring over in infinite filed k with the maximal ideal $\mathfrak{m} = \langle x, y \rangle$. Due to the theorem in [9] any monomial ideal in R has a two-generated minimal reduction, but the proof uses the same does not give any hint of how to compute such a reduction. We show the procedure for some cases.

Let $I \subset R$ be a monomial ideal. An ideal mJ is a reduction of mI' if and only if J is a reduction of I'. We recall that any monomial ideal I is on the form $I = mI'$, where m is a monomial and I' is either an $\mathfrak{m}$-primary monomial ideal or the ring R. The latter case is trivial. Thus we may assume that I is $\mathfrak{m}$-primary, that is, x^a and y^b belong to I for some a and b.

Definition 3.1. Let $I = \langle x^{A_i} y^{B_i} \rangle_{0 \le i \le s}$ with $0 = A_0 < A_1 < \ldots A_{s-1} < A_s$ and $B_0 > B_1 > \ldots > B_{s-1} > B_s = 0$ be minimally generated by the $x^{A_i} y^{B_i}$'s, that is, $\mu(I) = s + 1$. Define $I_{mmr} = \langle x^{A_{i_j}} y^{B_{i_j}} \rangle$ as follows:

$i_0 = 0$,

i_1 be the greatest i such that the minimal value of the expression $\frac{A_i}{B_0 - B_i}$ is obtained,

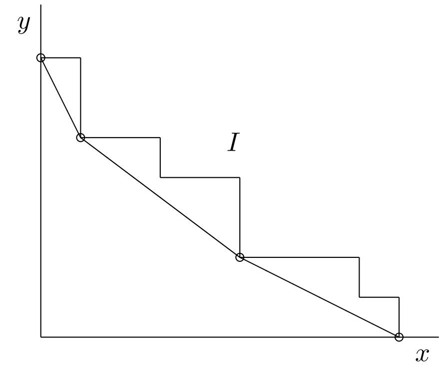
for $j \ge 2$ let $i_j = \max\{\, i > i_{j-1};\ \frac{A_i - A_{i_{j-1}}}{B_{i_{j-1}} - B_i} \text{ is minimal} \,\}$.

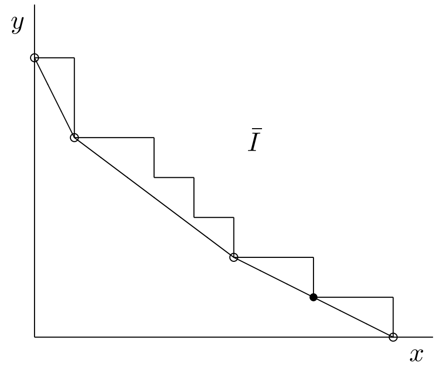
Graphically we define the generators of I_{mmr} in $\mathbb{N}^2$ recursively by starting with $(0, B_0)$ and choosing the greatest index i such that (A_i, B_i) gives the steepest slope between the two points. Taking this exponent as our new starting point we repeat the procedure. The ideal I_{mmr} has the same integral closure as I, in other words, $\text{conv}(\Gamma(I_{mmr})) = \Gamma(\bar{I})$.

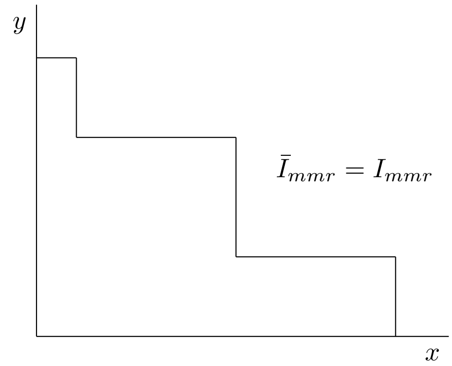
Example 3.2. Let $I = \langle y^7, xy^5, x^3y^4, x^5y^2, x^8y, x^9 \rangle$. Its integral closure is $\bar{I} = \langle y^7, xy^5, x^3y^4, x^4y^3, x^5y^2, x^7y, x^9 \rangle$. We have $I_{mmr} = \bar{I}_{mmr} = \langle y^7, xy^5, x^5y^2, x^9 \rangle$, the generators are marked by empty circles.

It is clear that among all monomial ideals with integral closure $\bar{I}$ the ideal I_{mmr} is the minimal one. Equivalently, among all monomial ideals which are reductions of $\bar{I}$, the ideal I_{mmr} is minimal. We call it the unique *minimal monomial reduction* of I. Moreover, I_{mmr} is the minimal monomial reduction of any ideal J such that $I_{mmr} \subset J \subset \bar{I}$.

This coincides with Proposition 2.1 [12], which is stated for n variables, but for the two-dimensional case.







Example 3.3. The ideal $\bar{I}$ in the previous example is a product of the blocks $\overline{\langle x, y^2\rangle}$, $\overline{\langle x^4, y^3\rangle}$, $\overline{\langle x^2, y\rangle}$, $\overline{\langle x^2, y\rangle}$. It is depicted by the vertices (7,0), (1,5), (5,2), (7,1) and (9,0), where we omit (7,1) to get the minimal monomial reduction.

Proposition 3.4. *Any power of a simple and* $\mathfrak{m}$*-primary integrally closed monomial ideal* $I = \langle y^{b_0}, \ldots, x^{a_i}y^{b_i}, \ldots, x^{a_r}\rangle \subset k[x, y]$ *has the ideal* $\langle y^{b_0}, x^{a_r}\rangle$ *as a reduction.*

Specially, if I *is an* $\mathfrak{m}$*-primary ideal generated by monomials of the same degree* d*, then* $\langle y^d, x^d\rangle$ *is a reduction of* I*.*

Proof. If I is x-tight, then $I = \langle x^i y^{b_i}\rangle_{0\leq i\leq r}$ with $b_i = \lceil \frac{r-i}{r} b_0\rceil$. By Theorem 2.3 we have $I^2 = \langle x^i y^{b_0+b_i}, x^{r+i}y^{b_i}\rangle$, and it is obvious that $I \cdot \langle y^{b_0}, x^r\rangle = I^2$.

If I is y-tight and $I = \langle x^{a_i}y^{r-i}\rangle_{0\leq i\leq r}$, where $a_i = \lceil i\frac{a_r}{r}\rceil$, then $I^2 = \langle x^{a_i}y^{2r-i}, x^{a_r+a_i}y^{r-i}\rangle$. The result follows similarly.

The last statement follows from $\langle y^d, x^d\rangle \subseteq I \subseteq \bar{I}$. □

Corollary 3.5. *A monomial ideal* $I = \langle y^{b_0}, \ldots, x^{a_i}y^{b_i}, \ldots, x^{a_r}\rangle \subset k[x, y]$ *with the integral closure being a power of some block ideal has a minimal reduction* $\langle y^{b_0}, x^{a_r}\rangle$*.*

Remark 3.6. The result generalizes one part of the main result in [2].

If the minimal monomial reduction is three-generated, then there is a way to determine a two-generated reduction too.

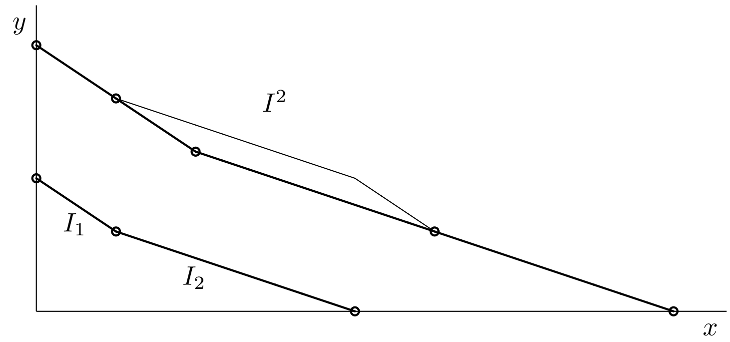
Proposition 3.7. *Let $I \subset k[x,y]$ be an $\mathfrak{m}$-primary monomial ideal with a three-generated minimal monomial reduction, say, $I_{mmr} = \langle y^b, x^c y^d, x^a\rangle$. Then the ideal $J = \langle y^b + x^a, x^c y^d\rangle$ is a reduction of I.*

Proof. We will show that if the integral closure of I, $\bar{I}$, is a product of some powers of two different block ideals, then there is a reduction on the desired form. There are three cases to consider:

(1) I is a product of y-tight ideals;
(2) I is a product of x-tight ideals;
(3) I is a product of an x-tight and a y-tight ideal.

We will prove the first case. The proofs of the remaining cases are based on the same idea.

Let $\bar{I}$ be a product of $I_1 = \overline{\langle y^b, x^{a_b}\rangle}^k$ and $I_2 = \overline{\langle y^d, x^{c_d}\rangle}^l$, where $\frac{a_b}{b} < \frac{c_d}{d}$. We introduce the notations $kb = r$, $A_r = ka_b$ and $ld = s$, $C_s = lc_d$. Clearly $\frac{A_r}{r} < \frac{C_s}{s}$. Then $I_1 = \langle x^{A_i} y^{r-i}\rangle_{i=0}^{r}$ with $A_i = \lceil i\frac{A_r}{r}\rceil$ and $I_2 = \langle x^{C_j} y^{s-j}\rangle_{j=0}^{s}$ with $C_j = \lceil j\frac{C_s}{s}\rceil$. With these notations and using Theorem 2.3 we have $I = y^s I_1 + x^{A_r} I_2$ and $I^2 = y^{2s+r} I_1 + x^{A_r} y^{2s} I_1 + x^{2A_r} y^s I_2 + x^{2A_r+C_s} I_2$. The minimal monomial reduction is $\langle y^{s+r}, x^{A_r} y^s, x^{A_r+C_s}\rangle$.



Our claim is $\bar{I}^2 = \langle y^{s+r} + x^{A_r+C_s}, x^{A_r} y^s\rangle \bar{I}$. We have

$$\bar{I}^2 = \langle x^{A_i} y^{2s+2r-i}\rangle + \langle x^{A_r+A_i} y^{2s+r-i}\rangle + \langle x^{2A_r+C_j} y^{2s-j}\rangle + \langle x^{2A_r+C_s+C_j} y^{s-j}\rangle$$

$$J\bar{I} = \langle x^{A_i} y^{2s+2r-i} + x^{A_r+C_s+A_i} y^{s+r-i}, x^{A_r+A_i} y^{2s+r-i}, \quad x^{A_r+C_j} y^{2s+r-j} + x^{2A_r+C_s+C_j} y^{s-j}, x^{2A_r+C_j} y^{2s-j}\rangle \tag{3.1}$$

We will show that $J\bar{I}$ is monomial with the same generating set as I^2.

Consider the first generator $x^{A_i} y^{2s+2r-i} + x^{A_r+C_s+A_i} y^{s+r-i}$. We will show the the second term is superfluous because it is a multiple of one of the two monomial generators of $J\bar{I}$.

- ***$0 \le i \le r - s$:*** Then $C_s + A_i = \lceil C_s\rceil + \lceil i\frac{A_r}{r}\rceil = \lceil C_s + i\frac{A_r}{r}\rceil \ge \lceil s\frac{A_r}{r} + i\frac{A_r}{r}\rceil = A_{s+i}$, thus $x^{A_r+A_{s+i}} y^{2s+r-(s+i)} \mid x^{A_r+C_s+A_i} y^{s+r-i}$.
- ***$r - s \le i \le r$:*** We have $C_s + A_i = \lceil C_s + r\frac{A_r}{r} - (r-i)\frac{A_r}{r}\rceil \ge \lceil s\frac{C_s}{s} - (r-i)\frac{C_s}{s} + A_r\rceil = C_{s-r+i} + A_r$; then $x^{A_r+C_s+A_i} y^{s+r-i}$ is a multiple of $x^{2A_r+C_{s-r+i}} y^{2s-(s-r+i)}$.

Similarly we show that the first term in $x^{A_r+C_j} y^{2s+r-j} + x^{2A_r+C_s+C_j} y^{s-j}$ is superfluous:

- ***$0 \le j \le r$:*** Then $C_j \ge A_j$ and hence $x^{A_r+A_j} y^{2s+r-j} \mid x^{A_r+C_j} y^{2s+r-j}$;

j > r: Then $C_j = \lceil j\frac{C_s}{s}\rceil = \lceil r\frac{C_s}{s}+(j-r)\frac{C_s}{s}\rceil \geq \lceil r\frac{A_r}{r}-(j-r)\frac{C_s}{s}\rceil = A_r+C_{j-r}$, that is, $x^{2A_r+C_{j-r}}y^{2s-(j-r)} \mid x^{A_r+C_j}y^{2s+r-j}$.

Thus $J\bar{I} = \bar{I}^2$ and J is a two-generated reduction of any ideal between J and $\bar{I}$, in particular, the ideal I.

The proofs of the remaining cases when I is a product of x-tight ideals or a product of an x-tight and a y-tight ideal are based on the same idea. □

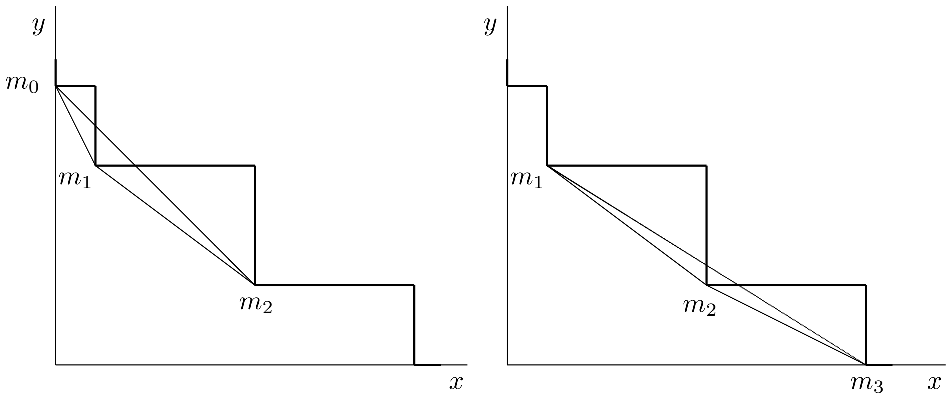
Remark 3.8. The statement in the proposition for the special case when I itself is x-tight is equivalent with Theorem 5.5 in [4], while our statement is valid for a larger class of ideals. However, their proof cannot be generalized to the case when the ideal I (or rather its integral closure $\bar{I}$) is a product of an x- and y- tight ideal.

Example 3.9. Consider the ideal $I = \langle y^{10}, x^2y^9 + x^3y^6, x^4y^4, x^7y^3 + x^8y^2, x^{10}\rangle = \langle p_i\rangle$. Let K be the ideal generated by the monomial in every p_i and $\bar{I}$ its integral closure. We have $K_{mmr} = \langle y^{10}, x^4y^4, x^{10}\rangle$ and a reduction of it $J = \langle y^{10} + x^{10}, x^4y^4\rangle$. Since $K_{mmr} \subset I \subset \bar{K}$, the ideal J is a reduction of I.

4. Reductions in local rings

The results in the previous section are valid in the local rings $k[x,y]_{\langle x,y\rangle}$ and $k[[x,y]]$ as well. In this section we take a closer look at the relations between the generators of the minimal monomial reduction of an ideal. It turns out that there is a condition, which in the local case can be used to determine a minimal reduction. In the sequel the considered rings are local.

Example 4.1. Let $I_{mmr} = \langle y^7, xy^5, x^5y^2, x^9\rangle = \langle m_i\rangle_{i=0}^3$ from Example 3.2.



It is clearly seen that if we translate m_1 either horizontally or vertically, then it will intersect the diagonal line from m_0 to m_2. The same is true for m_2 with respect to the line from m_1 to m_3. Algebraically we can describe these relations as, for example, $(xy^5)^4x \mid (y^7)^3(x^5y^2)$ and $(x^5y^2)^2y \mid (xy^5)(x^9)$. We formulate these pictorial relations between three consecutive generators with respect to x:

$$m_1^4x|m_0^3m_2 \text{ and } m_2^5x|m_1^2m_3^3. \tag{4.1}$$

In (4.1) we might as well choose y instead of x and get:

$$m_1^3y|m_0^2m_2 \text{ and } m_2^2y|m_1m_3. \tag{4.2}$$

The minimal monomial reduction I_{mmr} is constructed in such a way that there are relations similar to (4.2) and (4.1) between all the generators. Let $I_{mmr} =$

$\langle x^{A_i}y^{B_i}\rangle_{i=0}^r = \langle m_i\rangle_{i=0}^r$. We have $\frac{A_i-A_{i-1}}{B_{i-1}-B_i} < \frac{A_{i+1}-A_i}{B_i-B_{i+1}} \Leftrightarrow (A_{i+1}-A_{i-1})B_i < (A_{i+1}-A_i)B_{i-1} + (A_i - A_{i-1})B_{i+1}$. If we define $l_i = A_{i+1} - A_{i-1}$ and $c_i = A_{i+1} - A_i$, then

$$\left\{\begin{array}{rcl} l_iA_i & = & c_iA_{i-1} + (l_i - c_i)A_{i+1} \\ l_iB_i + 1 & \leq & c_iB_{i-1} + (l_i - c_i)B_{i+1}, \text{ that is, } m_i^{l_i}y | m_{i-1}^{c_i}m_{i+1}^{l_i-c_i}. \end{array}\right. \tag{4.3}$$

Most often we can choose smaller l_i's and c_i's than the ones defined in (4.3). In Example 4.1 we have $l_1 = 3$ and $c_1 = 2$ while the values suggested by (4.3) are 5 respectively 4.

The relations (4.3) can be expressed as

$$m_{i-1}^{c_i}m_{i+1}^{l_i-c_i} \in m_i^{l_i}\mathfrak{m}. \tag{4.4}$$

Letting $l = \mathrm{lcm}(l_i)$ we can assume that all l_i are equal.

Moreover, since $m_{i-1}^{c_i}m_{i+1}^{l-c_i} \in m_i^l\mathfrak{m}$ and $m_i^{c_{i+1}}m_{i+2}^{l-c_{i+1}} \in m_{i+1}^l\mathfrak{m}$ for any $0 < i < r-2$, we have

$$m_{i-1}^{c_il}(m_i^{c_{i+1}}m_{i+2}^{l-c_{i+1}})^{l-c_i} \in (m_{i-1}^{c_i}m_{i+1}^{l-c_i})^l\mathfrak{m} \subseteq m_i^{l^2}\mathfrak{m}. \tag{4.5}$$

The monomials m_i are nonzerodivisors; hence

$$m_{i-1}^{c'}m_{i+2}^{l'-c'} \in m_i^{l'}\mathfrak{m}. \tag{4.6}$$

Hence, the relations between three consecutive generators can be extended to any triplet of generators.

The argument that deduced (4.6) from (4.4) does not depend on the ring being $k[[x,y]]$ or $k[x,y]_{\langle x,y\rangle}$, and we formulate the result in a more general form.

Lemma 4.2. *Let $(R,\mathfrak{m})$ be a local integral domain and $I = \langle m_i\rangle_{i=0}^r$ an ideal in R. Suppose that for every $1 \leq i \leq r-1$ there are positive integers c_i and l_i, $c_i < l_i$, such that $m_{i-1}^{c_i}m_{i+1}^{l_i-c_i} \in m_i^{l_i}\mathfrak{m}$. Then, for each triplet of indices $i < i' < j$ there are positive integers c and l, $c < l$, such that*

$$m_i^cm_j^{l-c} \in m_{i'}^l\mathfrak{m}. \tag{4.7}$$

There is an alternative way to express (4.7). For any pair of indices i and j such that $j - i \geq 2$, there are positive integers c and l, $c < l$, such that $m_i^cm_j^{l-c} \in I^l\mathfrak{m}$. Multiplying by a proper power of some of these two generators yields the following result.

Proposition 4.3. *Let $(R,\mathfrak{m})$ be a local integral domain and $I = \langle m_i\rangle_{i=0}^r$ an ideal in R. Suppose that for every $1 \leq i \leq r-1$ there are positive integers c_i and l_i, $c_i < l_i$, such that $m_{i-1}^{c_i}m_{i+1}^{l_i-c_i} \in m_i^{l_i}\mathfrak{m}$. Then there is an integer l such that $m_i^lm_j^l \in I^{2l}\mathfrak{m}$ for any two indices i and j such that $j - i \geq 2$.*

Going back to our starting point of monomial ideals in $k[[x,y]]$ or $k[x,y]_{\langle x,y\rangle}$, the proposition above states that given a monomial ideal I and its $I_{mmr} = \langle m_i\rangle$, for any two indices i and j such that $j - i \geq 2$ there is an integer l such that $m_i^lm_j^l \in I^{2l}\mathfrak{m}$.

Next we determine minimal reductions for a class of ideals in any local commutative ring. Further we will show that the minimal monomial reductions we defined earlier belong to this class. That way we will be able to determine a minimal reduction J of any monomial ideal I in , since $J \subseteq I_{mmr} \subseteq I \subseteq \bar{I}$ where I_{mmr} is integral over J. Hence, J is a minimal reduction of any ideal between J and $\bar{I}$.

Let $(R,\mathfrak{m})$ be a local ring and $I=\langle m_i\rangle_{i=0}^r$ an ideal in it. Suppose that some of the generators satisfy the condition

$$m_i^l m_j^l \in I^{2l}\mathfrak{m} \tag{4.8}$$

for some integer l if $i\neq j$. A reduction of an ideal, the generators of which satisfy (4.8), can be expressed in a quite convenient way. Before showing that we need two lemmas. The first one is Lemma 2 on p.147 in [10].

Lemma 4.4. *Let $J\subseteq I$ be ideals. Then J is a reduction of I if and only if $J+\mathfrak{m}I$ is a reduction of I.*

Proof. If $JI^l=I^{l+1}$, then $(J+\mathfrak{m}I)I^l=JI^l+\mathfrak{m}I^{l+1}=I^{l+1}$.

If $(J+\mathfrak{m}I)I^l=I^{l+1}$, then we use Nakayama's lemma on $\mathfrak{m}(I^{l+1}/JI^l)=(JI^l+\mathfrak{m}I^{l+1})/JI^l=((J+\mathfrak{m}I)I^l)/JI^l=I^{l+1}/JI^l$ which gives us $I^{l+1}/JI^l=\bar{0}$ and hence $I^{l+1}=JI^l$. The proof is complete. □

Lemma 4.5. *Let $I=\langle m_i\rangle_{i=0}^r$ be an ideal. Assume that there is an ideal $J\subseteq I$ and an integer l such that $m_i^l\in JI^{l-1}+\mathfrak{m}I^l$ for all i. Then J is a reduction of I.*

Proof. Let $l'=(l-1)r$, then $I^{l'+1}=\langle\prod_{i=0}^r m_i^{l_i}\mid\sum_{i=0}^r l_i=l'+1\rangle$. For every generator (product) there is some index k such that $l_k\geq l$, according to the pigeon hole principle. Then $m_k^{l_k}\in(JI^{l-1}+\mathfrak{m}I^l)I^{l_k-l}$ and thus $\prod_{i=0}^r m_i^{l_i}=m_k^{l_k}\big(\prod_{i\neq k}m_i^{l_i}\big)\in JI^{l'}+\mathfrak{m}I^{l'+1}$. Thus, $I^{l'+1}\subseteq JI^{l'}+\mathfrak{m}I^{l'+1}$ and we are done due to Lemma 4.4. □

Theorem 4.6. *Let $I=\langle m_i\rangle_{i=0}^r$ be an ideal in a local ring $(R,\mathfrak{m})$. Assume that there is a partition of the index set $\{0,\dots,r\}=\cup S_\alpha$ such that if $i,j\in S_\alpha$ and $i\neq j$, then $m_i^l m_j^l\in I^{2l}\mathfrak{m}$ for some integer l (that depends on i and j).*

Let further $J=\big\langle\sum_{i\in S_\alpha}m_i\big\rangle$, that is every generator of J is the sum of all m_i with $i\in S_\alpha$ and there is a generator for each α. Then J is a reduction of I.

Proof. If $|S_\alpha|=1$ for all α, then the ideal $J=I$ is trivially a reduction.

Suppose that $|S_\alpha|\geq 2$ for some α. For that α define $p_\alpha=\sum_{i\in S_\alpha}m_i$ and fix some $k\in S_\alpha$. By assumption $m_k^l m_i^l\in I^{2l}\mathfrak{m}$ for all $k\neq i\in S_\alpha$. Let $l'=|S_\alpha|\cdot(l-1)$. Then for any t, such that $0\leq t\leq l+l'-1$, we have

$$\begin{aligned}&m_k^{l+l'-t}(p_\alpha-m_k)^{t+1}+m_k^{l+l'-t-1}(p_\alpha-m_k)^{t+2}=\\&m_k^{l+l'-t-1}(p_\alpha-m_k)^{t+1}p_\alpha\in I^{l+l'}J,\\&\text{that is, }\big[m_k^{l+l'-t}(p_\alpha-m_k)^{t+1}\big]=\big[-m_k^{l+l'-t-1}(p_\alpha-m_k)^{t+2}\big]\text{ in }R/JI^{l+l'}.\end{aligned}\tag{4.9}$$

Using this we rewrite the zero element in the quotient ring $R/JI^{l+l'}$ as follows:

$$\begin{aligned}&\big[m_k^{l+l'}p_\alpha\big]=\big[m_k^{l+l'+1}+m_k^{l+l'}(p_\alpha-m_k)\big]\overset{(4.9)}{=}\\&\big[m_k^{l+l'+1}-m_k^{l+l'-1}(p_\alpha-m_k)^2\big]=\ldots=\big[m_k^{l+l'+1}\pm m_k^l(p_\alpha-m_k)^{l'+1}\big]=\\&\big[m_k^{l+l'+1}\pm m_k^l\big(\sum_{\substack{\sum l_i=\\ l'+1}}\beta_{\dots}\big(\prod_{\substack{i\neq k,\\ i\in S_\alpha}}m_i^{l_j}\big)\big)\big],\end{aligned}\tag{4.10}$$

where the $\beta_{\dots}$'s are the multinomial coefficients. According to the pigeon hole principle there is some k' in each product $\prod m_i^{l_i}$ such that $l_{k'}\geq l$. For that k' we have

$$m_k^l\big(\prod_{\substack{i\neq k,\\ \sum l_i=l'+1}} m_i^{l_i}\big)\in I^{2l}\mathfrak{m}I^{l'+1-l}=I^{l+l'+1}\mathfrak{m}. \tag{4.11}$$

From (4.10) and (4.11) we deduce that $m_k^{l+l'+1}\in JI^{l+l'}+I^{l+l'+1}\mathfrak{m}$. Since the index k was chosen arbitrarily it follows that there is an integer L such that $m_i^L\in JI^{L-1}+I^L\mathfrak{m}$ for all i. Lemma 4.5 completes the proof. □

Example 4.7. Let $I=\langle x^3yz^2, x^2y^2z, xy^3z, y^4z^2\rangle=\langle m_i\rangle_{i=0}^3\subset R=k[[x,y,z]]$. We have $m_0m_2\in m_1^2\mathfrak{m}$ and $m_1m_3\in m_2^2\mathfrak{m}$. Hence we partition the indices into $\{0,2\}\cup\{1,3\}$. Theorem 4.6 gives us a minimal reduction of I equal to $\langle x^3yz^2+xy^3z, x^2y^2z+y^4z^2\rangle$.

Example 4.8. Let $I=\langle x^4,x^2y,y^4,y^2z,z^4,z^2x\rangle=\langle m_i\rangle_{i=0}^5\subset R=k[[x,y,z]]$. This is an $\mathfrak{m}$-primary ideal of dimension zero, hence the number of the generators of a minimal reduction of it must be at least three. We see that $(x^2y)^2z\,|\,(x^4)(y^2z)$, that is, $m_0m_3\in(x^2y^2)\mathfrak{m}$. By symmetry reasons there are equivalent relations for the other generators. We partition the indices into $\{0,3\}\cup\{2,5\}\cup\{1,4\}$ which gives a minimal reduction $\langle x^4+y^2z, y^4+z^2x, z^4+x^2y\rangle$.

We return to monomial ideals in a two-dimensional local ring R. A minimal reduction of an $\mathfrak{m}$-primary ideal is generated by two elements in this ring.

Corollary 4.9. *Let $I=\langle m_j\rangle_{j=0}^s$ be a monomial ideal in $k[[x,y]]$, with infinite residue field, and $I_{mmr}=\langle m_i\rangle_{i=0}^r$ its minimal monomial reduction where the generators are ordered in the lexicographic order. Then $J=\langle\sum_{\text{even } i}m_i, \sum_{\text{odd } i}m_i\rangle$ is a reduction of I.*

Moreover, J is a reduction of any ideal between J and $\bar{I}$.

Remark 4.10. Under the assumption that I has maximal analytic spread J is a minimal reduction of I.

Remark 4.11. This same result was obtained independently by Chan and Liu in [3], using a completely different technique. Namely, they investigate which monomial ideals have a given polynomial ideal as a (not necessarily minimal) reduction in the ring $k[[x,y]]$. Our result is a special case of Theorem 4.6. In fact, for some monomial ideals in Corollary 4.9 one can obtain other minimal reductions using different partitions from Theorem 4.6.

Proof. We have shown that the relation (4.8) is valid for all generators of I_{mmr} except for any two consecutive. Hence two consecutive generators must belong to different subsets of the partition. Thus the only possible choice of partition is to split the indices into odd ones and even ones. The rest follows from the theorem. □

Example 4.12. Consider Example 3.2 with the restriction that the ring is local. The ideal $J=\langle y^7+x^5y^2, xy^5+x^9\rangle$ is a minimal reduction of all ideals lying between J and $\bar{I}$.

Example 4.13 (A minimal reduction of the maximal ideal in monomial subrings)**.** Let $k[[x,y]]$ and let $k[[m_i]]\cong k[[A_i]]/I$ be a monomial subring in it. Let the defining relations in it , that is the generators of I, be all the binomials $A_{i-1}A_{i+1}-A_i^3$ and possibly other polynomials. We write down the first five such subrings:

$$
\begin{aligned}
R_1 &= k[[x^3, xy, y^3]] \cong k[[A_1, A_2, A_3]]/\langle A_1A_3 - A_2^3\rangle \\
R_2 &= k[[x^8, x^3y, xy^3, y^8]] \cong \\
&\cong k[[A_1, A_2, A_3, A_4]]/\langle A_1A_3 - A_2^3, A_2A_4 - A_3^3, A_1A_4 - (A_2A_3)^2\rangle \\
R_3 &= k[[x^{21}, x^8y, x^3y^3, xy^8, y^{21}]] \\
R_4 &= k[[x^{55}, x^{21}y, x^8y^3, x^3y^8, xy^{21}, y^{55}]] \\
R_5 &= k[[x^{144}, x^{55}y, x^{21}y^3, x^8y^8, x^3y^{21}, xy^{55}, x^{144}]].
\end{aligned}
\tag{4.12}
$$

The other generators of I except for the binomials $A_{i-1}A_{i+1} - A_i^3$, for example $A_1A_4 - A_2^2A_3^2$ in R_2, are irrelevant in this situation. Since the generators of the maximal ideal $\mathfrak{m} = \langle A_i\rangle$ satisfy (4.8), a minimal reduction of $\mathfrak{m}$ is equal to $\langle \sum_{\text{odd } i} A_i\rangle + \langle \sum_{\text{even } i} A_i\rangle$.

We can determine a minimal reduction of the maximal ideal in any monomial subring $k[[A_i]]/I \subset k[[x, y]]$ as long as the relations in I satisfy (4.8). For example, $\langle a + b, c\rangle$ is a minimal reduction of $\langle a, b, c\rangle$ in the ring $k[[a, b, c]]/\langle ac - b^3\rangle \cong k[[x^3, xy^2, y^6]]$.

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